AN INTRODUCTION TO $\mathcal{D}$-MODULES

by

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Introduction

This text gathers notes of a five hours course on $\mathcal{D}$-modules given for the *Winter School on Derived Categories, Weyl Algebras and Hodge Theory* organized by D. Rumynin and T. Stafford on March 16-20 in Warwick. During this week, a course on *Hodge theory and algebraic geometry* was given by L. Migliorini and a course on *Derived categories and constructible sheaves* was given by G. Williamson. The primary goal of the school was to explain to which extend topology, Hodge theory and $\mathcal{D}$-modules interact. The audience was not supposed to be familiar with $\mathcal{D}$-modules. The following theorem [**Kas75**] was used as a guideline for this course

Theorem 1. — *Let $\mathcal{M}$ be a holonomic $\mathcal{D}$-module on a complex manifold X. Then the de Rham complex $\mathrm{DR}\,\mathcal{M}$ of $\mathcal{M}$ is a perverse sheaf.*

A full proof[(1)] of the following theorem was given

Theorem 2. — *Let X be a complex manifold and let $\mathcal{M}$ be a complex of $\mathcal{D}_X$-modules with bounded and holonomic cohomology. Then $\mathrm{DR}\,\mathcal{M}$ has bounded and constructible cohomology.*

This theorem is a superb application of the machinery of derived categories and functorialities: trying to prove it for a single holonomic $\mathcal{D}_X$-module sticking to X does not lead anywhere whereas push-forward allows to argue by induction on the dimension of X. Since push-forward is not an exact functor, we are naturally led to use derived push-forward, thus producing complexes even if the input $\mathcal{M}$ is concentrated in degree 0. Hence, derived category is the right setting for both the statement and the proof of theorem 2.

Let us explain the content of each section of these notes. The first section introduces the notion of $\mathcal{D}$-modules on a complex manifold, DR and Sol for $\mathcal{D}$-modules and state

(1) Note that we don't claim originality in the proofs given in these notes.

the Riemann-Hilbert correspondence. Section 2 has to do with functorialities for $\mathcal{D}$-modules. We give a proof of Kashiwara theorem on direct images by a closed immersion and a proof of the commutativity of DR with push-forward. In section 3, we explain why the characteristic variety is of fundamental importance in the theory of $\mathcal{D}$-module and how it can be used. Section 4 gives a full proof of Kashiwara constructibility theorem, following [**LM93**]. Section 5 has to do with regularity. For meromorphic connections, we insist on the necessity of a meromorphic structure in the analytic setting already to state a definition, and then give some fundamental theorems leading to the algebraic Riemann-Hilbert correspondence for algebraic flat connections.

I thank the organizers of the Winter School *Derived Categories, Weyl Algebras and Hodge Theory* for giving me the opportunity to teach this course, as well as L. Migliorini and G. Williamson for advices on what should be and what should not be in a first course on $\mathcal{D}$-modules, saving the audience from a certain number of unpleasant computations. I also thank the students who attended the school for providing a very pleasant and stimulating work atmosphere during the week.

Contents

1. The formalism of $\mathcal{D}$-modules

Let X be a complex manifold and let d_X be the dimension of X. We denote by TX the tangent bundle of X and by T^*X the cotangent bundle of X. We denote by Θ_X (resp. Ω^1_X) the sheaf of sections of TX (resp. T^*X) on X. If $p : T^*X \longrightarrow X$ denotes the canonical projection, Θ_X is a subsheaf of $p_*\mathcal{O}_{T^*X}$.

Note that Θ_X can be viewed as the subsheaf of derivations of $\mathrm{End}_{\mathbb{C}}(\mathcal{O}_X)$. More generally $\mathrm{Sym}^k\,\Theta_X$ can be viewed as the subsheaf of $\mathcal{H}om_{\mathbb{C}}(\otimes^k_{\mathbb{C}}\mathcal{O}_X, \mathcal{O}_X)$ of symmetric k-linear forms restricting to derivations on each factor.

1.1. On the filtered sheaf $\mathcal{D}_X$. — A holomorphic function $f \in \mathcal{O}_X$ defines an element of $\mathrm{End}_{\mathbb{C}}(\mathcal{O}_X)$ by multiplication. We say that this element is a *differential operator of order 0* and we still denote it by f.

For $n \geqslant 1$, we say that $P \in \mathrm{End}_{\mathbb{C}}(\mathcal{O}_X)$ is a *differential operator of order* $\leqslant n$ if for every $f \in \mathcal{O}_X$, the commutator $[P, f] = Pf - fP$ is a differential operator of order $\leqslant n-1$.

Definition 1.1.1. — We denote by $\mathcal{D}_X$ the subsheaf of $\mathrm{End}_{\mathbb{C}}(\mathcal{O}_X)$ of differential operators with finite order.

The following filtration

$$F^n\mathcal{D}_X := \{\text{differential operators of order} \leqslant n\}$$

turns $\mathcal{D}_X$ into a sheaf of filtered algebras. We say that $P \in \mathcal{D}_X$ has *order n* if P has order $\leqslant n$ and P has not order $\leqslant n-1$.

In $X = \mathbb{C}^n$ endowed with coordinates $(x_1, \ldots, x_n)$, Θ_X is the free $\mathcal{O}_X$-module generated by the tangent vector fields $\frac{\partial}{\partial x_i}$ acting on $\mathcal{O}_X$ by differentiation. We have the following

$$\mathcal{D}_{\mathbb{C}^n} \simeq \bigoplus_{\alpha\in\mathbb{N}^n} \mathcal{O}_{\mathbb{C}^n}\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} \tag{1.1.2}$$

So for a general smooth X, $\mathcal{D}_X$ is an infinite dimensional vector bundle generated by Θ_X as an $\mathcal{O}_X$-algebra.

Definition 1.1.3. — A left (resp. right) $\mathcal{D}_X$*-module* is a sheaf of modules over $\mathcal{D}_X$ (resp. $\mathcal{D}_X^{\mathrm{op}}$).

