II Stokes data from classical to ∞-categorical

1) Good formal decomposition

$X/\mathbb{C}$ smooth, D sncd

$(\mathcal{M}, \nabla)$ meromorphic flat bundle with poles along D

$\mathcal{M}$ = loc free module of finite type over $\mathcal{O}_X(*D)$

$\nabla \quad \mathcal{M} \longrightarrow \Omega^1_X(*D) \underset{\mathcal{O}_X(*D)}{\otimes} \mathcal{M}$ flat

Ex $\quad a \in \mathcal{O}_X(*D)$, $\mathcal{E}^a = (\mathcal{O}_X(*D), d - da)$

Ex $\quad$ Regular singular (∇ has log poles in local basis)

Deligne

$$\begin{tikzcd}
\{\text{Regular singular}\} \arrow[r, "\sim"] & \mathrm{Loc}(U, \mathbb{C}) \\
(\mathcal{M}, \nabla) \arrow[r] & \mathrm{Ker}\, \nabla|_U \\
\bigcap & \bigcap \\
\left\{\begin{matrix}\text{"good" Meromorphic} \\ \text{flat bundle}\end{matrix}\right\} \arrow[r, "\sim"] & \left\{\begin{matrix}\text{Stokes filtered local} \\ \text{systems}\end{matrix}\right\}
\end{tikzcd}
\qquad U = X \setminus D$$

Goal $\quad$ explain the bottom line of the above square

Def $\quad$ We say that $(\mathcal{M}, \nabla)$ has good formal decomposition at $x \in D$ if

(1) $\quad \mathcal{O}_{\widehat{X,x}} \underset{\mathcal{O}_{X,x}}{\otimes} \mathcal{M} \simeq \underset{a \in \mathcal{O}_{X,x}(*D)}{\bigoplus} \mathcal{E}^a \otimes R_a$ $\quad$ regular singular

(2) $\quad$ If $D = \{z_1 \cdots z_k = 0\}$ near x,

the non zeros a's in (1) have the form $u/z_1^{m_1} \cdots z_k^{m_k}$, u unit

the non zeros $a - b$'s have the form $u/z_1^{m_1} \cdots z_k^{m_k}$, u unit

Def The a's contributing to (1) are the irregular values of $(\mathcal{M},\nabla)$ at x

Ex Good formal decomposition always holds for X curve

Good formal decomposition is not automatic in dim ≥ 2

However, we have the following amazing theorem

TH (Kedlaya, Mochizuki)

$\exists\ \pi : Y \longrightarrow X$ composition of blow-up above D such that $\pi^*(\mathcal{M},\nabla)$ has good formal decomposition at every point

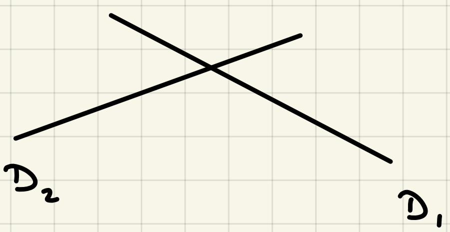
2) Stokes filtered local systems

To extract Stokes data from a flat bundle with good formal decomposit°
we need to work in sectors that emanate from points of D
The most convenient way to do this is to work on the real blow-up

