A COURSE ON IRREGULAR SINGULARITIES OF MEROMORPHIC CONNECTIONS

by

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Introduction

Let X be a smooth complex algebraic variety. Let $i : D \hookrightarrow X$ be a normal crossing divisor in X and let η be a codimension 1 point in D. Let $\mathcal{M}$ be a meromorphic connection on X with poles along D. Ignoring extension and ramification issues, Levelt-Turrittin decomposition theorem asserts that the restriction of $\mathcal{M}$ to the formal neighbourhood of η splits as a direct sum of differential modules which are easy to work with.

This decomposition may not hold at some other points of D, but when it does, we say[(1)] that $\mathcal{M}$ *has good formal decomposition along* D. A conjecture of Sabbah [**Sab00**], recently proved by Kedlaya [**Ked10**][**Ked11**] and Mochizuki [**Moc09**] [**Moc11**] independently, asserts the existence of a chain of blow-ups $p : Y \longrightarrow \overline{X}$ above D such that $p^*\mathcal{M}$ has good formal decomposition along $p^{-1}(D)$.

In a sense, this result is to meromorphic connections what Hironaka desingularization is to varieties. It has recently allowed ground-breaking progresses in our understanding of $\mathcal{D}$-modules. Let us mention an avatar of the decomposition theorem for semi-simple holonomic $\mathcal{D}$-modules [**Moc11**], containing as a particular case the decomposition theorem for an arbitrary semi-simple perverse sheaf, as conjectured by Kashiwara. Let us also mention the generalization of Grothendieck-Deligne comparison theorem to arbitrary algebraic flat connections [**Hie09**], and Kashiwara-d'Agnolo progress towards an irregular version of the Riemann-Hilbert correspondence [**DK13**].

The goal of this course is to introduce various incarnations of irregularity (analytic, algebraic, formal, cohomological) and to explain the concepts at stake in the statement of Kedlaya-Mochizuki theorem. In the first section, we explain the proof of

(1) See section 3 for a precise definition.

Malgrange formula [**Mal71**] for the irregularity number $\mathrm{irr}(P)$ of a differential operator P in one variable. In this situation, $\mathrm{irr}(P)$ measures the difference between the actions of P on convergent power series and on formal power series.

In section 2, we give a cohomological interpretation of $\mathrm{irr}(P)$ when $\mathcal{D}/\mathcal{D}P$ defines a germ of meromorphic connection. In this situation, irregularity is the obstruction to lift a formal solution of $P(f) = 0$ to an analytic solution defined on a small punctured disc.

Section 3 is devoted to the exegesis of Kedlaya-Mochizuki theorem. We also discuss a fundamental result of André [**And07**].

In section 4, we give an application to the existence of periods for arbitrary flat algebraic connections.

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1. Irregularity in dimension one

1.1. Notations. — Let $\mathbb{C}\{x\}$ be the space of germs at $0 \in \mathbb{C}$ of convergent power series, and let $\mathbb{C}[\![x]\!]$ be the ring of formal power series. We note

$$\mathcal{D} = \bigoplus_{k\in\mathbb{N}} \mathbb{C}\{x\} \left(\frac{d}{dx}\right)^k$$

the space of germs at $0 \in \mathbb{C}$ of finite order differential operators. The symbol $\frac{d}{dx} \in \mathcal{D}$ is subjected to the relations

$$\frac{d}{dx} f = f' + f \frac{d}{dx}$$

for every $f \in \mathbb{C}\{x\}$. An operator $P \in \mathcal{D}$ acts on $\mathbb{C}\{x\}$ and on $\mathbb{C}[\![x]\!]$ by usual differentiation. The main slogan of this section is

Slogan 1.1.1. — *The irregularity of P measures the difference between the action of P on $\mathbb{C}\{x\}$ and the action of P on $\mathbb{C}[\![x]\!]$.*

The aim of what follows is to give a precise meaning to 1.1.1. We write

$$P = \sum_{k=1}^{d} a_k \left(\frac{d}{dx}\right)^k$$

with $a_d \neq 0$ and we denote by $\mathrm{Ker}(P, \mathbb{C}\{x\})$ and $\mathrm{Coker}(P, \mathbb{C}\{x\})$ (resp. $\mathrm{Ker}(P, \mathbb{C}[\![x]\!]$ and $\mathrm{Coker}(P, \mathbb{C}[\![x]\!])$) the kernel and cokernel of the action of P on $\mathbb{C}\{x\}$ (resp. on $\mathbb{C}[\![x]\!]$).

We say that a linear morphism $\Phi : E_1 \longrightarrow E_2$ between $\mathbb{C}$-vector spaces has *finite index* if $\mathrm{Ker}\,\Phi$ and $\mathrm{Coker}\,\Phi$ are finite dimensional. In that case, the number

$$\chi(\Phi) = \dim \mathrm{Ker}\,\Phi - \dim \mathrm{Coker}\,\Phi$$

is called the *index* of Φ.

1.2. Analysis of the action of P on $\mathbb{C}[\![x]\!]$. — We prove the following

Proposition 1.2.1. — *The action of P on $\mathbb{C}[\![x]\!]$ has finite index, and*

$$\chi(P, \mathbb{C}[\![x]\!]) = \operatorname{Sup}_k (k - \operatorname{ord} a_k)$$

where $\operatorname{ord} a_k$ *denotes the maximal power of x dividing a_k.*

Proof. — The fact that $\mathrm{Ker}(P, \mathbb{C}[\![x]\!])$ is finite dimensional boils down to the fact that a solution $f = \sum a_k x^k$ of $P(f) = 0$ is determined by a finite number of a_i's which is independent of f.

Let us prove that $\mathrm{Coker}(P, \mathbb{C}[\![x]\!])$ is finite dimensional. Set $M = \operatorname{Sup}(k - \operatorname{ord} a_k)$. Let us denote by A the set of $k \in \mathbb{N}$ such that $k - \operatorname{ord} a_k = M$. For every k, we have $\operatorname{ord} a_k \geqslant k - M$ so we can write $a_k = x^{k-M} b_k$ where $b_k \in \mathbb{C}\{x\}$ and $b_k(0) \neq 0$ iff $k \in A$. We have

$$a_k(x) \frac{d^k}{dx^k} x^i = i(i-1)\dots(i-k+1) b_k(0) x^{i-M} + \text{terms of degree} > x^{i-M}$$

Thus,

$$P(x^i) = (\sum_{k\in A} i(i-1)\dots(i-k+1)b_k(0))x^{i-M} + \text{terms of degree} > x^{i-M} \tag{1.2.2}$$

The coefficient $c(i)$ of x^{i-M} is polynomial in i of degree $\operatorname{Max} A \geqslant 0$. If this degree is 0, this means that $A = \{0\}$ and then c constant to $b_0(0) \neq 0$. If not, c is non constant. In both case, one can find $i_0 > 0$ such that $c(i) \neq 0$ for every $i \geqslant i_0$. For $i \geqslant \operatorname{Max}(i_0, M)$, one can write

$$P(c(i)^{-1}x^i) = x^{i-M} + \text{terms of degree} > x^{i-M} \tag{1.2.3}$$

So

$$P(c(i)^{-1}x^i) = x^{i-M} + \alpha x^{i-M+1} + \text{terms of degree} > x^{i-M+1}$$

The relation (1.2.3) for $i+1$ thus gives

$$P(c(i)^{-1}x^i - c(i+1)^{-1}\alpha x^{i+1}) = x^{i-M} + \text{terms of degree} > x^{i-M+1}$$

By iterating this process, we see that for $i \geqslant \operatorname{Max}(i_0, M)$, we have $x^{i-M} \in \operatorname{Im} P$. Thus, $\operatorname{Coker}(P, \mathbb{C}[\![x]\!])$ is generated by the classes of $1, x, \dots, x^{i_0+M}$. So it is finite dimensional.