In the sequel, we denote by $\mathcal{D}_X$-mod (resp. $\mathcal{D}_X^{\mathrm{op}}$-mod) the category of left (resp. right) $\mathcal{D}_X$-modules. Right $\mathcal{D}_X$-modules are important since nature produces plenty of them (see 1.2.2) and they are used to define direct image for left $\mathcal{D}_X$-modules in a convenient way.

1.2. A short way to define a $\mathcal{D}$-module. — As a consequence of 1.1.2, a left $\mathcal{D}_X$-module is the same as an $\mathcal{O}_X$-module $\mathcal{M}$ endowed with an action

$$\begin{array}{rcl} \Theta_X \otimes_{\mathbb{C}_X} \mathcal{M} & \longrightarrow & \mathcal{M} \\ \xi \otimes m & \longrightarrow & \xi \cdot m \end{array}$$

satisfying for every $f \in \mathcal{O}_X$, $\xi, \eta \in \Theta_X$, $m \in \mathcal{M}$ the following identities:

(1) $f\xi \cdot m = f \cdot (\xi \cdot m)$
(2) $\xi \cdot fm = \xi(f)m + f\xi \cdot m$
(3) $[\xi, \eta] \cdot m = \xi \cdot (\eta \cdot m) - \eta \cdot (\xi \cdot m)$

In local coordinates, relation (3) says for example that the actions of $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial x_j}$ commute.

Example 1.2.1. — As a consequence of Schwarz theorem, $\mathcal{O}_X$ is a left $\mathcal{D}_X$-module.

Similarly, a right $\mathcal{D}_X$-module is the same as an $\mathcal{O}_X$-module $\mathcal{N}$ endowed with an action

$$\begin{array}{rcl} \mathcal{N} \otimes_{\mathbb{C}_X} \Theta_X & \longrightarrow & \mathcal{N} \\ m \otimes \xi & \longrightarrow & m \cdot \xi \end{array}$$

satisfying for every $f \in \mathcal{O}_X$, $\xi, \eta \in \Theta_X$, $n \in \mathcal{N}$ the following identities:

(1) $n \cdot f\xi = fn \cdot \xi$
(2) $f(n \cdot \xi) = \xi(f)n + n \cdot f\xi$
(3) $n \cdot [\xi, \eta] = (n \cdot \xi) \cdot \eta - (n \cdot \eta) \cdot \xi$

Example 1.2.2. — For $\omega \in \Omega_X^k$ and for $\xi \in \Theta_X$, recall that we have a contraction operation $\iota_\xi : \Omega_X^k \longrightarrow \Omega_X^{k-1}$ which associates to $\omega \in \Omega_X^k$ the $k-1$ form

$$\begin{array}{rcl} \iota_\xi \omega : \Lambda^{k-1}\Theta_X & \longrightarrow & \mathcal{O}_X \\ \xi_1 \wedge \cdots \wedge \xi_{k-1} & \longrightarrow & \omega(\xi, \xi_1, \ldots, \xi_{k-1}) \end{array}$$

The *Lie derivative* is the operator

$$\begin{array}{rcl} \mathcal{L}_\xi : \Omega_X^k & \longrightarrow & \Omega_X^k \\ \omega & \longrightarrow & \iota_\xi d\omega + d\iota_\xi \omega \end{array}$$

It satisfies the following identities

(1) $\mathcal{L}_{f\xi}\omega = f\mathcal{L}_\xi \omega + df \wedge \iota_\xi \omega$
(2) $\mathcal{L}_{[\xi_1, \xi_2]}\omega = \mathcal{L}_{\xi_1}\mathcal{L}_{\xi_2}\omega - \mathcal{L}_{\xi_2}\mathcal{L}_{\xi_1}\omega$

For $\omega \in \omega_X := \Omega_X^{d_X}$, we have in particular $\mathcal{L}_\xi \omega = d\iota_\xi \omega$. Thus, for such ω we have

$$\mathcal{L}_{f\xi}\omega = d\iota_{f\xi}\omega = d\iota_\xi f\omega = \mathcal{L}_\xi f\omega$$

Since ι_ξ is skew multiplicative,

$$\iota_\xi(df \wedge \omega) = \xi(f)\omega - df \wedge \iota_\xi\omega = 0$$

So (1) gives

$$\mathcal{L}_{f\xi}\omega = d\iota_{f\xi}\omega = d(f\iota_\xi\omega) = df \wedge \iota_\xi\omega + fd\iota_\xi\omega = \xi(f)\omega + f\mathcal{L}_\xi\omega$$

So the assignment $\omega \cdot \xi := -\mathcal{L}_\xi\omega$ for every $\omega \in \omega_X$ and every $\xi \in \Theta_X$ defines a right $\mathcal{D}_X$-module structure on ω_X.

1.3. Left and right. —

1.3.1. From left to right. — Let $\mathcal{M}$ (resp. $\mathcal{N}$) be a left (resp. right) $\mathcal{D}_X$-module. We see that the following formula

$$(n \otimes m) \cdot \xi = n\xi \otimes m - n \otimes \xi m$$

endows $\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{M}$ with a structure of $\mathcal{D}_X^{\mathrm{op}}$-module. We can thus define the $\mathcal{D}_X^{\mathrm{op}}$-module

$$\mathcal{M}^{\mathfrak{r}} := \omega_X \otimes_{\mathcal{O}_X} \mathcal{M} \tag{1.3.1}$$

1.3.2. From right to left. — Let $\mathcal{N}, \mathcal{N}'$ be $\mathcal{D}_X^{\mathrm{op}}$-modules. We see that the following formula

$$(\xi \cdot \varphi)(n) = \varphi(n \cdot \xi) - \varphi(n) \cdot \xi$$

endows $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{N}')$ with a structure of $\mathcal{D}_X$-module. We can thus define the $\mathcal{D}_X$-module

$$\mathcal{N}^{\mathfrak{l}} := \mathcal{H}om_{\mathcal{O}_X}(\omega_X, \mathcal{N}) = \omega_X^{-1} \otimes_{\mathcal{O}_X} \mathcal{N} \tag{1.3.2}$$

The functors $\mathfrak{r}$ and $\mathfrak{l}$ are equivalences of categories inverse to each other.