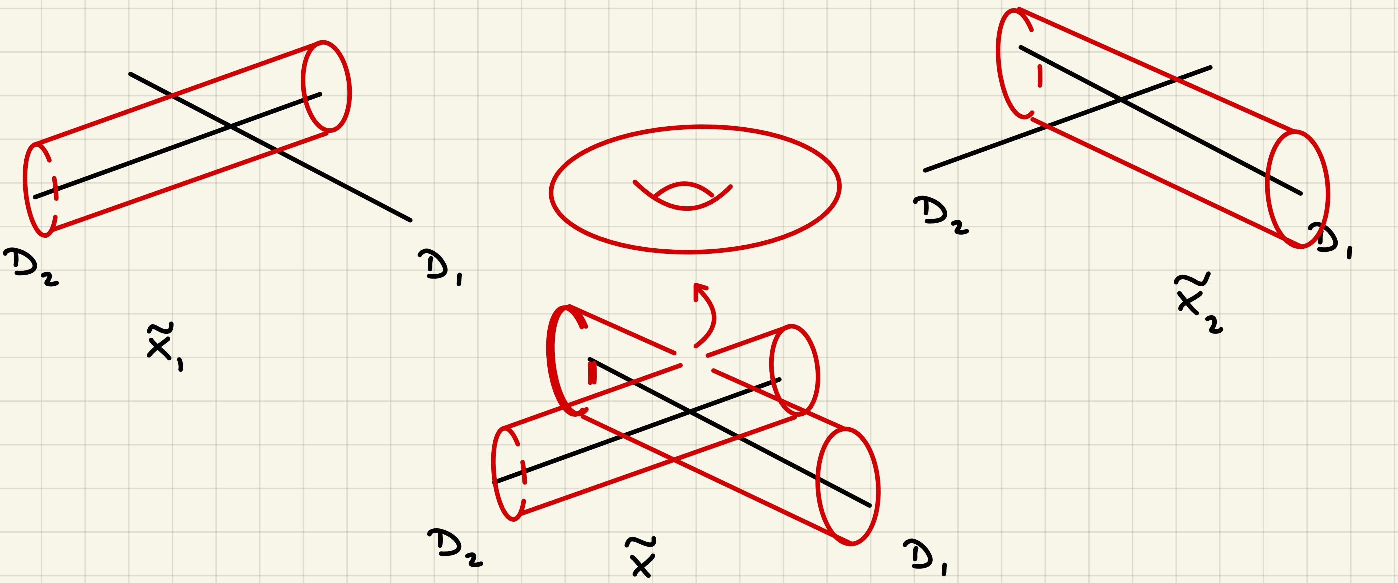
$D_1, \dots, D_m$ irreducible components of D

$\widetilde{X}_i \longrightarrow X$ real blow-up of X along D_i

$\partial \widetilde{X}_i$ is a fibration in circle over D_i



$\pi : \widetilde{X} = \widetilde{X}_1 \times_X \cdots \times_X \widetilde{X}_m \longrightarrow X$



Ex $\quad X = \mathbb{C}^2, \quad D = \{xy = 0\}$

$\quad \widetilde{X} \simeq \mathbb{R}_+^2 \times S^1 \times S^1$

In general, working on $\widetilde{X}$ amounts to work in polar coordinates

$$\mathcal{L} = \operatorname{Ker} \nabla|_U \rightsquigarrow L = j_* \mathcal{L} \in \operatorname{Loc}(\widetilde{X})$$

The topology of $\widetilde{X}$ is such that the germs of L are the sections of $\mathcal{L}$ on a product of small sectors

Our goal now is to define a filtration on the germs of L

It will be indexed by an ordered set cooked up with the irregular values

The order will vary when x varies

Assumption $\quad (\mathcal{M}, \nabla)$ good at every point $x \in D$

Fact $\quad$ The irregular values of $(\mathcal{M}, \nabla)$ organize as a subsheaf of finite sets

$$\mathcal{I} \subseteq \mathcal{O}_X(*D)/\mathcal{O}_X$$

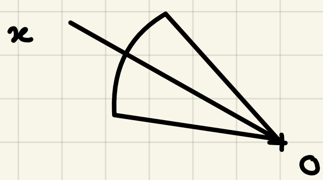
A subsheaf $\mathcal{I}$ as above is a good sheaf of irregular values

$$\mathcal{I} \in Sh(X, Set) \rightsquigarrow \pi^{-1}\mathcal{I} \in Sh(\widetilde{X}, Set)$$

We are going to promote $\pi^{-1}\mathcal{I}$ into a sheaf of posets

For $a, b \in (\pi^{-1}\mathcal{I})_x$, write

$$a \leq_x b \iff e^{a-b} \text{ has moderate growth in a small multi-sector containing the direction } x$$