The proof above gives slightly more than the finite dimensionality of $\operatorname{Coker}(P, \mathbb{C}[\![x]\!])$. Let $\mathcal{M} := (x)$ be the maximal ideal of $\mathbb{C}\{x\}$. For every $k \geqslant M$, we get from (1.2.2) that P sends $\mathcal{M}^k$ to $\mathcal{M}^{k-M}$, so induces a map

$$P_k : \mathbb{C}[\![x]\!]/\mathcal{M}^k \longrightarrow \mathbb{C}[\![x]\!]/\mathcal{M}^{k-M}$$

The proof above implies that for $k \gg 0$, the map $P : \mathcal{M}^k \longrightarrow \mathcal{M}^{k-M}$ is surjective. In particular, for $k \gg 0$

$$\operatorname{Coker}(P, \mathbb{C}[\![x]\!]) \simeq \operatorname{Coker} P_k$$

On the other hand, the sequence of spaces $E_k := \operatorname{Ker}(P, \mathbb{C}[\![x]\!]) \bigcap \mathcal{M}^k$ is a decreasing sequence of finite dimensional $\mathbb{C}$-vector spaces such that $\bigcap_k E_k = \{0\}$. So $E_k = \{0\}$ for $k \gg 0$. Thus

$$\operatorname{Ker}(P, \mathbb{C}[\![x]\!]) \simeq \operatorname{Ker} P_k$$

for $k \gg 0$. Choosing k big enough, we deduce

$$\begin{aligned}\chi(P, \mathbb{C}[\![x]\!]) &= \chi(P_k)\\ &= \dim \mathbb{C}[\![x]\!]/\mathcal{M}^k - \dim \mathbb{C}[\![x]\!]/\mathcal{M}^{k-M}\\ &= M\end{aligned}$$

□

1.3. Analysis of the action of P on $\mathbb{C}\{x\}$. — We prove the following

Proposition 1.3.1. — *The action of P on $\mathbb{C}\{x\}$ has finite index, and*

$$\chi(P, \mathbb{C}\{x\}) = d - \operatorname{ord} a_d$$

For $r > 0$ and $k \geqslant 0$, let us denote by $B^k(\overline{D}_r)$ the space of C^k functions on $\overline{D}_r$ which are holomorphic on D_r. It is a Banach space for the norm

$$\|f\|_k = \|f\|_\infty + \|f'\|_\infty + \cdots + \|f^{(k)}\|_\infty$$

where $\|f\|_\infty = \operatorname{Sup}_{x\in\overline{D}_r} |f(x)|$. For $r > 0$ small enough, P induces a continuous linear map

$$P_r : B^d(\overline{D}_r) \longrightarrow B^0(\overline{D}_r)$$

Let us prove the following

Lemma 1.3.2. — *The operator P_r has finite index $d - \operatorname{ord} a_d$.*

Proof. — Let us write $P_r = a_d \left(\frac{d}{dx}\right)^d + Q_r$ with Q_r of order $< d$ and let us prove that P_r has finite index equal to $d - \operatorname{ord} a_d$. Since multiplication by a_d has finite index $-\operatorname{ord} a_d$ and $\frac{d}{dx}$ has finite index 1, the composite $a_d \left(\frac{d}{dx}\right)^d$ has finite index $d - \operatorname{ord} a_d$. So it is enough to prove that Q_r is a compact operator. If $(f_n) \in B^d(\overline{D}_r)^{\mathbb{N}}$ is a bounded sequence, then the sequences $\|Q_r(f_n)\|_\infty$ and $\|Q_r(f_n)'\|_\infty$ are bounded. By Arzela-Ascoli theorem, the set formed by the f_n is relatively compact. Hence, Q_r is compact. □

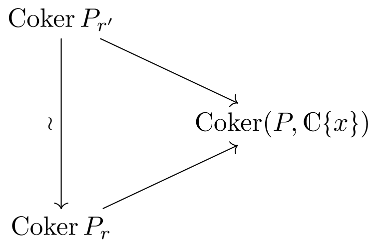
We are left to show how 1.3.2 implies 1.3.1. Taking the germ at 0 induces an injection

$$\operatorname{Ker} P_r \longrightarrow \operatorname{Ker}(P, \mathbb{C}\{x\}) \tag{1.3.3}$$

and $\operatorname{Ker}(P, \mathbb{C}\{x\})$ is the union of all the $\operatorname{Ker} P_r$. As a subspace of $\operatorname{Ker}(P, \mathbb{C}[\![x]\!])$, $\operatorname{Ker}(P, \mathbb{C}\{x\})$ is finite dimensional. So (1.3.3) is an isomorphism for r small enough. We are left to prove that the same holds for cokernels. Let us first assume that for $r \leqslant r'$ small enough, the restriction morphism

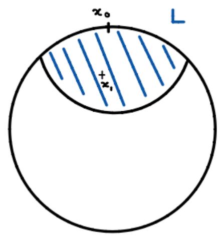
$$\operatorname{Coker} P_{r'} \longrightarrow \operatorname{Coker} P_r \tag{1.3.4}$$

is injective. Then 1.3.2 shows that (1.3.4) is an isomorphism for $r \leqslant r'$ small enough. From



we deduce that $\operatorname{Coker} P_r \simeq \operatorname{Coker}(P, \mathbb{C}\{x\})$ for $r \ll 1$, which finishes the proof.

Let us prove that (1.3.4) is injective for $r < r'$ small enough. Let $g \in B^0(\overline{D}_{r'})$ such that $g = P_r(f)$ on $\overline{D}_r$ with $f \in B^d(\overline{D}_r)$. We have to prove that f extends to $\widetilde{f} \in B^d(\overline{D}_{r'})$. Take r' such that a_d does not vanish on $\overline{D}_{r'}\backslash\{0\}$. By Cauchy theorem, f extends uniquely and holomorphically to $D_{r'}$.



Let us still denote by f this extension. Then, $\widetilde{f}$ is uniquely determined by f, so it is enough to extend f to a lens $L := D_{r'} \bigcap \overline{D(x_0, \epsilon)}$ where $x_0 \in S_{r'}$ and $\epsilon > 0$ is small enough. Let us denote by $B^0(L)$ (resp. $B^1(L)$) the space of continuous (resp. C^1) functions on L which are holomorphic on the interior of L. $B^0(L)$ is a Banach space for $\|\cdot\|_\infty$. To construct $\widetilde{f}$ on L is equivalent to find $\mathbf{u} \in B^1(L)^d$ satisfying

$$\left\{ \begin{array}{l} \mathbf{u}' = A\mathbf{u} + \mathbf{v} \\ \mathbf{u}(x_1) = \mathbf{w} \end{array} \right. \tag{1.3.5}$$

where

$$A = \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & \ddots & \ddots & \\ \vdots & & & \\ 0 & \cdots & 0 & 1 \\ -a_0/a_d & \cdots & -a_{d-2}/a_d & -a_{d-1}/a_d \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ g/a_d \end{pmatrix}$$

where x_1 is a point in the interior of L, and where $\mathbf{w}$ is the column vector $(f(x_1), \dots, f^{(d-1)}(x_1))$. If L is chosen small enough, the operator

$$\begin{array}{ccc} B^0(L) & \longrightarrow & B^0(L) \\ \mathbf{u} & \longrightarrow & \left(z \longrightarrow \mathbf{w} + \displaystyle\int_{x_1}^{x} (A\mathbf{u} + \mathbf{g}) \right) \end{array}$$

is contracting. By Banach fixed point theorem, it has a unique fixed point, which automatically satisfies (1.3.5).

