Homotopy theory and the derived moduli of Stokes data

(joint with Porta)

TH $X/\mathbb{C}$ smooth D normal crossing divisor

$\mathcal{I} \subseteq \mathcal{O}_X(*D)/\mathcal{O}_X$ good sheaf of irregular values

k simplicial commutative ring

Then, the derived stack

$$\mathrm{St}_{\mathcal{I}} \quad \mathrm{dAff}_k \longrightarrow \mathrm{Spc} = \infty\text{-groupoids}$$

$$A \longrightarrow \{\mathcal{I}\text{-Stokes functors with values in } \mathrm{Perf}\, A\}$$

is geometric

Goal Explain 1) what this means

2) why we care

3) ideas of proof

Plan I Exodromy

II Stokes data from classical to ∞-categorical

III $\mathrm{St}_{\mathcal{I}}$ is geometric

I Exodromy

(after MacPherson, Treumann, Lurie, Clausen - Jansen, Haine - Porta - T)

1) Exodromic spaces

$X \in \mathrm{Top}$

$\Pi_1(X)$ fundamental groupoid

$\mathrm{Ob}\, \Pi_1(X)$ = points of X

$\mathrm{Hom}_{\Pi_1(X)}(x,y) = \{$ homotopy classes of C^0 path γ $x \to y$ $\}$

Monodromy equivalence

$$\mathrm{Loc}(X, \mathrm{Set}) \simeq \mathrm{Fun}(\Pi_1(X), \mathrm{Set})$$

Remark $\Pi_1(X)$ ignores simplices of dimension > 1

To remedy this, we need to look at

$\Pi_\infty(X)$ = simplicial set of all C^0 simplices σ $|\Delta^n| \to X$

Then, the classical monodromy equivalence upgrades to the following

TH (Haine - Porta - T)

X locally weakly contractible. Then

$$\mathrm{Loc}(X, \mathrm{Spc}) \simeq \mathrm{Fun}(\Pi_\infty(X), \mathrm{Spc})$$

Question Is there a similar functor description for constructible sheaves?

To think about this question in a tractable way, let us introduce the

Def For $C \in \mathrm{Pr}^L$, say that $x \in C$ is atomic if

$$\mathrm{Map}_C(x, -) \quad C \to \mathrm{Spc}$$

commutes with colimits

$C^{at} \subseteq C$ full subcategory spanned by atomic objects

$$\begin{tikzcd} C^{at} \arrow[r, hook, "\iota"] \arrow[d, "\text{Yoneda}"'] & C \\ \mathrm{Fun}(C^{at,op}, \mathrm{Spc}) \arrow[ur, "\exists! \, L\iota \text{ colimit preserving}"'] & \end{tikzcd}$$

<u>Def</u> C is atomically generated if $L\iota$ is an equivalence. In which case

$$C \simeq \mathrm{Fun}(C^{at,op}, \mathrm{Spc})$$

<u>Upshot</u> asking that a presentable category C is of the form $\mathrm{Fun}(-, \mathrm{Spc})$ amounts to ask if C is atomically generated

<u>Ex</u> (Lurie) $C \in \mathrm{Cat}_\infty$. Then

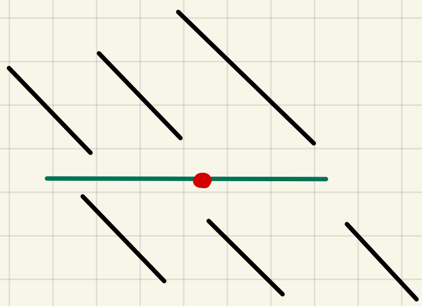
$$\mathrm{Fun}(C^{op}, \mathrm{Spc})^{at} = \{ \text{retracts of representable functors} \}$$

<u>Ex</u> $X \in \mathrm{Top}$ locally weakly contractible

$$\mathrm{Loc}(X, \mathrm{Spc})^{at} \simeq \Pi_\infty(X)^{op}$$

$$(X, P) \text{ stratified space } \begin{cases} X \in \mathrm{Top}, \ P \text{ poset} \\ f \ X \to P \ C^0 \end{cases}$$

where P is endowed with the topology whose open subsets are closed upwards subsets



$$\downarrow$$

$$\{ 0 \le 1 \le 2 \}$$

Def We say that (X,P) is exodromic if

(1) $\mathrm{Cons}_P(X, \mathrm{Spc})$ is atomically generated

(2) $\mathrm{Cons}_P(X, \mathrm{Spc}) \subseteq \mathrm{Sh}(X, \mathrm{Spc})$ is stable under limits and colimits