For $U \subseteq \tilde{X}$ open and $a, b \in (\pi^{-1}\mathcal{I})(U)$, write

$$a \leq_U b \iff a_x \leq_x b_x \quad \forall x \in U$$

From the orders $\leq_U$, we get $\pi^{-1}\mathcal{I} \in Sh(\tilde{X}, \mathrm{Poset})$

We can now define the Stokes filtration

For a section $a \in \pi^{-1}\mathcal{I}$, put

$$L_{\leq a} = \{ f \in L \mid e^{-a} f \text{ has moderate growth along } D \}$$

The $(L_{\leq a})_a$ is the Stokes filtration of $(\mathcal{M}, \nabla)$

The Stokes filtration satisfies two strong constrains

TH (Mochizuki)

(1) Splitting condition

$\forall x \in \tilde{X}$, $\exists$ a decomposition

$$L_x = \bigoplus_{a \in \mathcal{I}_{\pi(x)}} V_a$$

by subspaces V_a such that $\forall a \in \mathcal{I}_{\pi(x)}$,

$$L_{\leq a, x} = \bigoplus_{b \leq_x a} V_b$$

(2) Induction condition

" The Stokes filtration at x determines that at a close enough y "

More precisely, for $x \in D$ one shows that there is an open contractible neighbourhood U of x such that

$$(\pi^{-1}\mathcal{I}, \leq_U) \xrightarrow{\sim} (\mathcal{I}_{\pi(x)}, \leq_x)$$

For $y \in U$, we thus get a morphism of posets

$$\kappa \quad (\mathcal{I}_{\pi(x)}, \leq_x) \longrightarrow (\mathcal{I}_{\pi(y)}, \leq_y)$$

Since L is a local system on $\tilde{X}$, there is a canonical parallel transport map

$$\iota \quad L_x \xrightarrow{\sim} L_y$$

Then $\forall\, y \in U \ \forall\, a \in \mathcal{I}_{\pi(y)}$, we have

$$L_{\leq a, y} = \sum_{\substack{b \in \mathcal{I}_{\pi(x)} \\ \kappa(b) \leq_y a}} \iota(\mathcal{L}_{\leq b, x})$$

We now turn the above properties into a definition

Def The data of $(L, L_{\leq a})$ where

- $L \in \mathrm{Loc}(\tilde{X})$
- $(L_{\leq a})_a$ is a filtration indexed by $\mathcal{I}$ satisfying the splitting and induction conditions

is a $\mathcal{I}$- Stokes filtered local systems

TH (Deligne - Malgrange dim 1, Mochizuki dim > 1)

$$\left\{ \begin{array}{c} \mathcal{I}\text{- good meromorphic} \\ \text{flat bundles} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \mathcal{I}\text{- Stokes filtered} \\ \text{local systems} \end{array} \right\}$$

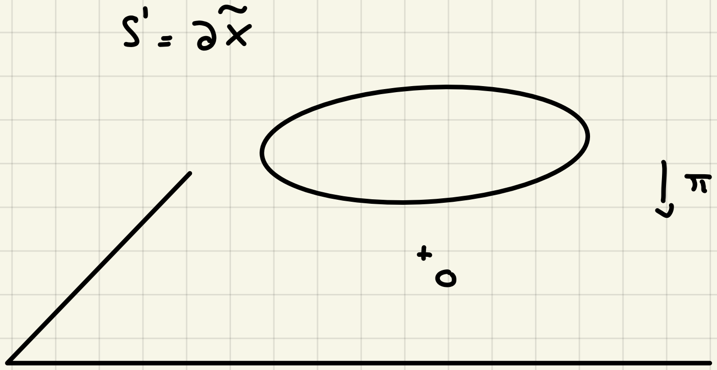
$$(\mathcal{M}, \nabla) \longrightarrow (L, (L_{\leq a})_a)$$