1.4. Solution and de Rham functors. —

Let $\mathcal{M} \in \mathcal{D}_X$-mod. We define the *de Rham complex* $\mathrm{DR}\,\mathcal{M}$ *of* $\mathcal{M}$ as

$$\mathcal{M} \longrightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow \cdots \longrightarrow \Omega_X^{d_X} \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow 0$$

where $\mathcal{M}$ sits in degree $-d_X$, and where the differentials are defined in local coordinates $(x_1, \dots, x_n)$ as follows

$$\omega \otimes m \longrightarrow d\omega \otimes m + \sum dx_i \wedge \omega \otimes \frac{\partial}{\partial x_i} m$$

We define *the solution complex of* $\mathcal{M}$ as

$$\mathrm{Sol}\,\mathcal{M} := R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \tag{1.4.1}$$

For holonomic[(2)] $\mathcal{D}_X$-modules, $\mathrm{DR}\,\mathcal{M}$ and $\mathrm{Sol}\,\mathcal{M}$ are related by the following

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[(2)] Let us say for now that holonomy is the right condition to put on $\mathcal{D}_X$-modules to have good finiteness properties.

Theorem 1.4.2. — *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. There is a canonical isomorphism*

$$\mathrm{Sol}\,\mathcal{M} \simeq R\mathcal{H}om(\mathrm{DR}\,\mathcal{M}, \mathbb{C}_X)[d_X]$$

For two different proofs of this theorem, let us mention [**Meb79**] and [**KK81**].

Example 1.4.3. — Let us take $P \in \mathcal{D}_X$ and let us "compute" $\mathrm{Sol}(\mathcal{D}_X/\mathcal{D}_X P)$. Homological algebra says we can compute it using an injective resolution of $\mathcal{O}_X$ or a projective resolution of $\mathcal{D}_X/\mathcal{D}_X P$. The first option is less than tempting, since we don't have any favourite injective resolution of $\mathcal{O}_X$ as a $\mathcal{D}_X$-module. However, we have the following projective resolution

$$0 \longrightarrow \mathcal{D}_X \xrightarrow{\cdot P} \mathcal{D}_X \longrightarrow \mathcal{D}_X/\mathcal{D}_X P \longrightarrow 0$$

Hence $\mathrm{Sol}(\mathcal{D}_X/\mathcal{D}_X P)$ is the complex obtained by applying $\mathcal{H}om(\,\cdot\,, \mathcal{O}_X)$ to

$$\mathcal{D}_X \xrightarrow{\cdot P} \mathcal{D}_X$$

Since a morphism of $\mathcal{D}_X$-modules with source $\mathcal{D}_X$ is uniquely determined by the image of 1, $\mathrm{Sol}(\mathcal{D}_X/\mathcal{D}_X P)$ is computed as

$$\mathcal{O}_X \xrightarrow{P} \mathcal{O}_X$$

Hence,

$$\mathcal{H}^0\,\mathrm{Sol}(\mathcal{D}_X/\mathcal{D}_X P) \simeq \{f \in \mathcal{O}_X \text{ such that } P(f) = 0\}$$

and

$$\mathcal{H}^1\,\mathrm{Sol}(\mathcal{D}_X/\mathcal{D}_X P) \simeq \mathcal{O}_X/P(\mathcal{O}_X)$$

This explains why (1.4.1) should be called "the solution complex".

Example 1.4.4. — By holomorphic Poincaré lemma, $\mathrm{DR}\,\mathcal{O}_X$ is quasi-isomorphic to $\mathbb{C}_X[d_X]$.

Example 1.4.5. — A *flat connection* on a complex manifold X is a $\mathcal{D}_X$-module whose underlying sheaf E is locally free of finite rank r. Such a $\mathcal{D}_X$-module is usually denoted by (E, ∇), where ∇ is the first differential in the de Rham complex. Take X to be a ball B in X endowed with coordinates $(x_1, \dots, x_n)$. Let $e_1, \dots, e_r$ be a trivialization of E on B. Then

$$\begin{array}{rcl} \nabla : E & \longrightarrow & \Omega^1_B \otimes_{\mathcal{O}_B} E \\ e_j & \longrightarrow & \displaystyle\sum_{i=1}^{r} \gamma_{ij} \otimes e_i \end{array}$$

where $\Gamma := (\gamma_{ij}) \in \Gamma(B, \mathrm{Mat}_r(\Omega^1_X))$. Hence, we see that $\mathrm{Ker}\,\nabla$ (also called the sheaf of horizontal sections of E) identifies to the sheaf of r-uples of column vectors $\underline{f} \in \mathcal{O}_B^r$ satisfying

$$\frac{\partial \underline{f}}{\partial x_i} = -(\iota_{\frac{\partial}{\partial x_i}} \Gamma)\underline{f} \tag{1.4.6}$$

for every $i = 1, \dots, n$. By Cauchy theorem, the spaces of solutions of (1.4.6) defined on B is finite dimensional and evaluation at $0 \in B$ yields an isomorphism

$\Gamma(B, \operatorname{Ker} \nabla) \xrightarrow{\ \sim\ } \mathbb{C}^r$. Hence, a basis $s_1, \dots, s_r \in \Gamma(B, \operatorname{Ker} \nabla)$ gives rise to an isomorphism of $\mathcal{D}_B$-modules $\mathcal{O}_B^r \xrightarrow{\ \sim\ } (E, \nabla)_{|B}$. From 1.4.4, we deduce that $\operatorname{DR}(E, \nabla)$ is concentrated in degree $-d_X$ and that $\mathcal{H}^{-d_X} \operatorname{DR}(E, \nabla)$ is a local system.

