III Derived moduli of Stokes data

1) A last example

Before moving to moduli, let us make the following

Remark Stokes functors make sense for every cocartesian fibration in posets $\begin{array}{c}\mathcal{I}\\ \downarrow\\ \mathcal{X}\end{array}$ over an abstract ∞-categorical base!

Ex $\mathcal{X} = *$ Then $\mathcal{I}$ is just a poset, and the induction condition is empty since our base category has only one object.
Let $\mathcal{I}^{set}$ be the underlying set of $\mathcal{I}$, viewed as a trivial poset.
Let $\iota : \mathcal{I}^{set} \to \mathcal{I}$ be the identity map viewed as a map of posets.
Then

$$\mathrm{St}_{\mathcal{I},\mathcal{E}} = \text{essential image of } \iota_! : \mathrm{Fun}(\mathcal{I}^{set}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{I}, \mathcal{E})$$

This category is poorly behaved!
In particular, it has no limits nor colimits. Let us see why.

Take $F_\bullet : A \to \mathrm{St}_{\mathcal{I},\mathcal{E}}$ be a diagram and let us perform the naive limit $F = \lim\limits_{\alpha} F_\alpha$ in $\mathrm{Fun}(\mathcal{I}, \mathcal{E})$

$$\alpha \longmapsto F_\alpha$$

By definition, $\forall \alpha \in A$, $\exists V_\alpha : \mathcal{I}^{set} \to \mathcal{E}$ | $F_\alpha \xrightarrow[\varphi_\alpha]{\sim} \iota_!(V_\alpha)$

⚠ The φ_α are unrelated! In particular they are not compatible with the morphisms of A. So $\lim\limits_{A} \varphi_\alpha$ does not make sense.
So there is a priori no reason for $F = \lim\limits_{A} F_\alpha$ to split too.

☹

2) The derived stack of Stokes functors

TH $X/\mathbb{C}$ smooth D normal crossing divisor

$\mathcal{I} \subseteq \mathcal{O}_X(*D)/\mathcal{O}_X$ good sheaf of irregular values

k simplicial commutative ring

Then, the derived stack

$$\mathrm{St}_{\mathcal{I}} \quad \mathrm{dAff}_k \longrightarrow \mathrm{Spc}$$
$$A \longrightarrow \{\text{Stokes functors } F \quad \mathcal{I} \to \mathrm{Perf}\, A\}^{\simeq}$$

is geometric

Remark (for the experts) The geometricity holds also for ramified irregular values

The proof is modeled on that for the stack of constructible sheaves

So the steps are the same We have to show that

Step 1 $\quad \mathrm{St}_{\mathcal{I},\mathrm{Mod}\,k} \in \mathrm{Pr}^{L,\omega}_k$ stable

This is already a highly non-trivial statement since this says in particular that $\mathrm{St}_{\mathcal{I},\mathrm{Mod}\,k}$ has every limits and colimits

Step 2 $\quad \mathrm{St}_{\mathcal{I},\mathrm{Mod}\,k}$ is compact in $\mathrm{Pr}^{L,\omega}_k$

Step 3 $\quad \mathrm{St}_{\mathcal{I}} \simeq$ Toen-Vaquié moduli of objects for $\mathcal{C} = \mathrm{St}_{\mathcal{I},\mathrm{Mod}\,k}$

Like for $\mathrm{Cons}_P(X)$, the combination of these steps with Toen-Vaquié geometricity result does the job

Our goal in these notes is now to

- Explain the proof of Step 1
- Sketch an approach of Step 2 using Exodromy

Step 1

It will follow from the following

TH In situation from the above theorem, take $\mathcal{E} \in \mathrm{Pr}^L$ stable
Then $\mathrm{St}_{\mathcal{I},\mathcal{E}} \subseteq \mathrm{Fun}(\mathcal{I},\mathcal{E})$ is stable under limits and colimits.

As we have seen in the above example, this is completely false over a point
So this is a kind of miracle of the theory
In fact, this is a global phenomenon that pertains to the way the order on the irregular values interact with the stratification P

To make use of the above theorem, let us invoke the following

TH (Ragimov - Schlank)
Let $\mathcal{C} \in \mathrm{Pr}^L$ and let $\mathcal{D} \subseteq \mathcal{C}$ be a full subcategory stable under limits and colimits Then $\mathcal{D} \in \mathrm{Pr}^L$

Combining both theorems gives

$$\mathrm{St}_{\mathcal{I},\mathcal{E}} \in \mathrm{Pr}^L$$

By the adjoint functor theorem applied to the inclusion $\mathrm{St}_{\mathcal{I},\mathcal{E}} \hookrightarrow \mathrm{Fun}(\mathcal{I},\mathcal{E})$ we thus get for free a left and right adjoints

$$\begin{tikzcd} \mathrm{St}_{\mathcal{I},\mathcal{E}} \arrow[r, hook] & \mathrm{Fun}(\mathcal{I},\mathcal{E}) \arrow[l, bend right, "R"'] \arrow[l, bend left, "L"] \end{tikzcd}$$