3) Example

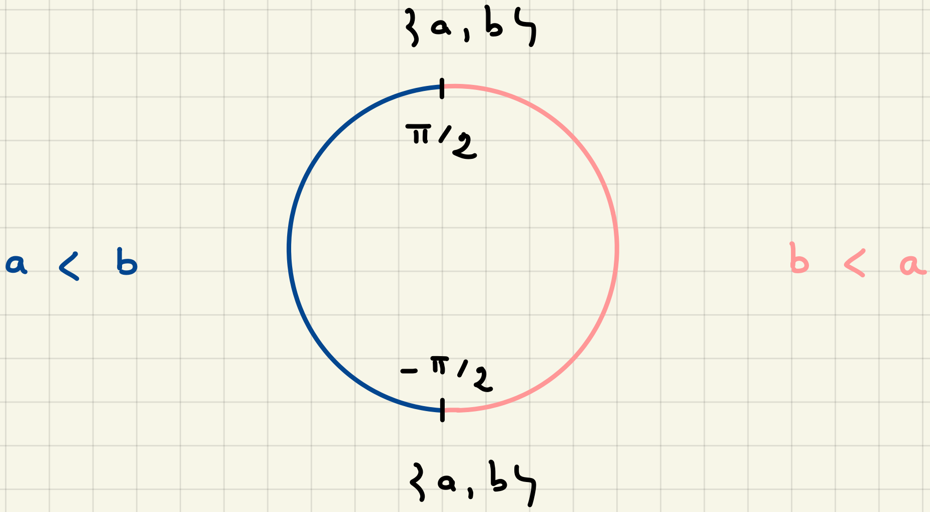
$X = \mathbb{C}$, $D = \{0\}$

$\mathcal{I} = \{ \underbrace{\tfrac{1}{x}}_{a} , \underbrace{-\tfrac{1}{x}}_{b} \}$

$e^{a-b} = e^{2/x} \sim e^{(2\cos\theta)/r}$



So as a sheaf of posets, $\pi^{-1}\mathcal{I}$ looks like



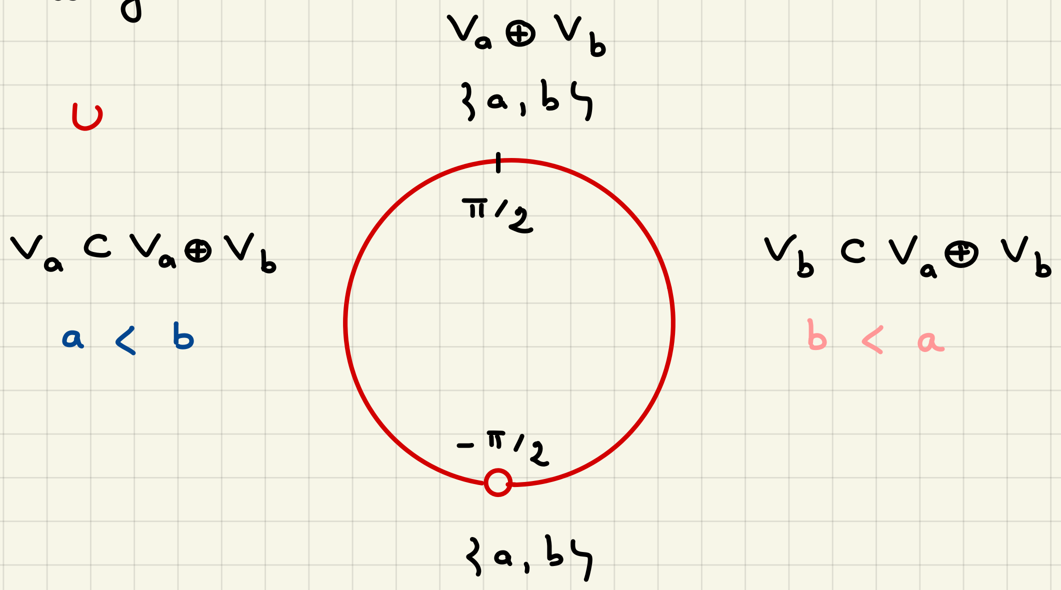
To understand a Stokes filtered local system for this $\mathcal{I}$, it is enough to understand what happens on an open cover and then see how they glue. We are going to use the cover $U = S^1 \setminus \{-i\}$ and $U' = S^1 \setminus \{i\}$