1.5. The Riemann-Hilbert correspondence. — It can be summarized by the following diagram

(1.5.1)
$$\begin{array}{ccc}
\operatorname{Conn}(X) & \xrightarrow[\mathcal{H}^{-d_X}\operatorname{DR}]{\sim} & \operatorname{Loc}_{\mathbb{C}}(X) \\
\big\downarrow & & \big\downarrow \\
\operatorname{Reg}(X) & \xrightarrow[\operatorname{DR}]{\sim} & \operatorname{Perv}(X) \\
\big\downarrow & & \big\downarrow \\
D^b_{rh}(\mathcal{D}_X) & \xrightarrow[\operatorname{DR}]{\sim} & D^b_c(X, \mathbb{C})
\end{array}$$

In this diagram, $\sim$ means "is an equivalence of categories" and

- $\operatorname{Conn}(X)$ is the category of flat connections 1.4.5 on X.
- $\operatorname{Loc}_{\mathbb{C}}(X)$ is the category of local systems of $\mathbb{C}$-vector spaces on X.
- $\operatorname{Reg}(X)$ is the category of regular holonomic $\mathcal{D}_X$-modules (see 3.3.1 and 5.2.1).
- $\operatorname{Perv}(X)$ is the category of perverse sheaves [**BBD82**] on X.
- $D^b_{rh}(\mathcal{D}_X)$ is the derived category of complexes of $\mathcal{D}_X$-modules with bounded and holonomic regular cohomology.
- $D^b_c(X, \mathbb{C})$ is the derived category of complexes of sheaves of $\mathbb{C}$-vector spaces with bounded and constructible cohomology.

An inverse functor to the top horizontal arrow of (1.5.1) is easy to construct. From $\mathcal{L} \in \operatorname{Loc}_{\mathbb{C}}(X)$, one defines $E := \mathcal{O}_X \otimes_{\mathbb{C}_X} \mathcal{L}$ and

$$\begin{array}{rcl}
\nabla : E & \longrightarrow & \Omega^1_X \otimes_{\mathcal{O}_X} E \\
f \otimes s & \longrightarrow & df \otimes s
\end{array}$$

Let us mention that the first equivalence appeared for the first time in [**Del70**]. It's interest lies in the fact that $\operatorname{Loc}_{\mathbb{C}}(X)$ is equivalent to the category of $\mathbb{C}$-representations of $\pi_1(X)$, whereas if X is algebraic, $\operatorname{Conn}(X)$ admits a purely algebraic interpretation. Thus, Deligne's correspondance provides a bridge between topology and algebraic geometry.

The general Riemann-Hilbert correspondence was proved by Kashiwara [**Kas79**] [**Kas84**] and Mebkhout [**Meb80**][**Meb84**] using two different (but non trivially equivalent) approaches to regularity. Note that in Kashiwara's approach, the preservation of regularity by duality is a triviality, but the fully-faithfullness in the Riemann-Hilbert correspondence is a theorem. Moreover, an explicit inverse functor is constructed.

In Mebkhout's approach, preservation of regularity by duality is a hard fact, but the fully faithfulness is almost tautological. This approach also provides a very flexible sheaf-theoretic measure of the failure of regularity, that is the *irregularity sheaf* [**Meb90**].

The interest of such a correspondence is not to be proved: it led to the discovery of perverse sheaves which are now ubiquitous in topology and representation theory. A purely sheaf-theoretic description of a (possibly irregular) $\mathcal{D}$-module that would enhance the Riemann-Hilbert correspondence to all holonomic $\mathcal{D}_X$-modules is a very active area. For a recent breakthrough, let us mention [**dK13**].

2. Functorialities

2.1. Inverse image. — Let $f : X \longrightarrow Y$ be a morphism of smooth manifolds and let $\mathcal{M} \in \mathcal{D}_Y$-mod. Let us denote by $f^+M \in \mathcal{D}_X$-mod the inverse image of $\mathcal{M}$ by f in the sense of $\mathcal{D}$-modules. By definition, the underlying $\mathcal{O}_X$-module of $\mathcal{M}$ is

$$f^*\mathcal{M} := \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{M}$$

where f^{-1} is the topological inverse image. The $\mathcal{D}_X$-module structure is given by the following formula. Let $x \in X$ and $y = f(x)$. Let $\mathbf{x} = (x_1, \ldots, x_n)$ (resp. $\mathbf{y} = (y_1, \ldots, y_p)$) be local coordinates on X centred at x (resp. on Y centred at y). Let $f = (f_1, \ldots, f_p)$ be the components of f in $\mathbf{y}$, and let $g \in \mathcal{O}_{X,x}$ and $m \in \mathcal{M}_y$. Then for every $i = 1, \ldots, n$, we define

$$\frac{\partial}{\partial x_i}(g \otimes m) = (\frac{\partial g}{\partial x_i}) \otimes m + \sum_{j=1}^{p} g \frac{\partial f_j}{\partial x_i} \otimes \frac{\partial}{\partial x_j} m$$

Thus $f^+ : \mathcal{D}_Y\text{-mod} \longrightarrow \mathcal{D}_X\text{-mod}$ is right exact. Let $\mathrm{for}_X : \mathcal{D}_X\text{-mod} \longrightarrow \mathcal{O}_X\text{-mod}$ be the functor forgetting the action of $\mathcal{D}_X$. It is exact and we have

$$\mathrm{for}_X\, f^+ = f^*\, \mathrm{for}_Y$$

Since $\mathcal{D}_Y$-mod has enough flat objects, the left derived functor

$$\mathbb{L}f^+ : D^-(\mathcal{D}_Y\text{-mod}) \longrightarrow D^-(\mathcal{D}_X\text{-mod})$$

associated to f^+ is well defined. Since $\mathcal{D}_X$ is locally free as a $\mathcal{O}_X$-module, it is flat. Thus, any $\mathcal{D}_X$-flat complex is also $\mathcal{O}_X$-flat. Hence

$$\mathbb{L}(f^*\, \mathrm{for}_Y) \simeq \mathbb{L}f^*\mathbb{L}\, \mathrm{for}_Y \simeq \mathbb{L}f^*\, \mathrm{for}_Y$$

So

$$\mathrm{for}_X\, \mathbb{L}f^+ = \mathbb{L}f^*\, \mathrm{for}_Y$$

Let us define the *transfer bimodule for* f by $\mathcal{D}_{X\to Y} := f^+\mathcal{D}_Y$. This is a $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$-module. Another way to look at f^+ is to notice that

$$\begin{aligned} f^+\mathcal{M} &:= \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{M} \\ &\simeq \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{M} \\ &\simeq \mathcal{D}_{X\to Y} \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{M} \end{aligned}$$

Through this identification, the action of $\mathcal{D}_X$ on $f^+\mathcal{M}$ is inherited from the action of $\mathcal{D}_X$ on $\mathcal{D}_{X\to Y}$. Thus

$$\mathbb{L}f^+\mathcal{M} \simeq \mathcal{D}_{X\to Y} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{M}$$

In the sequel, we write f^+ for $\mathbb{L}f^+$.