Important condition but we are not going to use it

In that case, if we put

$$\Pi_\infty(X,P) = \mathrm{Cons}_P(X, \mathrm{Spc})^{at, op}$$

we have

$$\mathrm{Cons}_P(X, \mathrm{Spc}) \simeq \mathrm{Fun}(\Pi_\infty(X,P), \mathrm{Spc}) \qquad (*)$$

<u>Terminology</u> $\Pi_\infty(X,P)$ is the stratified homotopy type of (X,P)

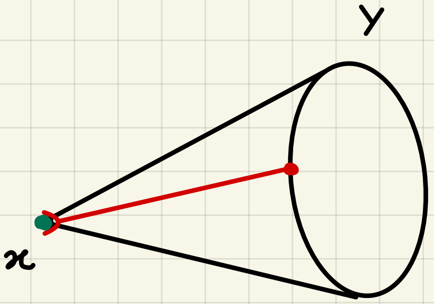
The equivalence $(*)$ is the Exodromy equivalence

2) Examples

$f : Y \to Q$ stratified space

$Q^{\triangleleft} = Q \sqcup \{-\infty\}$

↳ smallest element



$C(Y)$ = cone of Y

$$C(Y) \longrightarrow Q^{\triangleleft}$$

$$x \longrightarrow -\infty$$

$$(t \neq 0, y) \longrightarrow f(y)$$

Def (Lurie) (X,P) is conical if every point of X admits an open neighbourhood U such that there is an homeomorphism of stratified spaces

$$(U, P|_U) \simeq (Z \times C(Y), Q^{\triangleleft})$$

In a conical situation, the atomic objects of $\mathrm{Cons}_P(X, \mathrm{Spc})$ admit a very concrete description

<u>Def</u> (MacPherson-Lurie)

Define a sub-simplicial set $\mathrm{Exit}(X,P) \subseteq \Pi_\infty(X)$ by

- $\mathrm{Exit}(X,P)_0 = \Pi_\infty(X)_0 = $ points of X

- $\mathrm{Exit}(X,P)_1 \subseteq \Pi_\infty(X)_1$
$= \{ C^0$ path γ $x \to y$ whose image away from x sits in a less deeps stratum than that of x

For the higher dimensional cells, see Lurie Higher Algebra

The paths in $\mathrm{Exit}(X,P)_1$ are the exit paths of (X,P)

exit paths

Not exit paths

<u>TH</u> (Lurie) Every (X,P) conical is exodromic and

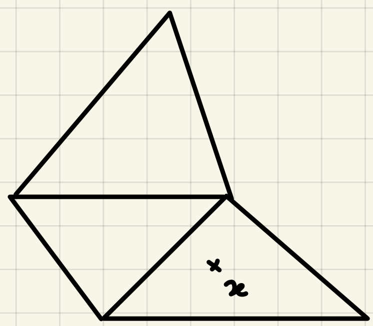
$$\Pi_\infty(X,P) \simeq \mathrm{Exit}(X,P)$$

<u>Ex 1</u> $X \in \mathrm{Top}$ + triangulation

$V \subset X$ set of vertices

$K \subset \mathcal{P}^f(V)$ set of faces such that

↪ finite subset in V



$\forall \sigma \in K,\ \tau \subseteq \sigma \Rightarrow \tau \in K$

K is a poset via the inclusion $\subset$

$X \longrightarrow K$

$x \longmapsto$ Face in the interior of which x lies

$\rightsquigarrow$ stratification of X whose strata are the interior of each face

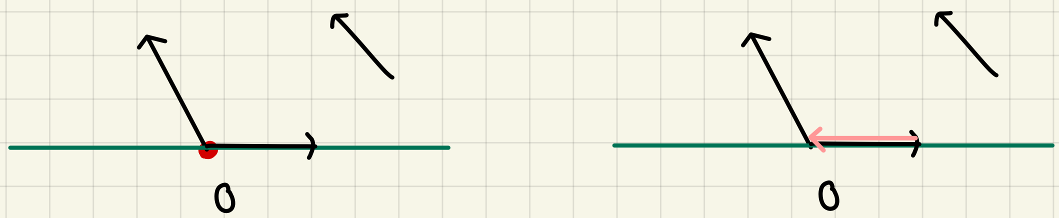
<u>Fact</u> (X, K) is conical and $\mathrm{Exit}(X, K) \xrightarrow{\sim} K$