- At $x = i$, a and b cannot be compared, so the splitting condition just amounts to the data of a decomposition of L_x into two subspaces V_a and V_b

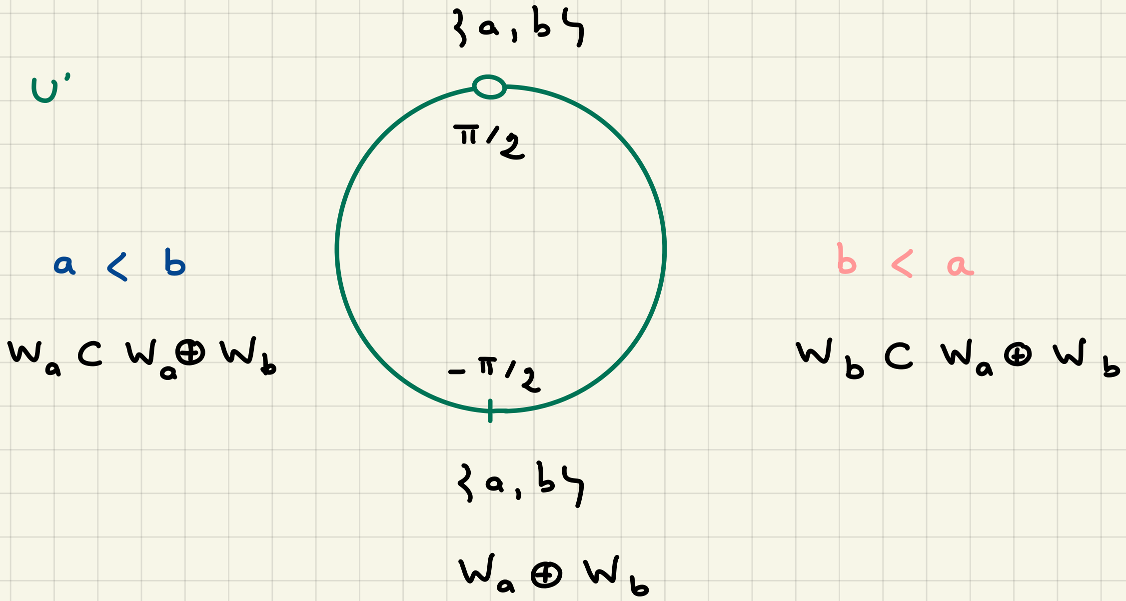
- For $y \in)$, the induction condition expresses the Stokes filtration at y in terms of that at x. In this case, the morphism κ is

$$\{a, b\} \longrightarrow \{a < b\}$$

We thus get



We proceed similarly on V



Since the sheaf of poset $\pi^{-1}\mathcal{I}$ is constant on (and), a glueing on (and) amounts to morphisms of filtered spaces

$$\begin{array}{c c} & \begin{array}{cc} V_a & V_b \end{array} \\ \begin{array}{c} W_a \\ W_b \end{array} & \begin{pmatrix} \square & \square \\ 0 & \square \end{pmatrix} \end{array} \qquad\qquad \begin{array}{c c} & \begin{array}{cc} V_a & V_b \end{array} \\ \begin{array}{c} W_a \\ W_b \end{array} & \begin{pmatrix} \square & 0 \\ \square & \square \end{pmatrix} \end{array}$$

Upshot For the above $\mathcal{I}$, a $\mathcal{I}$-Stokes filtered local system amounts to the data of two splittings of a vector space + 2 invertible block triangular matrices as above

4) ∞-categorical Stokes data

$X/\mathbb{C}$ smooth, D sncd

$\mathcal{I} \subset \mathcal{G}_X(*D)/\mathcal{G}_X$ good sheaf of irregular values