1.4. The irregularity space. — We can now give a precise meaning to 1.1.1. Define $\mathcal{Q} := \mathbb{C}[\![x]\!]/\mathbb{C}\{x\}$. Applying $\mathrm{RHom}(\mathcal{D}/\mathcal{D}P, \cdot\,)$ to the exact sequence of $\mathcal{D}$-modules

$$0 \longrightarrow \mathbb{C}\{x\} \longrightarrow \mathbb{C}[\![x]\!] \longrightarrow \mathcal{Q} \longrightarrow 0$$

gives a distinguished triangle

$$\mathrm{RHom}(\mathcal{D}/\mathcal{D}P, \mathbb{C}\{x\}) \longrightarrow \mathrm{RHom}(\mathcal{D}/\mathcal{D}P, \mathbb{C}[\![x]\!]) \longrightarrow \mathrm{RHom}(\mathcal{D}/\mathcal{D}P, \mathcal{Q}) \tag{1.4.1}$$

As a by-product of the proof of the finite dimensionality of $\mathrm{Ker}(P, \mathbb{C}[\![x]\!])$ in 1.2.1, we have $\mathrm{Coker}(P, \mathcal{Q}) \simeq 0$. So the long exact sequence associated to (1.4.1) reads

(1.4.2)

$$0 \longrightarrow \mathrm{Ker}(P, \mathbb{C}\{x\}) \longrightarrow \mathrm{Ker}(P, \mathbb{C}[\![x]\!]) \longrightarrow \mathrm{Ker}(P, \mathcal{Q}) \longrightarrow \mathrm{Coker}(P, \mathbb{C}\{x\}) \longrightarrow \mathrm{Coker}(P, \mathbb{C}[\![x]\!]) \longrightarrow 0$$

Thus, the obstruction that the analytic and formal Kernel and Cokernel of P coincide lies in the non vanishing of $\mathrm{Ker}(P, \mathcal{Q})$. Putting together 1.2.1 and 1.3.1 gives the following theorem, due to Malgrange **[Mal71]**

Theorem 1.4.3. — *The space* $\mathrm{Ker}(P, \mathcal{Q})$ *is finite dimensional, and we have*

$$\dim_{\mathbb{C}} \mathrm{Ker}(P, \mathcal{Q}) = \mathrm{Sup}_k(k - \mathrm{ord}\, a_k) - (d - \mathrm{ord}\, a_d)$$

We define the *irregularity number of* P as

$$\mathrm{irr}(P) := \dim_{\mathbb{C}} \mathrm{Ker}(P, \mathcal{Q})$$

Definition 1.4.4. — We say that P is regular if $\mathrm{irr}(P) = 0$.

Example 1.4.5. — Malgrange formula predicts that $\mathrm{irr}(\mathcal{D}/\mathcal{D}(x^2\frac{d}{dx} - 1)) = 1$. On the other hand, $f := \sum n! x^{n+1}$ satisfies

$$x^2\frac{df}{dx} - f = -x = 0 \text{ in } \mathcal{Q}$$

So $\mathrm{Ker}(x^2\frac{d}{dx} - 1, \mathcal{Q}) \simeq \mathbb{C}f$.

1.5. From differential operators to differential modules. —

Definition 1.5.1. — We define a $\mathbb{C}\{x\}[x^{-1}]$-differential module as the data of a finite dimensional $\mathbb{C}\{x\}[x^{-1}]$ vector space endowed with a $\mathbb{C}$-linear map $\nabla : M \longrightarrow M$ satisfying the Leibniz rule

$$\nabla(fm) = f'm + f\nabla m$$

The Leibniz rule turns M into a module over $\mathcal{D}$. Set $d := \dim_{\mathbb{C}\{x\}[x^{-1}]} M$. Let $A \in M_d(\mathbb{C}\{x\}[x^{-1}])$ be the matrix of ∇ in a basis e. Let e' be another basis of M and let P be the matrix of coefficients of e' in e. Then, the matrix of ∇ in e' is

$$P^{-1}P' + P^{-1}AP$$

One can prove [**Sab93**, 4.2.8] that as a $\mathcal{D}$-module, $M \simeq \mathcal{D}/\mathcal{D}P$ where $P \in \mathcal{D}$. In particular, $\mathrm{irr}(M)$ is well-defined. We have

Theorem 1.5.2. — *The following conditions are equivalent*
(1) $\mathrm{irr}(M) = 0$.
(2) M admits a basis in which the entries of the matrix of $x\nabla$ have no poles.

Note that description (2) is quite concrete and subtle at the same time, since one requires that *there exists* a basis such that etc. For a randomly given basis, the poles of the matrix of ∇ may be arbitrary high even if M is regular. Also note that condition (2) is algebraic so it can be generalized to other differential fields.

2. A cohomological interpretation

2.1. Motivation. — Let $P \in \mathcal{D}$ be a finite order differential operator. As seen in the previous section, if P is regular, the canonical comparison map

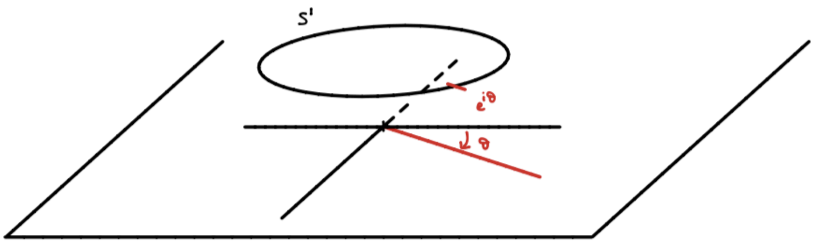
$$\mathrm{Ker}(P, \mathbb{C}\{x\}) \longrightarrow \mathrm{Ker}(P, \mathbb{C}[\![x]\!])$$

is an isomorphism. In general it is not, as the following example[2] shows: take $P = x^2\frac{d}{dx} + (3x-1)\frac{d}{dx} + 1$. Then the serie $\sum n!x^n$ is a formal solution for P.

This is however not the end of the story. Suppose that $\mathcal{D}/\mathcal{D}P$ is a differential module as defined in 1.5.1, and choose a direction d. Then, the main asymptotic existence theorem allows to lift a formal solution F to an analytic solution f_d defined on a small sector containing d. The collection of f_d gives rise to a 1-cocycle with value in the sheaf of solutions of $P(f) = 0$ with rapid decay at 0. If this cocycle vanishes, one can produce an analytic lift to F defined on a small punctured disc. We will in see in 2.5.1 how this vanishing question relates to $\mathrm{irr}(P)$. The main upshot will be the

Slogan 2.1.1. — *Irregularity is the obstruction to lift a formal solution of $P(f) = 0$ to an analytic solution defined on a small punctured disc.*

2.2. Asymptotic expansions. — Let us define the *real oriented blow up* $\widetilde{D}$ of D as the closure in $D \times S^1$ of the graph of the map $D^* \longrightarrow S^1$ given by $x \longrightarrow x/|x|$. The morphism $\pi : \widetilde{D} \longrightarrow D$ induced by first projection is an isomorphism above D^*. Let $j : D^* \longrightarrow \widetilde{D}$ be the associated open immersion. On the other hand $(0, e^{i\theta}) \in \{0\} \times S^1$ is the limit of $(e^{i\theta}/n, e^{i\theta})_{n \in \mathbb{N}^*}$, so $\pi^{-1}\{0\} = \{0\} \times S^1$. In what follows, we identify $\pi^{-1}\{0\}$ with S^1 and note θ for $(0, e^{i\theta})$.



One can show that $\widetilde{D}$ inherits the structure of a C^∞-manifold with boundary S^1. The topology of $\widetilde{D}$ is that induced by $D \times S^1$, so $\theta \in S^1$ has a fundamental system of neighbourhood of the form $U_r(a,b) := \widetilde{D} \cap D_r \times (a,b)$ where (a,b) is the open interval in S^1 between a and b. So if we define the sector

$$S_r(a,b) = \{x \in D_r^* \text{ with } x/|x| \in (a,b)\}$$

we have set theoretically

$$U_r(a,b) = (a,b) \bigsqcup S_r(a,b) \qquad (2.2.1)$$

In particular, the space of germs of $j_*\mathcal{O}_{D^*}$ at θ is the inverse limit of the $\Gamma(S_r(a,b), \mathcal{O}_D)$ where r and the amplitude of (a,b) become smaller and smaller.

[2] which I owe to P. Jossen.

Definition 2.2.2. — We say that $f \in \Gamma(S_r(a,b), \mathcal{O}_D)$ has $c = \sum_{n\geqslant 0} c_n x^n \in \mathbb{C}[\![x]\!]$ as *asymptotic expansion at* 0 if for every $N > 0$ and every closed sector W in $S_r(a,b)$, one can find $C(N,W) > 0$ such that

$$|f(x) - \sum_{n=0}^{N-1} c_n x^n| \leqslant C(N,W)|x|^n \tag{2.2.3}$$

for every $x \in W$.

Note that in case c as in 2.2.2 exists, it is unique. We denote it by $J(f)$.

Definition 2.2.4. — Let $\mathcal{A}$ be the subsheaf of $i^{-1}j_*\mathcal{O}_{D^*}$ whose space of germs at $\theta \in S^1$ is the space of $f \in (j_*\mathcal{O}_{D^*})_\theta$ admitting an asymptotic expansion in a small enough sector centred at θ.