Example 2.1.1. — Let $i : \mathbb{C}^p \longrightarrow \mathbb{C}^n$ be the canonical inclusion $(x_1, \dots, x_p) \longrightarrow (x_1, \dots, x_p, 0, \dots, 0)$. From 1.1.2, we see that

$$i^+\mathcal{D}_{\mathbb{C}^n} \simeq \bigoplus_{\alpha \in \mathbb{N}^{[\![p+1,n]\!]}} \mathcal{D}_{\mathbb{C}^p} \left(\frac{\partial}{\partial x_{p+1}}\right)^{\alpha_{p+1}} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

Hence, inverse image for $\mathcal{D}$-modules does not preserve coherence.

2.2. Direct image. — Let $f : X \longrightarrow Y$ be a morphism of smooth manifolds. We define a direct image functor f_+ for right $\mathcal{D}$-modules and deduce a direct image functor (also denoted by f_+) for left $\mathcal{D}_X$-modules by the formula

$$\mathcal{M} \longrightarrow (f_+\mathcal{M}^{\mathfrak{r}})^{\mathfrak{l}}$$

Let us recall that $\mathcal{D}_{X\to Y} := f^+\mathcal{D}_Y$ is a $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$-module. Hence, for every right $\mathcal{D}_X$-module $\mathcal{N}$, the sheaf $\mathcal{N} \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}$ is a right $f^{-1}\mathcal{D}_Y$-module. Hence

$$f_*(\mathcal{N} \otimes_{\mathcal{D}_X} f^{-1}\mathcal{D}_Y) \tag{2.2.1}$$

is a right $f_*f^{-1}\mathcal{D}_Y$-module. Thus, the adjunction morphism $\mathcal{D}_Y \longrightarrow f_*f^{-1}\mathcal{D}_Y$ endows (2.2.1) with a structure of right $\mathcal{D}_Y$-module. We define[(3)]

$$\begin{array}{rcl} f_+ : D^+(\mathcal{D}_X^{\mathrm{op}}) & \longrightarrow & D^+(\mathcal{D}_Y^{\mathrm{op}}) \\ \mathcal{N} & \longrightarrow & Rf_*(\mathcal{N} \otimes^{\mathbb{L}}_{\mathcal{D}_X} \mathcal{D}_{X\to Y}) \end{array}$$

2.3. The Spencer complex. — Let X be a complex manifold. The Spencer complex $\mathrm{Sp}_X^\bullet$ is a resolution of $\mathcal{O}_X$ as a $\mathcal{D}_X$-module by locally free $\mathcal{D}_X$-modules. Hence, it is adapted to compute $\otimes^{\mathbb{L}}_{\mathcal{D}_X} \mathcal{O}_X$. We denote by $\mathrm{Sp}_X^\bullet$ the complex whose term in degree $-k$ is $\mathcal{D}_X \otimes_{\mathcal{O}_X} \Lambda^k\Theta_X$ and whose differentials are given by[(4)]

$$P \otimes \boldsymbol{\xi} \longrightarrow \sum_{i=1}^{k} (-1)^{i-1} P\xi_i \otimes \widehat{\boldsymbol{\xi}}^i + \sum_{i<j} (-1)^{i+j} P \otimes [\xi_i, \xi_j] \wedge \widehat{\boldsymbol{\xi}}^{i,j}$$

$(\mathrm{Sp}_X^\bullet, \delta_X)$ is a complex of $\mathcal{D}_X$-modules. This is the *Spencer complex of X*. Let us notice that $\mathcal{H}^0\,\mathrm{Sp}_X^\bullet$ identifies canonically to $\mathcal{O}_X$.

Lemma 2.3.1. — *The canonical map*

$$\mathrm{Sp}_X^\bullet \longrightarrow \mathcal{O}_X[0] \tag{2.3.2}$$

is a quasi-isomorphism of $\mathcal{D}_X$-modules.

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(3) Note that some care would be needed here since we derive a left exact and a right exact functor.

(4) where $\boldsymbol{\xi}$ stands for $\xi_1 \wedge \cdots \wedge \xi_k$, $\widehat{\boldsymbol{\xi}}^i$ stands for $\xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \xi_k$ and

$$\widehat{\boldsymbol{\xi}}^{i,j} = \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \widehat{\xi_j} \wedge \cdots \wedge \xi_k$$

Proof. — We have to prove that $\mathrm{Sp}_X^\bullet$ is acyclic in degree < 0. Let $F_p \mathrm{Sp}_X$ be the subcomplex of $\mathrm{Sp}_X^\bullet$ whose term in degree $-k$ is

$$F_{p-k}\mathcal{D}_X \otimes_{\mathcal{O}_X} \Lambda^k \Theta_X$$

From 6.3 and since Θ_X is flat over $\mathcal{O}_X$, the graded complex $\mathrm{gr}^F \mathrm{Sp}_X^\bullet := \oplus_k \mathrm{gr}_k^F \mathrm{Sp}_X$ identifies to the complex

$$\longrightarrow \mathrm{Sym}\,\Theta_X \otimes_{\mathcal{O}_X} \Lambda^k \Theta_X \longrightarrow \mathrm{Sym}\,\Theta_X \otimes_{\mathcal{O}_X} \Lambda^{k-1}\Theta_X \longrightarrow$$

with differential given by

$$\overline{P} \otimes \boldsymbol{\xi} \longrightarrow \sum_{i=1}^{k} (-1)^{i-1} \overline{P}\xi_i \otimes \widehat{\boldsymbol{\xi}}^i$$

We have a canonical identification

$$\mathrm{gr}^F \mathrm{Sp}_X^k \simeq \Lambda^k_{\mathrm{Sym}\,\Theta_X}(\mathrm{Sym}\,\Theta_X \otimes_{\mathcal{O}_X} \Theta_X)$$