The existence of L combined with some standard facts from Higher Topos theory ensures that $\mathrm{St}_{\mathcal{I},\mathcal{E}} \in \mathrm{Pr}^{L,\omega}_{k}$ stable as soon as $\mathcal{E}$ is

From now on, put $\mathcal{E} = \mathrm{Mod}\, k$

Step 2

It will follow from

a) Reduction to show that $\mathrm{St}_{\mathcal{I}|_{\pi^{-1}(x)},\mathcal{E}}$ is compact in $\mathrm{Pr}^{L,\omega}_{k}$ $\forall\, x \in X$

b) Reduction to show that $\mathrm{St}_{\mathcal{I}|_{\pi^{-1}(x)},\mathcal{E}}$ is compact in $\mathrm{Pr}^{L,\omega}_{k}$ when the poles order of the irregular values are all the same

c) Proof that $\mathrm{St}_{\mathcal{I}|_{\pi^{-1}(x)},\mathcal{E}}$ is compact in $\mathrm{Pr}^{L,\omega}_{k}$ when the poles order of the irregular values are all the same

Here we will

- Explain the reduction a) using Exodromy
- Admit b)
- Explain c) in a particular case

To do this we will make use of the following lemma twice

Lemma $F \quad A \to \mathrm{Pr}^{L,\omega}_{k}$ finite diagram whose transition functors are both left and right adjoints Then

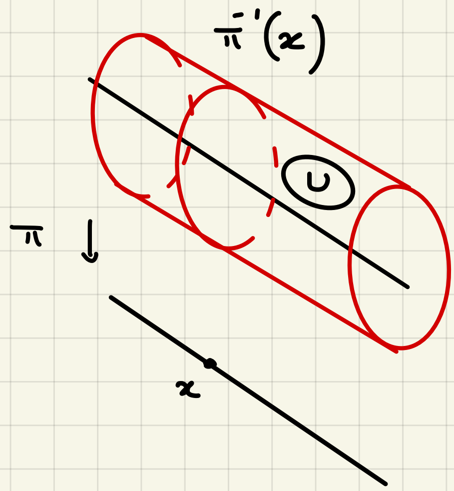
F_α is compact in $\mathrm{Pr}^{L,\omega}_{k}$ $\forall \alpha \Longrightarrow$ $\lim\limits_{A} F$ computed in Cat_∞ is also compact in $\mathrm{Pr}^{L,\omega}_{k}$

a) Reduction to show that $\mathrm{St}_{\mathcal{I}|_{\pi^{-1}(x)},\mathcal{E}}$ is compact

For an open subset $U \subseteq \tilde{X}$, we have $\mathrm{St}_{\mathcal{I}|_U,\mathcal{E}} \in \mathrm{Cat}_\infty$

We are going to put all these categories in a single categorical sheaf called the <u>Stokes sheaf</u>

$$\begin{aligned} \mathfrak{St} \quad \mathrm{Op}(\tilde{X})^{op} &\longrightarrow \mathrm{Cat}_\infty \\ U &\longrightarrow \mathrm{St}_{\mathcal{I}|_U,\mathcal{E}} \end{aligned}$$



In particular,

$$\boxed{\mathrm{St}_{\mathcal{I},\mathcal{E}} = (\pi_* \mathfrak{St})(x)}$$

<u>Fact 1</u> $\quad \mathfrak{St} \in \mathrm{Cons}_P(\tilde{X}, \mathrm{Cat}_\infty)$

<u>Fact 2</u> $\quad \pi_* \mathfrak{St} \in \mathrm{Cons}_Q(X, \mathrm{Cat}_\infty)$ for some $X \to Q$ finite analytic

Fact 2 and Exodromy imply that $\pi_* \mathfrak{St}$ corresponds to a functor

$$F \quad \Pi_\infty(X, Q) \longrightarrow \mathrm{Cat}_\infty$$

Via the exodromy equivalence

$$\begin{aligned} \mathrm{Cons}_P(X, \mathrm{Cat}_\infty) &\xleftrightarrow{\ \sim\ } \mathrm{Fun}(\Pi_\infty(X,Q), \mathrm{Cat}_\infty) \\ \text{global section} &\longleftrightarrow \lim \\ \text{germ at } x \in X &\longleftrightarrow \text{evaluation at } x \end{aligned}$$

Hence

$$St_{\mathcal{G},\mathcal{E}} \simeq (\pi_* \mathcal{S}t)(x) \simeq \lim_{x \in \Pi_\infty(X,Q)} F(x)$$

$$\simeq \lim_{x \in \Pi_\infty(X,Q)} (\pi_* \mathcal{S}t)_x$$

$$\simeq \lim_{x \in \Pi_\infty(X,Q)} St_{\mathcal{G}|_{\pi^{-1}(x)},\mathcal{E}}$$ base change