Still denoting by $\mathbb{C}[\![x]\!]$ the constant sheaf on S^1 with value $\mathbb{C}[\![x]\!]$, asymptotic expansion defines a morphism of sheaves on S^1

$$J : \mathcal{A} \longrightarrow \mathbb{C}[\![x]\!] \tag{2.2.5}$$

The following is known as the Borel-Ritt lemma

Lemma 2.2.6. — *The morphism* (2.2.5) *is surjective.*

We denote by $\mathcal{A}^0$ the kernel of J. The sections of $\mathcal{A}^0$ are called *holomorphic functions with rapid decay at* 0.

2.3. A few examples. —

(1) Complex analysis shows that $\Gamma(S^1, \mathcal{A}) \simeq \mathbb{C}\{x\}$ and $\Gamma(S^1, \mathcal{A}^0) \simeq 0$.

(2) Let $\varphi = a_n x^{-n} + \cdots + a_1 x^{-1} \in x^{-1}\mathbb{C}[x^{-1}]$ with $a_i \in \mathbb{C}$ and $a_n \neq 0$. Then $e^\varphi \in j_*\mathcal{O}_{D^*}$ and one can ask for which direction θ the function e^φ falls in $\mathcal{A}_\theta$. We have to analyse the growth of $r \longrightarrow |e^{\varphi(re^{i\theta})}| = |e^{\operatorname{Re}\varphi(re^{i\theta})}|$ when $r \to 0$.

- If $\operatorname{Re}(a_n e^{-in\theta}) > 0$, the function $|e^{\varphi(re^{i\theta})} - P|$ diverges to ∞ when $r \to 0$ for every polynomial P, so (2.2.3) cannot hold on a close sector containing θ.
- If $\operatorname{Re}(a_n e^{-in\theta}) = 0$, for any angle $\theta^+ > \theta$ close enough to θ, we have $\operatorname{Re}(a_n e^{-in\theta^+}) > 0$, so again (2.2.3) cannot hold on a close sector containing θ.
- Suppose $\operatorname{Re}(a_n e^{-in\theta}) < 0$, that is θ belongs to a segment of type

$$S(k,n,\theta_0) := \left(\theta_0 - \frac{\pi}{2n} + \frac{2k\pi}{n}, \theta_0 + \frac{\pi}{2n} + \frac{2k\pi}{n}\right)$$

where $0 \leqslant k \leqslant n-1$ and where $\theta_0 \in [0, 2\pi[$ is the argument of a_n. Then (2.2.3) holds with $c = 0$ for any close sector W in a sector of type

$$S_r(k,n,\theta_0) := S_r\left(\theta_0 - \frac{\pi}{2n} + \frac{2k\pi}{n}, \theta_0 + \frac{\pi}{2n} + \frac{2k\pi}{n}\right)$$

So e^φ defines a section of $\mathcal{A}$ over $\bigsqcup_k S(k,n,\theta_0)$.

(3) Take $f \in \Gamma(S_r(a,b), \mathcal{O}_D)$ with rapid decay at 0. Choose $\theta \in (a,b)$, $\rho < r$ and let $\gamma : [0,1] \longrightarrow \mathbb{C}$ be the segment joining 0 to $\rho e^{i\theta}$. The function

$$g : z \longrightarrow \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$$

is well defined on $D_\rho \backslash [0, \rho e^{i\theta}[$.

Fact 2.3.1. — *The function g admits*

$$\widehat{g} := \sum_{n \geqslant 0} \left(\int_\gamma f(\zeta)\zeta^{-n-1} d\zeta \right) x^n \in \mathbb{C}[\![x]\!]$$

as asymptotic expansion at 0 *on $D_\rho \backslash [0, \rho e^{i\theta}[$. In particular, g induces an element of $\Gamma(S^1 \backslash \{\theta\}, \mathcal{A})$.*

Proof. — Let W be a close sector in $D_\rho \backslash [0, \rho e^{i\theta}[$ and let $N \geqslant 0$. We have $M := \inf_{]0,\rho e^{i\theta}[\times W} |1 - x/\zeta| > 0$. By definition of f, one can find $C \geqslant 0$ such that $|f(\zeta)| \leqslant C|\zeta|^{N+1}$ for $\zeta \in [0, \rho e^{i\theta}[$. Thus

$$\left| g(x) - \sum_{n=0}^{N-1} \left(\int_\gamma f(\zeta)\zeta^{-n-1} d\zeta \right) x^n \right| = \left| \int_\gamma \frac{f(\zeta)(x/\zeta)^N}{\zeta(1 - x/\zeta)} d\zeta \right| \leqslant \frac{\rho C}{M} |x|^N$$

□

(4) Let $f \in \Gamma(S_r(a,b), \mathcal{O}_D)$ admitting an asymptotic expansion at 0. Set $J(f) = \sum_{n \geqslant 0} c_n x^n$. We prove the

Fact 2.3.2. — *The function f' admits $J(f)'$ as asymptotic expansion at 0.*

Proof. — Let $W \subset S_r(a,b)$ be a closed sector, and $N \geqslant 0$. Let $W' \subset S_r(a,b)$ be a closed sector whose radius and amplitude is strictly bigger than that of W. There exists $\delta > 0$ such that for any $x \in W \backslash \{0\}$, the closed disc $\overline{D}(x, \delta|x|)$ centred at x with radius $\delta|x|$ is included in W'. For $x \in W \backslash \{0\}$, define $\gamma_x : t \longrightarrow x + \delta|x|e^{2i\pi t}$. Cauchy formula reads

$$f(y) = \int_{\gamma_x} \frac{f(\zeta)}{\zeta - y} d\zeta$$

for every $y \in D(x, \delta|x|)$. Derivating and evaluating at x gives

$$f'(x) = \int_{\gamma_x} \frac{f(\zeta)}{(\zeta - x)^2} d\zeta$$

The same holds for $J(f)_N := \sum_{n=0}^{N-1} c_n x^n$. So

$$\begin{aligned} |f'(x) - J(f)'_N(x)| &\leqslant \frac{1}{(\delta|x|)^2} \int_{\gamma_x} |f(\zeta) - J(f)_N(\zeta)| d\zeta \\ &\leqslant 2\pi \frac{C(N+1, W')}{\delta} |x|^{N-1} \end{aligned}$$

□

As a corollary of 2.3.2, we obtain that P acts on $\mathcal{A}$ and $\mathcal{A}^0$.

2.4. The main asymptotic existence theorem. — In this section, $P \in \mathcal{D}$ such that $\mathcal{D}/\mathcal{D}P$ is a differential module in the sense of 1.5.1. The main asymptotic development theorem is the following

Theorem 2.4.1. — *Let $\theta \in S^1$ and let $g \in \mathcal{A}_\theta$. Suppose that g=$P(F)$ with $F \in \mathbb{C}[\![x]\!]$. Then, one can find $f \in \mathcal{A}_\theta$ such that $J(f) = F$ and $g = P(f)$.*

This theorem is a consequence of the following

Theorem 2.4.2. — *The morphism of sheaves $P : \mathcal{A}^0 \longrightarrow \mathcal{A}^0$ is surjective.*

Let us prove that 2.4.2 implies 2.4.1. We have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{A}^0_\theta & \longrightarrow & \mathcal{A}_\theta & \xrightarrow{J} & \mathbb{C}[\![x]\!] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{A}^0_\theta & \longrightarrow & \mathcal{A}_\theta & \xrightarrow{J} & \mathbb{C}[\![x]\!] & \longrightarrow & 0 \end{array}$$

where the vertical maps are given by the action of P. By Borel-Ritt lemma 2.2.6, the lines of this diagram are exact. The first square is trivially commutative. The commutativity of the second square comes from 2.3 (4). The first vertical morphism is surjective by 2.4.2. One gets the sought after lift by a diagram chase.