Let $(x_1, \ldots, x_n)$ be local coordinates centred at $x \in X$. Then, $\Theta_{X,x}$ is a free $\mathcal{O}_{X,x}$-module with base given by the germs at x of the $\frac{\partial}{\partial x_i}$. If we denote by τ_i the class of $\frac{\partial}{\partial x_i}$ in $(\mathrm{Sym}^1 \Theta_X)_x$, the complex $(\mathrm{gr}^F \mathrm{Sp}_X^\bullet)_x$ identifies with the complex

$$\longrightarrow \Lambda^k_{\mathcal{O}_{X,x}[\tau]} \mathcal{O}_{X,x}[\tau]^n \longrightarrow \Lambda^{k-1}_{\mathcal{O}_{X,x}[\tau]} \mathcal{O}_{X,x}[\tau]^n \longrightarrow$$

with differential given by

$$e_{i_1} \wedge \cdots \wedge e_{i_k} \longrightarrow \sum_{j=1}^{k} (-1)^{j-1} \tau_i e_1 \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_k}$$

So $(\mathrm{gr}^F \mathrm{Sp}_X^\bullet)_x$ is the Koszul complex of $\mathcal{O}_{X,x}[\tau]$ for the regular sequence $\tau_1, \ldots, \tau_n$. We deduce that the projection morphism

$$\mathrm{gr}^F \mathrm{Sp}_X^\bullet \longrightarrow \mathrm{gr}_0^F \mathrm{Sp}_X^\bullet = \mathcal{O}_X[0]$$

is a quasi-isomorphism. Hence, $\mathrm{gr}_p^F \mathrm{Sp}_X^\bullet$ is acyclic for every $p > 0$. Since we have exact sequences of complexes

$$0 \longrightarrow F_{p-1} \mathrm{Sp}_X^\bullet / F_0 \mathrm{Sp}_X^\bullet \longrightarrow F_p \mathrm{Sp}_X^\bullet / F_0 \mathrm{Sp}_X^\bullet \longrightarrow \mathrm{gr}_p^F \mathrm{Sp}_X^\bullet \longrightarrow 0$$

a recursion on p shows that $F_p \mathrm{Sp}_X^\bullet / F_0 \mathrm{Sp}_X^\bullet$ is acyclic. Since filtered colimits are exact, $\mathrm{Sp}_X^\bullet / F_0 \mathrm{Sp}_X^\bullet$ is acyclic, so $\mathrm{Sp}_X^\bullet$ is acyclic in degree < 0 and 2.3.1 is proven. □

2.4. DR **commutes with direct image.** — Let X be a manifold and let $\mathcal{M}$ be a $\mathcal{D}_X$-module. We denote by $D(X, \mathbb{C})$ the derived category of sheaves of $\mathbb{C}$-vector spaces on X. We prove the following

Proposition 2.4.1. — *There is a canonical isomorphism in* $D(X, \mathbb{C})$

$$\mathrm{DR}\,\mathcal{M} \xrightarrow{\sim} \mathcal{M}^{\mathrm{r}} \otimes^{\mathbb{L}}_{\mathcal{D}_X} \mathcal{O}_X$$

Proof. — By 2.3.1, we have in $D(X,\mathbb{C})$

$$\begin{aligned}\mathcal{M}^{\mathrm{r}} \otimes^{\mathbb{L}}_{\mathcal{D}_X} \mathcal{O}_X &\simeq \mathcal{M}^{\mathrm{r}} \otimes_{\mathcal{D}_X} \mathrm{Sp}^{\bullet}_X \\ &\simeq \mathcal{M}^{\mathrm{r}} \otimes_{\mathcal{O}_X} \Theta^{-\bullet}_X\end{aligned}$$

The differential in degree $-k$ is given by

$$\omega\otimes m\otimes\boldsymbol{\xi} \longrightarrow \sum_{i=1}^{k}(-1)^{i-1}(-\mathcal{L}_{\xi_i}\omega\otimes m\otimes\widehat{\boldsymbol{\xi}}^{i} - \omega\otimes\xi_i m\otimes\widehat{\boldsymbol{\xi}}^{i}) + \sum_{i<j}(-1)^{i+j}\omega\otimes m\otimes[\xi_i,\xi_j]\wedge\widehat{\boldsymbol{\xi}}^{i,j}$$

Moreover we have an isomorphism of $\mathcal{O}_X$-modules

$$\begin{aligned}\omega_X \otimes_{\mathcal{O}_X} \mathcal{M} \otimes_{\mathcal{O}_X} \Lambda^k\Theta_X &\xrightarrow{\ \sim\ } \Omega^{n-k}_X \otimes_{\mathcal{O}_X} \mathcal{M} \\ \omega\otimes m\otimes\boldsymbol{\xi} &\longrightarrow (-1)^{k(k-1)/2}\omega(\boldsymbol{\xi}\wedge\cdot)\otimes m\end{aligned}$$

The right hand side is the term of degree $-k$ of $\mathrm{DR}\,\mathcal{M}$. Let us check that the following diagram

$$\begin{array}{ccc} \omega_X \otimes_{\mathcal{O}_X} \mathcal{M} \otimes_{\mathcal{O}_X} \Lambda^k\Theta_X & \longrightarrow & \Omega^{n-k}_X \otimes_{\mathcal{O}_X} \mathcal{M} \\ \downarrow & & \downarrow \\ \omega_X \otimes_{\mathcal{O}_X} \mathcal{M} \otimes_{\mathcal{O}_X} \Lambda^{k+1}\Theta_X & \longrightarrow & \Omega^{n-k-1}_X \otimes_{\mathcal{O}_X} \mathcal{M} \end{array} \tag{2.4.2}$$

commutes. Taking local coordinates $(x_1,\dots,x_n)$, it is enough to prove the compatibility for

$$\boldsymbol{\xi} = \frac{\partial}{\partial x_{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial x_{i_k}}$$

where $I\subset[\![1,n]\!]$. For $J\subset[\![1,n]\!]$ with elements $j_1<\cdots<j_m$, let us define

$$dx_J := dx_{j_1}\wedge\cdots\wedge dx_{j_m}$$

Let us write $\omega = f dx_I\wedge dx_{I^c}$ and define $\eta := \omega(\boldsymbol{\xi}\wedge\cdot)$. Then, the composite of the left and down arrows of (2.4.2) gives

$$\omega\otimes m\otimes\boldsymbol{\xi} \longrightarrow (-1)^{k(k+1)/2}\sum_{j=1}^{k}(-1)^j\big((\mathcal{L}_{\xi_j}\omega)(\widehat{\boldsymbol{\xi}}^{j}\wedge\cdot)\otimes m + \omega(\widehat{\boldsymbol{\xi}}^{j}\wedge\cdot)\otimes\xi_j m\big) \tag{2.4.3}$$