We have seen in Lecture 1 that $\Pi_\infty(X,Q)$ is a finite category

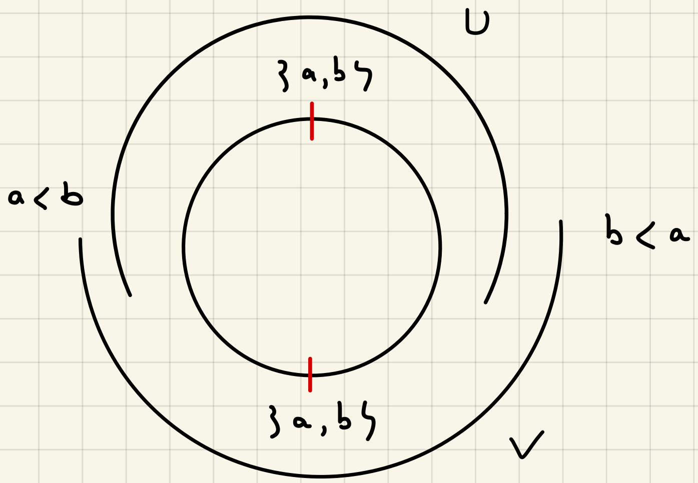
Using Step 1, one can show that each $St_{\mathcal{G}|_{\pi^{-1}(x)},\mathcal{E}}$ is an objet of $Pr^{L,\omega}_k$, is stable and that the transition functors are both left and right adjoints

By the above Lemma, we deduce that $St_{\mathcal{G},\mathcal{E}}$ is compact in $Pr^{L,\omega}_k$ if each $St_{\mathcal{G}|_{\pi^{-1}(x)},\mathcal{E}}$ is

b) Reduction to the case where the $a-b$, $a,b \in I$ have the same pole order Admitted

c) Single pole order case We illustrate it in the most simple case detailed at the end of II 3)

$X = \mathbb{C}$, $D = \{0\}$, $I = \{a,b\}$ and $a-b$ has a pole of order 2



By using that the Stokes sheaf is a sheaf, we have a pullback square

$$\begin{tikzcd} \mathrm{St}_{\mathcal{I},\mathcal{E}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{St}_{\mathcal{I}|_U,\mathcal{E}} \arrow[d] \\ \mathrm{St}_{\mathcal{I}|_V,\mathcal{E}} \arrow[r] & \mathrm{St}_{\mathcal{I}|_{U\cap V},\mathcal{E}} \end{tikzcd}$$

Since $\mathrm{St}_{\mathcal{I}|_{U\cap V},\mathcal{E}} \hookrightarrow \mathrm{Fun}(\mathcal{I}|_{U\cap V},\mathcal{E})$ is fully faithfull, the following square is also a pullback

$$\begin{tikzcd} \mathrm{St}_{\mathcal{I},\mathcal{E}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{St}_{\mathcal{I}|_U,\mathcal{E}} \arrow[d] \\ \mathrm{St}_{\mathcal{I}|_V,\mathcal{E}} \arrow[r] & \mathrm{Fun}(\mathcal{I}|_{U\cap V},\mathcal{E}) \end{tikzcd}$$

We can show that each arrow in this square is a morphism in $\mathrm{Pr}^{L,\omega}_k$ that admits both a left and right adjoint

By the lemma, to show that $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ is compact we are thus left to show that

$$\mathrm{St}_{\mathcal{I}|_U,\mathcal{E}},\ \mathrm{St}_{\mathcal{I}|_V,\mathcal{E}} \text{ and } \mathrm{Fun}(\mathcal{I}|_{U\cap V},\mathcal{E})$$

are compact

In a classical setting, we saw that a Stokes filtered local system on U and V amounts to the data of a splitting by two subobjects

Here we show that the evaluation at (i,a) and (i,b) yields an equivalence

$$\mathrm{St}_{\mathcal{I}|_U,\mathcal{E}} \xrightarrow{\sim} \mathcal{E}\times\mathcal{E}$$

which are compact

Similarly $\mathrm{St}_{\mathcal{I}|_V,\mathcal{E}} \xrightarrow{\sim} \mathcal{E}\times\mathcal{E}$

For $\mathrm{Fun}(\mathcal{I}|_{U\cap V}, \mathcal{E})$, note that

$$\{-1, 1\} \hookrightarrow U \cap V$$

is a homotopy equivalence

Thus $\mathcal{I}|_{U\cap V}$ and $\mathcal{I}_{-1} \sqcup \mathcal{I}_1$ are the same from a homotopical viewpoint

The posets $\mathcal{I}_{-1}$ and $\mathcal{I}_1$ are both $\Delta^1 = \{0 < 1\}$ Hence

$$\mathrm{Fun}(\mathcal{I}|_{U\cap V}, \mathcal{E}) \simeq \mathrm{Fun}(\Delta^1, \mathcal{E})^2$$

Now we can show that for every $C \in \mathrm{Cat}_\infty$ compact, $\mathrm{Fun}(C, \mathcal{E})$ is a compact object of $\mathrm{Pr}^{L,\omega}_k$ Since Δ^1 is compact, so is $\mathrm{Fun}(\Delta^1, \mathcal{E})$ ☻