2.5. Cohomological interpretation of irregularity. — Recall that $i : S^1 \longrightarrow \widetilde{D}$ is the inclusion of the boundary of $\widetilde{D}$. We denote by $\mathrm{Ker}(P, \mathcal{A}^0)$ the kernel of the action of P on $\mathcal{A}^0$. The goal of this section is to prove the following

Theorem 2.5.1. — *Let $P \in \mathcal{D}$ be a germ of differential operator such that $\mathcal{D}/\mathcal{D}P$ is a differential module in the sense of* 1.5.1. *Then*

$$\mathrm{irr}(P) = \dim H^1(S^1, \mathrm{Ker}(P, \mathcal{A}^0))$$

Theorem 2.5.1 is a consequence of the following

Lemma 2.5.2. — *The canonical map $H^1(S^1, \mathcal{A}^0) \longrightarrow H^1(S^1, \mathcal{A})$ is zero.*

Let us first see why 2.5.2 implies 2.5.1. From the Borel-Ritt lemma 2.2.6, we have an exact sequence of sheaves on S^1

$$0 \longrightarrow \mathcal{A}^0 \longrightarrow \mathcal{A} \longrightarrow \mathbb{C}[\![x]\!] \longrightarrow 0 \tag{2.5.3}$$

From 2.5.2 and 2.3 (1), it induces an exact sequence

$$0 \longrightarrow \mathbb{C}\{x\} \longrightarrow \mathbb{C}[\![x]\!] \longrightarrow H^1(S^1, \mathcal{A}^0) \longrightarrow 0$$

Hence $H^1(S^1, \mathcal{A}^0) \simeq \mathcal{Q}$. From 2.4.2, the following sequence of sheaves

$$0 \longrightarrow \mathrm{Ker}(P, \mathcal{A}^0) \longrightarrow \mathcal{A}^0 \xrightarrow{P} \mathcal{A}^0 \longrightarrow 0$$

is exact. So the associated exact sequence in cohomology gives the short exact sequence

$$0 \longrightarrow H^1\left(S^1, \mathrm{Ker}(P, \mathcal{A}^0)\right) \longrightarrow \mathcal{Q} \xrightarrow{P} \mathcal{Q} \longrightarrow 0$$

and 2.5.1 is proved. We are left to prove 2.5.2.

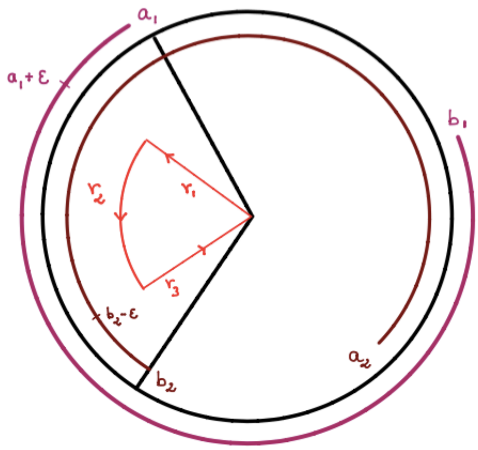
From [**Har77**, III 4. Ex 4.4], we know that for any sheaf of abelian groups $\mathcal{F}$ on S^1, the canonical morphism

$$\varinjlim \check{\mathrm{H}}^1(\mathcal{U}, \mathcal{F}) \longrightarrow H^1(S^1, \mathcal{F})$$

is an isomorphism, where the limit is taken over open covers $\mathcal{U}$ of S^1. Elements in $\varinjlim \check{\mathrm{H}}^1(\mathcal{U}, \mathcal{F})$ can be represented by a cocycle defined on a finite cover of S^1 by successive intervals. So we are left to prove that

$$\varinjlim \check{\mathrm{H}}^1(\mathcal{U}, \mathcal{A}^0) \longrightarrow \varinjlim \check{\mathrm{H}}^1(\mathcal{U}, \mathcal{A})$$

is zero, where the limits are taken over finite covers by successive intervals. We consider the case where $\mathcal{U}$ is made of two intervals (a_1, b_1) and (a_2, b_2), the general case being a consequence of this one. Then, $\check{\mathrm{H}}^1(\mathcal{U}, \mathcal{A}^0)$ is generated by classes of the form $(f, 0), (0, g) \in \mathcal{A}^0(a_1, b_2) \times \mathcal{A}^0(a_2, b_1)$. By symmetry, we are left to prove that for $f \in \mathcal{A}^0(a_1, b_2)$, the class of $(f, 0)$ in $\check{\mathrm{H}}^1(\mathcal{U}, \mathcal{A})$ is 0. At the cost of shrinking the amplitude of the intervals of $\mathcal{U}$ (which amounts to refining the cover), we can suppose that $f \in \Gamma(S_r(a_1, b_2), \mathcal{O}_D)$ for some $r > 0$.



Take $\rho < r$ and define on $S_\rho(a_1 + \epsilon, b_1)$

$$g_1 : x \longrightarrow \int_{\gamma_1} \frac{f(\zeta)}{\zeta - x} d\zeta$$

and on $S_\rho(a_2, b_2 - \epsilon)$

$$g_2 : x \longrightarrow -\int_{\gamma_2 + \gamma_3} \frac{f(\zeta)}{\zeta - x} d\zeta$$

If $\mathcal{V} := \{(a_1+\epsilon, b_1), (a_2, b_2-\epsilon)\}$, example 2.3 (3) shows $(g_1, g_2) \in \check{\mathcal{C}}^0(\mathcal{V}, \mathcal{A})$, where $(\check{\mathcal{C}}^\bullet, d_{\check{\mathcal{C}}})$ denotes the Cech complex construction. Thus

$$d_{\check{\mathcal{C}}}(g_1, g_2) = ((g_1 - g_2)_{|(a_1+\epsilon, b_2-\epsilon)}, (g_1 - g_2)_{|(a_2, b_1)}) \tag{2.5.4}$$

By Cauchy formula, the first term in the right hand side of (2.5.4) is $f_{|(a_1+\epsilon, b_2-\epsilon)}$. Since γ_1 and $-\gamma_2-\gamma_3$ are homotopic, the second term in the right hand side of (2.5.4) is 0. We conclude by noticing that $\mathcal{V}$ refines $\mathcal{U}$.

3. Kedlaya-Mochizuki theorem

3.1. Formal differential modules. — Let k be a field of characteristic 0.

Definition 3.1.1. — A $k((x))$-differential module M is the data of a finite dimensional $k((x))$-vector space endowed with a k-linear operator $\nabla : M \longrightarrow M$ satisfying the Leibniz rule

$$\nabla(fm) = f'm + f\nabla m$$

for every $f \in k((x))$ and every $m \in M$.

A basic example of such a module is a regular module (take 3.1.1 (2) as a definition of regularity). Another example is as follows: take $\varphi \in x^{-1}k[x^{-1}]$ and define $\mathcal{E}^\varphi$ as the $k((x))$-differential module whose underlying space is $k((x))$ with $\nabla : f \longrightarrow f' + f\varphi'$. Such a module is called an *exponential module.*

Regular modules and exponential modules are essentially the building blocs of all $k((x))$-differential modules, due to the theorem of Levelt-Turrittin [**Sv00**, 3.1].

Theorem 3.1.2. — *There exists a finite extension k'/k and an integer $n \geqslant 1$ such that*

$$k'((x^{1/n})) \otimes_{k((x))} M \simeq \bigoplus_{\varphi \in x^{-1/n}k[x^{-1/n}]} \mathcal{E}^\varphi \otimes \mathcal{R}_\varphi \tag{3.1.3}$$

where the $\mathcal{R}_\varphi$ are regular $k'((x^{1/n}))$-differential modules and where the induced connection $\nabla^{1/n}$ on the left hand side of (3.1.3) *is $\nabla^{1/n}(f \otimes m) = f' \otimes m + f \otimes \nabla m$.*

For k' and n realizing (3.1.3), the set of φ contributing to (3.1.3) is unique, as well as decomposition (3.1.3). In case $k = \mathbb{C}$ and $M = \mathbb{C}((x)) \otimes_{\mathbb{C}\{x\}[x^{-1}]} M_0$ where M_0 is a $\mathbb{C}\{x\}[x^{-1}]$ module, one can show

$$\operatorname{irr}(M_0) = \sum_{\varphi \in x^{-1/n}\mathbb{C}[x^{-1/n}]} \frac{\operatorname{ord}\varphi}{n} \operatorname{rk}\mathcal{R}_\varphi \tag{3.1.4}$$

Formula (3.1.4) is algebraic and makes sense for arbitrary k. Thus, we have a well-defined irregularity number for a $k((x))$-differential module M.