Since

$$\begin{aligned}\omega(\widehat{\boldsymbol{\xi}}^{j}\wedge\cdot) &= (-1)^{k-j} f dx_{I\setminus\{i_j\}}\wedge dx_{i_j}\wedge dx_{I^c}(\widehat{\boldsymbol{\xi}}^{j}\wedge\cdot) \\ &= (-1)^{k-j} f dx_{i_j}\wedge dx_{I^c} \\ &= (-1)^{k-j} dx_{i_j}\wedge\eta\end{aligned}$$

Thus

$$\begin{aligned}\sum_{j=1}^{k}(-1)^j\omega(\widehat{\boldsymbol{\xi}}^{j}\wedge\cdot)\otimes\xi_j m &= (-1)^k\sum_{i\in I} dx_i\wedge\eta\otimes\partial_i m \\ &= (-1)^k\sum_{i=1}^{k} dx_i\wedge\eta\otimes\partial_i m\end{aligned}$$

On the other hand

$$\begin{aligned}(\mathcal{L}_{\xi_j}\omega)(\widehat{\boldsymbol{\xi}}^j \wedge \cdot) &= (d\iota_{\xi_j}\omega)(\widehat{\boldsymbol{\xi}}^j \wedge \cdot)\\ &= (-1)^{j-1}(df \wedge dx_{I\setminus\{i_j\}} \wedge dx_{I^c})(\widehat{\boldsymbol{\xi}}^j \wedge \cdot)\\ &= (-1)^{j+k-2}\frac{\partial f}{\partial x_{i_j}}(dx_{I\setminus\{i_j\}} \wedge dx_{i_j} \wedge dx_{I^c})(\widehat{\boldsymbol{\xi}}^j \wedge \cdot)\\ &= (-1)^{j+k}\frac{\partial f}{\partial x_{i_j}} dx_{i_j} \wedge dx_{I^c}\end{aligned}$$

Thus

$$\sum_{j=1}^{k}(-1)^j(\mathcal{L}_{\xi_j}\omega)(\widehat{\boldsymbol{\xi}}^j \wedge \cdot) \otimes m = (-1)^k d\eta \otimes m$$

and the commutativity of (2.4.2) is proved. □

Corollary 2.4.4. — *Let $f : X \longrightarrow Y$ be a morphism of complex manifolds, and let $\mathcal{M}$ be a $\mathcal{D}_X$-module. There is a canonical isomorphism in $D(X, \mathbb{C})$*

$$\mathrm{DR}\, f_+\mathcal{M} \simeq Rf_* \,\mathrm{DR}\,\mathcal{M}$$

Proof. — From 2.4.1, we have

$$\begin{aligned}\mathrm{DR}\, f_+\mathcal{M} &\simeq Rf_*(\mathcal{M}^{\mathfrak{r}} \otimes^{\mathbb{L}}_{\mathcal{D}_X} \mathcal{D}_{X\to Y}) \otimes^{\mathbb{L}}_{\mathcal{D}_Y} \mathcal{O}_Y\\ &\simeq Rf_*(\mathcal{M}^{\mathfrak{r}} \otimes^{\mathbb{L}}_{\mathcal{D}_X} \mathcal{D}_{X\to Y} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{O}_Y)\\ &\simeq Rf_*(\mathcal{M}^{\mathfrak{r}} \otimes^{\mathbb{L}}_{\mathcal{D}_X} \mathcal{O}_X)\\ &\simeq Rf_*\,\mathrm{DR}\,\mathcal{M}\end{aligned}$$

□

Note that 2.4.4 is no longer true for Sol. The reason for this is that duality does not commute with Rf_*. However, it commutes in case f is *proper*. Thus, Sol commutes (modulo a shift) with proper direct image.

2.5. Kashiwara's theorem.

— The goal of this section is to prove the following theorem due to Kashiwara:

Theorem 2.5.1. — *Let $i : X \longrightarrow Y$ be a closed immersion of complex manifolds. Then, i_+ induces an equivalence of categories*

$$\{\mathcal{D}_X - \mathit{mod}\} \xrightarrow{\sim} \{\mathcal{D}_Y - \mathit{modules\ with\ support\ included\ in}\ X\}$$

Let us remark that this is false for coherent $\mathcal{O}_X$-modules. Take the inclusion $0 \hookrightarrow \mathbb{C}$. A coherent module on 0 is essentially the data of a positive number. However, for $k, n \in \mathbb{N}$, the $\mathcal{O}_{\mathbb{C}}^k/x^n\mathcal{O}_{\mathbb{C}}^k$ are two by two non isomorphic $\mathcal{O}_{\mathbb{C}}$-modules with support 0.