Fact $\exists$ finite analytic stratification $\tilde{X} \to P$ such that

$$\pi^{-1}\mathcal{I} \in \mathrm{Cons}_P(\tilde{X}, \mathrm{Poset})$$

Since $\mathrm{Poset} \subseteq \mathrm{Cat}_\infty$, a variant of the exodromy theorem for coefficients in Cat_∞ allows us to review $\pi^{-1}\mathcal{I}$ as a functor

$$\pi^{-1}\mathcal{I} \quad \Pi_\infty(\tilde{X}, P) \longrightarrow \mathrm{Cat}_\infty$$
$$x \longrightarrow (\pi^{-1}\mathcal{I})_x \in \mathrm{Poset}$$

Applying the Grothendieck construction to this functor gives rise to a cocartesian fibration over $\Pi_\infty(\tilde{X}, P)$

$$\begin{array}{c} \mathcal{J} \\ \downarrow \\ \Pi_\infty(\tilde{X}, P) \end{array}$$

By definition, the fiber of this fibration over $x \in \Pi_\infty(\tilde{X}, P)$ is the poset $(\pi^{-1}\mathcal{I})_x$ So in a nutshell, what $\mathcal{J}$ does is to organize all the irregular values and their orders in a single category

On the other hands, what a Stokes filtration does is to assign a vector space to each irregular values

So for any $\mathcal{E} \in \mathrm{Pr}^L$, we are going to realize an ∞-categorical Stokes data as functors from $\mathcal{J}$ to $\mathcal{E}$

To do this, we need to reinterpret Mochizuki's splitting and induction conditions as properties of objects of $\mathrm{Fun}(\mathcal{I}, \mathcal{E})$

(1) Splitting

$x \in \Pi_\infty(\tilde{X}, P)$, put $\mathcal{I}_x := (\pi^{-1}\mathcal{I})_x \in \mathrm{Poset}$

Let $\mathcal{I}_x^{set}$ be the underlying set of $\mathcal{I}_x$, reviewed as a trivial poset

Then the identity map is a morphism of posets

$$\iota_x \quad \mathcal{I}_x^{set} \longrightarrow \mathcal{I}_x$$

Since $\mathcal{E}$ has every colimits, the pull-back

$$\iota_x^* \quad \mathrm{Fun}(\mathcal{I}_x, \mathcal{E}) \longrightarrow \mathrm{Fun}(\mathcal{I}_x^{set}, \mathcal{E})$$

admits a left adjoint

$$\iota_{x!} \quad \mathrm{Fun}(\mathcal{I}_x^{set}, \mathcal{E}) \longrightarrow \mathrm{Fun}(\mathcal{I}_x, \mathcal{E})$$

explicitly computed as

$$V = (V_a)_a \longrightarrow i_{x!}(V) \quad \begin{aligned} \mathcal{I}_x &\longrightarrow \mathcal{E} \\ a &\longrightarrow \bigoplus_{b \leq_x a} V_b \end{aligned}$$

Def We say that $F \quad \mathcal{I} \longrightarrow \mathcal{E}$ split at x if $F|_{\mathcal{I}_x} \quad \mathcal{I}_x \longrightarrow \mathcal{E}$ lies in the essential image of $\iota_{x!}$

Remark $\iota_{x!}$ is not essentially surjective in general

Take $\mathcal{I}_x = \{0 < 1\}$, then $\iota_{x!}(V_0, V_1) \in \mathrm{Fun}(\Delta^1, \mathcal{E})$ is

$$V_0 \longrightarrow V_0 \oplus V_1$$

Not every morphism in $\mathcal{E}$ looks like this! ☒

(2) Induction

let $\gamma : x \to y$ be a morphism in $\Pi_\infty(\tilde{X}, P)$

Let $\kappa_\gamma : \mathcal{I}_x \to \mathcal{I}_y$ be its image under $\pi^{-1}\mathcal{I} : \Pi_\infty(\tilde{X}, P) \to \mathrm{Cat}_\infty$