3.2. Statement of Kedlaya-Mochizuki theorem. — The analytico-formal approach to irregularity introduced in section 1 generalizes to holonomic $\mathcal{D}$-modules in any dimension, see 4.1. For a normal crossing divisor D, The cohomological approach of section 2 generalizes to holonomic $\mathcal{D}$-modules into a relation between the De Rham complex with rapid decay along D and the irregularity sheaf along D. See [**Sab00**, 1.1.19]. What about the formal approach 3.1 for meromorphic connections in any dimension? It requires first a generalization of Levelt-Turrittin decomposition. Let us proceed by analogy with the one dimensional case to see what is the best to be expected.

dim 1	dim > 1
$k[\![x]\!]$	$k[\![x_1,\dots,x_n]\!]$
$k((x)) := k[\![x]\!][x^{-1}]$	$R_{n,m} := k[\![x_1,\dots,x_n\ ]\!][x_1^{-1},\dots,x_m^{-1}]$
$k((x))$-differential module	$R_{n,m}$-differential module: free $R_{n,m}$ module of finite type M endowed with commuting k-linear maps $\nabla_i : M \longrightarrow M$, $i=1,\dots,n$ satisfying the Leibniz rule $\nabla_i(fm) = \frac{\partial f}{\partial x_i}m + f\nabla_i m$ for every $f \in R_{n,m}$ and $m \in M$.
Regular module	M admits a basis in which the matrix of $x_i\nabla_i$ has no poles for every $i=1,\dots,n$.
Module $\mathcal{E}^\varphi$ for $\varphi \in k((x))$	For $\varphi \in R_{n,m}$ admissible, that is $\varphi = 0$ or $\varphi = u/x_1^{a_1}\cdots x_m^{a_m}$ with $u \in R_{n,0}$ such that $u(0) \neq 0$, the differential module $\mathcal{E}^\varphi := (R_{n,m}, \nabla_i f = \frac{\partial f}{\partial x_i} + f\frac{\partial \varphi}{\partial x_i})$
Unramified Levelt-Turrittin decomposition	Admissible decomposition: there exists an isomorphism of $R_{n,m}$-differential modules $$M \simeq \bigoplus_{\varphi \text{ admissible}} \mathcal{E}^\varphi \otimes \mathcal{R}_\varphi$$ where the $\mathcal{R}_\varphi$ are regular.
Levelt-Turrittin decomposition	Ramified admissible decomposition: there exists k'/k finite extension and an integer $n \in \mathbb{N}$ such that if $R_{n,m}(k',n) := k'[\![x_1,\dots,x_n]\!][x_1^{-1/n},\dots,x_m^{-1/n}]$, the module $R_{n,m}(k',n) \otimes_{R_{n,m}} M$ admits an admissible decomposition.

Note that

Ramified admissible decompositions may not exist !!

To understand where the obstruction comes from, let us consider a $R_{2,1}$-differential module M. We have $R_{2,1} = k[\![x_2]\!]((x_1))$. We denote by s the closed point of $\operatorname{Spec} k[\![x_2]\!]$ and by η its generic point. The restriction of $\mathcal{M}$ at η is

$$M_\eta = k((x_2))((x_1)) \otimes_{k[\![x_2]\!]((x_1))} M$$

We are here in a one-dimensional situation with residue field $k((x_2))$. In particular, the Levelt-Turrittin decomposition is available. Ignoring ramification and extension issues, M_η decomposes as

$$M_\eta \simeq \bigoplus_{\varphi \in x_1^{-1} k((x_2))[x_1^{-1}]} \mathcal{E}^\varphi \otimes \mathcal{R}_\varphi \tag{3.2.1}$$

The coefficients of the φ lies in $k((x_2))$. They may have poles.

On the other hand, to say that M_0 has an admissible decomposition is to say that

$$M \simeq \bigoplus_{\psi \text{ admissible}} \mathcal{E}^\psi \otimes \mathcal{R}_\psi \tag{3.2.2}$$

where $\psi = u/x_1^{a_\psi}$ with $u \in k[\![x_1, x_2]\!]$ such that $u(0) \neq 0$. If such a decomposition exists, the unicity of the exponential factors appearing in (3.1.2) gives that the set

$$\{\varphi \in x_1^{-1} k((x_2))[x_1^{-1}] \text{ contributing to (3.2.1)}\}$$

is equal to the set

$$\{\psi \in k[\![x_2]\!]((x_1)) \text{ contributing to (3.2.2)}\}$$

So again ignoring extension and ramification issues, if M admits an admissible decomposition, we have the following

Condition 3.2.3. — *The coefficients of a φ contributing to the Level-Turrittin decomposition of M_η have no poles at 0, and if $\varphi \neq 0$, the coefficient of smallest degree of φ does not vanish at 0.*

The reciproque cannot hold for the following heuristical reason: even if the right hand-side of (3.2.1) makes sense as a potential admissible decomposition for M, it may happen that the isomorphism (3.2.1) cannot be chosen to be defined over $k[\![x_2]\!]((x_1))$. This obstruction is hard to control in practice, since the isomorphism (3.2.1) is not explicit and there are many choices for it. However, as a consequence of a theorem of André [**And07**]

Theorem 3.2.4. — *The following conditions are equivalent*(3)

(1) M and End M *admits an admissible decomposition.*

(2) Condition (3.2.3) *is satisfied for M and* End M.

This result is breathtaking: it makes the non explicitness problem of the isomorphism in (3.2.1) disappear. Let us note that theorem 3.2.4 is used in Mochizuki's proof [**Moc09**] of Sabbah's conjecture for surfaces.

Let us now pass to a global situation. Suppose to simplify that $k = \overline{k}$. Let X be

(3) For a finer statement valid in the ramified case with general coefficients, see [**And07**].

a smooth algebraic variety over k and let D be a normal crossing divisor in X. By point of X, we mean a close point.

Definition 3.2.5. — A flat meromorphic connection on X with poles along D is the data of a locally free $\mathcal{O}_X(*D)$-module of finite rank $\mathcal{M}$ endowed with a k-linear operator

$$\nabla : \mathcal{M} \longrightarrow \Omega^1_X(*D) \otimes \mathcal{M}$$

satisfying the Leibniz rule

$$\nabla(fm) = df \otimes m + f\nabla m$$

for every $f \in \mathcal{O}_X(*D)$ and $m \in \mathcal{M}$. Furthermore, if for $i = 1, \dots, n$ we denote by $\nabla_i : \mathcal{M} \longrightarrow \mathcal{M}$ the k-linear morphism induced by ∇ and the contraction by $\frac{\partial}{\partial x_i}$, we require that the ∇_i commute with each other.

For a $A \in D$, we have $\widehat{\mathcal{O}}_{X,A} \simeq k[\![x_1, \dots, x_n]\!]$ where $x_1 \cdots x_m = 0$ is an equation of D in a neighbourhood of A. Restricting $\mathcal{M}$ to this formal neighbourhood yields a $R_{n,m}$-differential module denoted by $\widehat{\mathcal{M}}_A$. The module $\widehat{\mathcal{M}}_A$ may not have an admissible decomposition.

Definition 3.2.6. — We say that $\mathcal{M}$ has *good formal decomposition* at $A \in D$ if $\widehat{\mathcal{M}}_A$ and $\widehat{\mathcal{E}nd(\mathcal{M})}_A$ admit a ramified admissible decomposition.

The complements in D of the set of good formal decomposition points is called the *turning point locus* of $\mathcal{M}$.

We say that $\mathcal{M}$ has *good formal decomposition* if the turning point locus of $\mathcal{M}$ is empty.