⚠ Most stratified spaces are not conical

<u>Fact</u> (Jansen, Haine-Porta-T)

If
$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "\text{conical}"'] & & P \\ & R \arrow[ur] & \end{tikzcd}$$
admits a conical refinement then (X, P) is exodromic.

Furthermore, there is a concrete relation between $\Pi_\infty(X, P)$ and $\mathrm{Exit}(X, R)$



The new exit-path $\leftarrow$ that appears in $\mathrm{Exit}(X, P)$ is inverse to an exit path in $\mathrm{Exit}(X, R)$ lying in a single P-stratum

This is a general phenomenon

Put $W := \{ \gamma \in \mathrm{Exit}(X, R)$ in a P-stratum $\}$

Then
$$\Pi_\infty(X, P) \simeq \mathrm{Exit}(X, R)[W^{-1}]$$

<u>Ex</u> $P = *$, then $\Pi_\infty(X) \simeq \mathrm{Exit}(X, R)[\text{all arrows}^{-1}]$

<u>Ex 2</u> X analytic space, $X \to P$ locally finite analytic stratification

Then (X, P) is refined by a triangulation

<u>Ex 1</u> + <u>Fact</u> $\Rightarrow$ (X, P) is exodromic

3) $\underline{\mathrm{Cons}}_P$ is geometric

k ring, (X,P) stratified space

$\underline{\mathrm{Cons}}_P(X) \quad \mathrm{dAff}_k \longrightarrow \mathrm{Spc}$ ↪ inner groupoid

$\qquad A \longrightarrow \mathrm{Cons}_{P,\omega}(X, \mathrm{Mod}\,A)^{\simeq}$ (just keep the equivalences)

↳ compact germs

TH (Haine - Porta - T)

Assume that (X,P) is exodromic and that $\Pi_\infty(X,P) \in \mathrm{Cat}_\infty$ is compact

Then $\underline{\mathrm{Cons}}_P(X)$ is geometric

Why would we care?

- This implies the geometricity of the derived stack of perverse sheaves
- The proof of the geometricity of Stokes data is modeled on that for for $\underline{\mathrm{Cons}}_P$. So by understing the method in this simple case, we know what to look for in the Stokes situation

The proof is a combination of Exodromy with Toen - Vaquié's work

$\mathcal{C} \in \mathrm{Pr}_k^{L,\omega}$ stable

$\mathcal{M}_{\mathcal{C}} \quad \mathrm{dAff}_k \longrightarrow \mathrm{Spc}$

$\qquad A \longrightarrow \mathrm{Fun}_k^{st}(\mathcal{C}^{\omega,op}, \mathrm{Perf}\,A)^{\simeq}$

st = exact functors

k = k- linear functors

Def $\mathcal{M}_{\mathcal{C}}$ is Toen - Vaquié's moduli of objects

TH If C is a compact object in $\mathrm{Pr}^{L,\omega}_{k}$, then $\mathcal{X}_C$ is geometric

The geometricity of $\underline{\mathrm{Cons}}_P$ is then an immediate consequence of

TH Let (X,P) exodromic stratified space. Then

1) $C = \mathrm{Cons}_P(X, \mathrm{Mod}\,k) \in \mathrm{Pr}^{L,\omega}_{k}$ and is stable

2) If $\Pi_\infty(X,P)$ is compact in Cat_∞, then

$$C = \mathrm{Cons}_P(X, \mathrm{Mod}\,k)$$

is compact in $\mathrm{Pr}^{L,\omega}_{k}$

3) $\mathcal{X}_C \simeq \underline{\mathrm{Cons}}_P(X)$ $\blacksquare$

Here is a handy exodromic case where 2) holds

TH Let (X,P) analytic with P finite and X compactifiable

Then $\Pi_\infty(X,P)$ is finite

Ex For X algebraic and $P = *$, this gives back Lefschetz's finiteness for the homotopy type of an algebraic variety