Thus κ_γ is the morphism κ from Mochizuki's induction condition

Then, the pull-back

$$\kappa_\gamma^* : \mathrm{Fun}(\mathcal{I}_y, \mathcal{E}) \longrightarrow \mathrm{Fun}(\mathcal{I}_x, \mathcal{E})$$

admits a left adjoint

$$\kappa_{\gamma !} : \mathrm{Fun}(\mathcal{I}_x, \mathcal{E}) \longrightarrow \mathrm{Fun}(\mathcal{I}_y, \mathcal{E})$$

explicitly computed as

$$G : \mathcal{I}_x \to \mathcal{E} \quad \longrightarrow \quad \kappa_{\gamma !}(G) : \mathcal{I}_y \longrightarrow \mathcal{E}$$

$$a \longrightarrow \operatorname*{colim}_{\substack{b \in \mathcal{I}_x \\ \kappa_\gamma(b) \leq_y a}} G(b)$$

In Mochizuki's setting, the above colim is $\displaystyle\sum_{\substack{b \in \mathcal{I}_{\pi(x)} \\ \kappa(b) \leq_y a}} \iota(\mathcal{L}_{\leq b, x})$

Def we say that $F : \mathcal{I} \to \mathcal{E}$ is <u>cocartesian at γ</u> if the canonical map

$$\iota_{\gamma !}(F|_{\mathcal{I}_x}) \longrightarrow F|_{\mathcal{I}_y}$$

is an isomorphism

Like for Stokes filtered local systems, combining the splitting and induction conditions give rise to the notion of <u>Stokes functors</u>

$$\boxed{\mathrm{St}_{\mathcal{I},\mathcal{E}} = \{ F : \mathcal{I} \to \mathcal{E} \text{ split } \forall x \text{ and cocartesian } \forall \gamma \} \subseteq \mathrm{Fun}(\mathcal{I}, \mathcal{E})}$$

Ex Assume that $\mathcal{I}$ is the 0-sheaf

From a flat bundle perspective, this means that the flat bundles we look at only has 0 as irregular values

This means that it is regular singular

Then, Deligne's correspondence predicts that in this situation, Stokes functor should give back local systems on X

Let us prove it!

We know that the fibers of the fibration $\begin{array}{c}\mathcal{I}\\ \downarrow\\ \Pi_\infty(\widetilde{X}, P)\end{array}$ above a point x are the irregular values at x.

Here the fibers are singletons $\{0\}$! So

$$\begin{array}{c}\mathcal{I}\\ s\big\downarrow \\ \Pi_\infty(\widetilde{X}, P)\end{array}$$

Hence a Stokes functor in this case is a functor

$$F : \Pi_\infty(\widetilde{X}, P) \longrightarrow \mathcal{E}$$

A filtration indexed by a singleton automatically split $\Rightarrow$ the splitting condition is empty in this case

The induction condition at $\gamma : x \to y$ collapses to the condition that

$$F(\gamma) : F(x) \longrightarrow F(y)$$

is an equivalence. Hence

$$St_{\mathcal{I}, \mathcal{E}} = \{F : \Pi_\infty(\widetilde{X}, P) \to \mathcal{E} \text{ sending every morphisms to equivalences}\}$$

$$\simeq \mathrm{Fun}(\Pi_\infty(\widetilde{X}, P)[\text{all morphisms}^{-1}], \mathcal{E})$$ Def of localization

The map of exodromic stratified spaces

$$(\tilde{X}, P) \longrightarrow (X, *)$$

is a refinement Then we have in the first lecture that

$$\Pi_\infty(\tilde{X}, P)[\text{all morphisms}^{-1}] \simeq \Pi_\infty(\tilde{X})$$

Furthermore, one shows that

$$\Pi_\infty(\tilde{X}) \simeq \Pi_\infty(X)$$

By the Exodromy equivalence, we deduce

$$\mathrm{St}_{\mathcal{I}, \mathcal{E}} \simeq \mathrm{Fun}(\Pi_\infty(X), \mathcal{E}) \simeq \mathrm{Loc}(X, \mathcal{E})$$

Summary

Topology	Category
Local systems	$\mathrm{Fun}(\Pi_\infty(X), \mathrm{Spc})$
Constructible sheaves	$\mathrm{Fun}(\Pi_\infty(X, P), \mathrm{Spc})$
Stokes filtered local systems	Stokes functors $\subseteq \mathrm{Fun}(\mathcal{I}, \mathrm{Spc})$