As shown in [**Ked11**], the turning point locus of $\mathcal{M}$ is a close subset of D. If D is smooth, André proves that it is either empty or of pure codimension 1 in D. Working at good formal decomposition points is very pleasant since exponential modules and regular modules are easy to handle. One can often reduce to this situation, thanks to Kedlaya-Mochizuki theorem:

Theorem 3.2.7. — *Let $\mathcal{M}$ be a flat meromorphic connection on X with poles along D. There exists a finite composite $\pi : Y \longrightarrow X$ of blow-up above D such that $\pi^*\mathcal{M}$ has good formal decomposition.*

Example 3.2.8. — Take $X = \mathbb{A}^2_k$ with coordinate (x, y) and $\mathcal{M} = \mathcal{E}^{x/y}$. The origin is a turning point for $\mathcal{M}$. Let $\pi : \widetilde{X} \longrightarrow X$ be the blow-up of X at the origin. In the chart U_0 where $x = uv$ et $y = v$, the pole locus of $\pi^*\mathcal{E}^{x/y}$ is $v = 0$, and

$$(\pi^*\mathcal{E}^{x/y})_{|U_0} \simeq \mathcal{E}^{(uv)/v} = \mathcal{E}^u$$

So $\pi^*\mathcal{E}^{x^n/y}$ has good formal decomposition along on U_0. In the chart U_1 where $x = u$ et $y = uv$, the pole locus of $\pi^*\mathcal{E}^{x/y}$ is $uv = 0$, and

$$(\pi^*\mathcal{E}^{x/y})_{|U_1} \simeq \mathcal{E}^{u/(uv)} = \mathcal{E}^{1/v}$$

4. Application to periods of algebraic flat connections

4.1. Irregularity in any dimension. — Let X be a complex manifold and let $i : Z \longrightarrow X$ be an analytic subspace of X. Set $\mathcal{O}_{X|Z} := i^{-1}\mathcal{O}_X$ and let $\mathcal{O}_{\widehat{X|Z}}$ be the formalization of $\mathcal{O}_X$ along Z. Mimicking the 1-dimensional case, we consider the exact sequence of $\mathcal{D}_X$-modules

$$0 \longrightarrow \mathcal{O}_{X|Z} \longrightarrow \mathcal{O}_{\widehat{X|Z}} \longrightarrow \mathcal{Q}_Z \longrightarrow 0$$

For an holonomic $\mathcal{D}_X$-module $\mathcal{M}$, we deduce a distinguished triangle

$$i^{-1}\operatorname{Sol}(\mathcal{M}) \longrightarrow R\mathcal{H}om(\mathcal{M}, \mathcal{O}_{\widehat{X|Z}}) \longrightarrow R\mathcal{H}om(\mathcal{M}, \mathcal{Q}_Z) \xrightarrow{+1}$$

Following [**Meb90**], we define the *irregularity sheaf of* $\mathcal{M}$ *along* Z as $\operatorname{Irr}^*_Z \mathcal{M} := R\mathcal{H}om(\mathcal{M}, \mathcal{Q}_Z)[-1]$ viewed as a complex on X with support in Z.

Definition 4.1.1. — We say that $\mathcal{M}$ is regular if $\operatorname{Irr}^*_Z \mathcal{M} \simeq 0$ for every analytic subspace Z of X.

Example 4.1.2. — If $\mathcal{M}$ is a meromorphic connection with poles along a divisor D, and $i : D \longrightarrow X$ the inclusion. One can show

$$\operatorname{Irr}^*_D \mathcal{M} = i_* i^{-1}\operatorname{Sol}(\mathcal{M})$$

where $\operatorname{Sol}(\mathcal{M}) := R\mathcal{H}om(\mathcal{M}, \mathcal{O}_X)$.

The following theorem generalizes theorem 1.5.2 to any dimension

Theorem 4.1.3. — *Let $\mathcal{M}$ be a meromorphic connection with poles along a divisor D, and $i : D \longrightarrow X$ the inclusion. The following conditions are equivalent*

(1) $\mathcal{M}$ is regular.

(2) The irregularity sheaf of $\mathcal{M}$ along D vanishes on a dense open subset of D.

(3) For every $A \in D$, let $(x_1, \ldots, x_n)$ be local coordinates with D given by $x_1 \cdots x_m = 0$ in a neighbourhood of A. The $R_{n,m}$ differential module $\widehat{\mathcal{M}}_A$ is regular in the sense of 3.2*. That is, $\widehat{\mathcal{M}}_A$ admits a basis in which for every $i = 1, \ldots, n$, the matrix of $x_i \nabla_{\frac{\partial}{\partial x_i}}$ has no poles.*

Note that the equivalence between (1) and (2) does not require D to be a normal crossing. See [**Meb04**].

4.2. Grothendieck-Deligne comparison theorem. — Let X be a smooth complex algebraic variety of dimension d. Let $(\mathcal{E}, \nabla)$ be a flat connection on X. The *De Rham complex* of $\mathcal{E}$ is the complex $\operatorname{DR}\mathcal{E}$

$$0 \longrightarrow \mathcal{E} \xrightarrow{\nabla} \Omega^1_X \otimes \mathcal{E} \longrightarrow \cdots \longrightarrow \Omega^d_X \otimes \mathcal{E} \longrightarrow 0$$

where $\mathcal{E}$ lies in degree 0 and where the $\mathbb{C}$-linear morphism $d : \Omega^k_X \otimes \mathcal{E} \longrightarrow \Omega^{k+1}_X \otimes \mathcal{E}$ is given by

$$d(\omega \otimes m) = d\omega \otimes m + \omega \wedge \nabla m$$

In particular, there is a comparison morphism for every $k \geqslant 0$

$$H^k(X, \mathrm{DR}\,\mathcal{E}) \longrightarrow H^k(X^{\mathrm{an}}, \mathrm{DR}\,\mathcal{E}^{\mathrm{an}}) \tag{4.2.1}$$

By Poincaré lemma, $\mathrm{DR}\,\mathcal{E}^{\mathrm{an}}$ is acyclic in degree > 0, so

$$H^k(X^{\mathrm{an}}, \mathrm{DR}\,\mathcal{E}^{\mathrm{an}}) \simeq H^k(X^{\mathrm{an}}, \mathrm{Ker}\,\nabla^{\mathrm{an}})$$

By Cauchy theorem, $\mathrm{Ker}\,\nabla^{\mathrm{an}}$ is a local system on X^{an} of rank $\mathrm{rk}\,\mathcal{E}$ and (4.2.1) reads

$$H^k(X, \mathrm{DR}\,\mathcal{E}) \longrightarrow H^k(X^{\mathrm{an}}, \mathrm{Ker}\,\nabla^{\mathrm{an}}) \tag{4.2.2}$$

The morphism (4.2.2) has no reason to be an isomorphism in general.

Example 4.2.3. — Take $X = \mathbb{A}^1_{\mathbb{C}}$ and

$$\mathcal{E}^{x^2/2} = (\mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}, f \longrightarrow df - xf dx)$$

The only possibly non zero De Rham cohomology spaces are H^0 and H^1, and we have

$$H^0(\mathbb{A}^1_{\mathbb{C}}, \mathrm{DR}\,\mathcal{E}^{x^2/2}) \simeq 0$$
$$H^1(\mathbb{A}^1_{\mathbb{C}}, \mathrm{DR}\,\mathcal{E}^{x^2/2}) \simeq \mathbb{C}dx$$

Since $\mathrm{Ker}\,\nabla^{\mathrm{an}} \simeq \mathbb{C}e^{x^2/2}$,

$$H^0(\mathbb{C}, \mathrm{Ker}\,\nabla^{\mathrm{an}}) \simeq \mathbb{C}e^{x^2/2}$$
$$H^1(\mathbb{C}, \mathrm{Ker}\,\nabla^{\mathrm{an}}) \simeq 0$$

We have however the following positive result, proven by Grothendieck [**Gro66**] for the trivial connection $\mathcal{O}_X$ and in general by Deligne [**Del70**].

Theorem 4.2.4. — *If* $(\mathcal{E}, \nabla)$ *is regular*[(4)]*, the comparison morphism* (4.2.2) *is an isomorphism.*

Grothendieck and Deligne's proofs are global. A local proof of 4.2.4 is given in [**Meb89**]. It lies in the existence of a triangle

$$R\Gamma(X, \mathrm{DR}\,\mathcal{E}) \longrightarrow R\Gamma(X^{\mathrm{an}}, \mathrm{Ker}\,\nabla^{\mathrm{an}}) \longrightarrow R\Gamma(X^{\mathrm{an}}, \mathrm{Irr}^*_{D^{\mathrm{an}}}(j_*\mathcal{E})^{\mathrm{an}})[1]$$

If $(\mathcal{E}^*, \nabla^*)$ denotes the connection dual to $(\mathcal{E}, \nabla)$, theorem 4.2.4 can be equivalently formulated by saying that the pairing

$$H^k(X, \mathrm{DR}\,\mathcal{E}) \times H_k(X^{\mathrm{an}}, \mathrm{Ker}\,\nabla^{*\,\mathrm{an}}) \longrightarrow \mathbb{C} \tag{4.2.5}$$

induced by

$$((\omega \otimes e), (\gamma \otimes \varphi)) \longrightarrow \int_\gamma \varphi(e)\omega \tag{4.2.6}$$

is perfect if $(\mathcal{E}, \nabla)$ is regular.

---

(4) Regularity in the algebraic context means the following: for a (and actually, any) smooth compactification $j : X \longrightarrow \overline{X}$ of X with $D := \overline{X} \backslash X$ a normal crossing divisor, the meromorphic connection $(j_*\mathcal{E})^{\mathrm{an}}$ with poles along D is regular in the sense of 4.1.3.

4.3. Rapid decay homology. — Let us raise the following

Question. — *How to generalize the perfect pairing* (4.3.2) *to possibly irregular flat connections?*

In example 4.2.3, the pairing (4.3.2) fails to be perfect because the algebraic H^1 is non zero whereas the analytic H^1 is 0. This means that there are not enough cycles on X^{an} to be integrated on. An easy way to remedy this is to look for cycles drawn in a smooth compactification $\overline{X}$ of X with $D := \overline{X}\backslash X$ a normal crossing divisor. However, a flat section $\varphi \in \operatorname{Ker} \nabla^{*\,\mathrm{an}}$ may diverge at infinity. So integrating $\varphi(e)$ along γ does not make sense. To solve this issue, Deligne [**Del07**] proposed to consider only pairs $\gamma \otimes \varphi$ for which φ *has rapid decay along* γ. For a precise sheaf theoretic definition using the language of rapid decay functions on oriented blow-up introduced in section 2, we refer to [**Hie09**, 5.1].

The idea of Deligne leads to *rapid decay homology* $H_k^{\mathrm{rd}}(X, \mathcal{E}^*, \nabla^*)$ of $(\mathcal{E}^*, \nabla^*)$. This homology restores the perfectness of the period pairing (4.3.2), due to the following

Theorem 4.3.1. — *The canonical pairing*

$$H^k(X, \mathrm{DR}\,\mathcal{E}) \times H_k^{\mathrm{rd}}(X, \mathcal{E}^*, \nabla^*) \longrightarrow \mathbb{C} \tag{4.3.2}$$

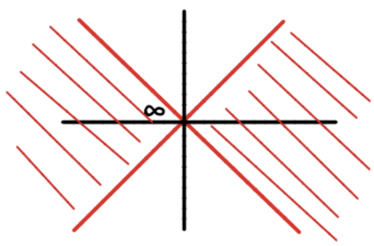
induced by (4.2.6) *is perfect.*

This theorem was proved by Bloch-Esnault [**BE04**] in dimension 1 and by Hien [**Hie09**] in dimension > 1. Hien's proof reduces essentially to the case of exponential modules using Kedlaya-Mochizuki theorem 3.2.7.

Example 4.3.3. — Let us compute the H_1^{rd} of the dual

$$\mathcal{E}^{-x^2/2} = (\mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}, f \longrightarrow df + xf dx)$$

of the connexion $\mathcal{E}^{x^2/2}$ introduced in example 4.2.3. We have $\operatorname{Ker} \nabla^{*\,\mathrm{an}} \simeq \mathbb{C}e^{-x^2/2}$. Taking $\mathbb{P}^1_{\mathbb{C}}$ as compactification, we have to look for segments γ drawned on $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}}$ such that $e^{-x^2/2}$ has rapid decay at ∞ along γ. In the local coordinate $\tau = 1/x$ of ∞, the function $e^{-x^2/2}$ takes the form $e^{-1/2\tau^2}$. The rapid decay condition is achieved for cycles approaching ∞ along directions in $(-\frac{3\pi}{4}, \frac{5\pi}{4})$ or $(-\frac{\pi}{4}, \frac{\pi}{4})$.



In particular, $e^{-x^2/2}$ has rapid decay along the real line $[-\infty, +\infty]$. Thus, $[-\infty, +\infty]\otimes e^{-x^2/2}$ is a rapid decay cycle. The corresponding integral is

$$\int_{-\infty}^{+\infty} e^{-x^2/2}dx = \sqrt{2\pi}$$

It is non zero. Since $H^1(\mathbb{A}^1_{\mathbb{C}}, \mathrm{DR}\,\mathcal{E}^{x^2/2})$ is one dimensional, theorem 4.3.1 implies that $H_1^{\mathrm{rd}}(X, \mathcal{E}^{-x^2/2})$ is the line generated by the class of $[-\infty, +\infty] \otimes e^{-x^2/2}$.

References

[And07] Y. André, *Structure des connexions méromorphes formelles de plusieurs variables et semi-continuité de l'irrégularité*, Invent. math. **170** (2007).

[BE04] S. Bloch and H. Esnault, *Homology for irregular connections*, Journal de Théorie des Nombres de Bordeaux **16** (2004).

[Del70] P. Deligne, *Equations différentielles à points singuliers réguliers*, Lecture Notes in Mathematics, vol. 163, Springer-Verlag, 1970.

[Del07] ———, *Singularités irrégulières: correspondance et documents*, Documents mathématiques, vol. 5, ch. Letter to Malgrange, 16/12/1976, 2007.

[DK13] A. D'Agnolo and M. Kashiwara, *Riemann-Hilbert correspondence for holonomic $\mathcal{D}$-modules*, Preprint, 2013.

[Gro66] A. Grothendieck, *On the De Rham cohomology of algebraic varieties*, Publications Mathématiques de l'IHES **29** (1966).

[Har77] R. Hartshorn, Graduate Texts in Mathematics, vol. 52, 1977.

[Hie09] M. Hien, *Periods for flat algebraic connections*, Invent. Math. **178** (2009).

[Ked10] K. Kedlaya, *Good formal structures for flat meromorphic connections I: Surfaces*, Duke Math.J. **154** (2010).

[Ked11] ———, *Good formal structures for flat meromorpohic connexions II: excellent schemes*, J. Amer. Math. Soc. **24** (2011).

[Mal71] B. Malgrange, *Sur les points singuliers des équations différentielles*, Séminaire d'équations aux dérivées partielles (Polytechnique) (1971).

[Meb89] Z. Mebkhout, *Le théorème de comparaison entre cohomologie de De Rham d'une variété algébrique complexe et le théorème d'existence de Riemann*, Publ. Math. de l'IHES **69** (1989).

[Meb90] ———, *Le théorème de positivité de l'irrégularité pour les $\mathcal{D}_X$-modules*, The Grothendieck Festschrift III, vol. 88, Birkhäuser, 1990.

[Meb04] ———, *Le théorème de positivité, le théorème de comparaison et le théorème d'existence de Riemann*, Éléments de la théorie des systèmes différentiels géométriques, Cours du C.I.M.P.A., Séminaires et Congrès, vol. 8, SMF, 2004.

[Moc09] T. Mochizuki, *Good formal structure for meromorphic flat connections on smooth projective surfaces.*, Algebraic analysis and around in honor of Professor Masaki Kashiwara's 60th birthday, Tokyo: Mathematical Society of Japan, 2009.

[Moc11] ———, *Wild Harmonic Bundles and Wild Pure Twistor $\mathcal{D}$-modules*, Astérisque, vol. 340, SMF, 2011.

[Sab93] C. Sabbah, *Introduction to algebraic theory of linear systems of differential equations*, Eléments de la théorie des systèmes différentiels, Les cours du CIMPA (Paris Hermann, ed.), 1993.

[Sab00] C. Sabbah, *Equations différentielles à points singuliers irréguliers et phénomène de Stokes en dimension 2*, Astérisque, vol. 263, SMF, 2000.

[Sv00] M.T Singer and M. van der Put, *Galois Theory of Linear Differential Equations*, Grundlehren der mathematischen Wissenschaften, vol. 328, Springer, 2000.

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