Let us prove 2.5.1. Because $\mathcal{D}_X$-mod is equivalent to $\mathcal{D}_X^{\mathrm{op}}$-mod, it is enough to prove it for right $\mathcal{D}_X$-modules, that is that i_+ induces an equivalence of categories

$$\{\mathcal{D}_X^{op} - \mathrm{mod}\} \xrightarrow{\sim} \{\mathcal{D}_Y^{op} - \text{module with support included in } X\}$$

For a $\mathcal{D}_Y^{op}$-module $\mathcal{N}$ with support included in X, we define

$$i^!\mathcal{N} := \mathcal{H}om_{i^{-1}\mathcal{D}_Y}(\mathcal{D}_{X\to Y}, i^{-1}\mathcal{N}) \tag{2.5.2}$$

For $\mathcal{M} \in \mathcal{D}_X^{op}$-mod, we have

$$i^!i_+\mathcal{M} = \mathcal{H}om_{i^{-1}\mathcal{D}_Y}(\mathcal{D}_{X\to Y}, i^{-1}i_*(\mathcal{M} \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}))$$

and there is a canonical map

$$\begin{aligned} (*)_i : \mathcal{M} &\longrightarrow i^!i_+\mathcal{M} \\ m &\longrightarrow (P \to m \otimes P) \end{aligned}$$

For a $\mathcal{D}_Y^{op}$-module $\mathcal{N}$ with support included in X, we have

$$i_+i^!\mathcal{N} = i_*(\mathcal{H}om_{i^{-1}\mathcal{D}_Y}(\mathcal{D}_{X\to Y}, i^{-1}\mathcal{N}) \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}) \tag{2.5.3}$$

We thus have a canonical map

$$\begin{aligned} (*)'_i : i_+i^!\mathcal{N} &\longrightarrow \mathcal{N} \\ \varphi \otimes P &\longrightarrow \varphi(P) \end{aligned}$$

We want to prove that $(*)_i$ and $(*)'_i$ are isomorphisms. This is a local statement, so one can take local coordinates $(x_1, \dots, x_n, t_1, \dots, t_d)$ such that X is given by $t_1 = \dots = t_n = 0$. We thus have a chain of inclusions

$$X \xrightarrow{i_d} \{t_1 = \dots = t_{d-1} = 0\} \xrightarrow{i_{d-1}} \cdots \longrightarrow \{t_1 = 0\} \xrightarrow{i_1} Y$$

and canonical isomorphisms $i_+ \simeq i_{1+} \cdots i_{d+}$ and $i^! \simeq i_d^! \cdots i_1^!$. Thus, it is enough to prove that the $(*)_{i_j}$ and $(*)_{i'_j}$ are isomorphisms. Hence, one can suppose that X is a hypersurface given by the equation $t = 0$. We have

$$\begin{aligned} \mathcal{D}_{X\to Y} &\simeq \bigoplus_{k\in\mathbb{N}} \mathcal{D}_X \left(\frac{\partial}{\partial t}\right)^k \\ &\simeq i^{-1}(\mathcal{D}_Y / t\mathcal{D}_Y) \end{aligned}$$

Hence,

$$\begin{aligned} i^!\mathcal{N} &= \mathcal{H}om_{i^{-1}\mathcal{D}_Y}(i^{-1}(\mathcal{D}_Y/t\mathcal{D}_Y), i^{-1}\mathcal{N}) \\ &\simeq i^{-1}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_Y/t\mathcal{D}_Y, \mathcal{N}) \\ &\simeq i^{-1}\mathcal{N}^\sharp \end{aligned}$$

where $\mathcal{N}^\sharp$ is the subsheaf of $\mathcal{N}$ of sections killed by t. Hence,

$$\begin{aligned} i^!i_+\mathcal{M} &\simeq (\mathcal{M} \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y})^\sharp \\ &\simeq \left(\bigoplus_{k\in\mathbb{N}} \mathcal{M} \left(\frac{\partial}{\partial t}\right)^k\right)^\sharp \end{aligned}$$

Noticing that $\mathcal{M}^\sharp = \mathcal{M}$ tautologically, $(*)_i$ identifies $\mathcal{M}$ with the term $k = 0$ of $i^!i_+\mathcal{M}$. On the other hand, if $m \in \mathcal{M}$ and if $k > 0$ is such that $m\left(\frac{\partial}{\partial t}\right)^k t = 0$, we have

$km\left(\frac{\partial}{\partial t}\right)^{k-1} = 0$, so $m = 0$. So $(*)_i$ is an isomorphism.

Moreover, $(*)_i'$ is the morphism

$$\begin{array}{ccc} \bigoplus_{k\in\mathbb{N}} i_*\left(i^{-1}\mathcal{N}^\sharp \otimes_{\mathcal{D}_X} \left(\frac{\partial}{\partial t}\right)^k\right) & \longrightarrow & \mathcal{N} \\ n\otimes\left(\frac{\partial}{\partial t}\right)^k & \longrightarrow & n\cdot\left(\frac{\partial}{\partial t}\right)^k \end{array}$$

Let us prove that $(*)_i'$ is an isomorphism at the level of the germs at $x \in X$. We prove the surjectivity. The injectivity is left as an exercise. Let $n \in \mathcal{N}_x$. Since the support of n is included in X, one can choose $k > 0$ such that $nt^k = 0$. If $k = 1$, there is nothing to do. Let us suppose $k > 1$ and let us argue by induction on k. Applying $\frac{\partial}{\partial t}$ to $nt^k = 0$ gives

$$\left(n\frac{\partial}{\partial t}t - kn\right)t^{k-1} = 0$$

So the recursion hypothesis applies to

$$n_1 = n\frac{\partial}{\partial t}t - kn = (1-k)n + nt\frac{\partial}{\partial t} \tag{2.5.4}$$

It also applies to $n_2 = nt$. From this and (2.5.4), we deduce that n is in the image of $(*)_i'$. Modulo the injectivity of $(*)_i'$ left to the reader, Kashiwara theorem is proved.

3. What is the characteristic variety good for?

3.1. What it is. — The characteristic variety of a coherent $\mathcal{D}_X$-module $\mathcal{M}$ is a conic subvariety $\mathrm{Char}(\mathcal{M})$ of T^*X attached to $\mathcal{M}$. It is local above X and measures how far $\mathcal{M}$ is from being a flat connection. This is due to the following

Proposition 3.1.1. — *$\mathcal{M}$ is a flat connection if and only if $\mathrm{Char}(\mathcal{M})$ is the zero section of T^*X.*

For a formal definition, we refer to any book on $\mathcal{D}$-modules. The characteristic variety is a very special type of subvariety of T^*X: it is involutive with respect to the canonical symplectic structure on T^*X. For an analytic proof of this result, let us mention [**SKK73**]. For algebraic inclined readers, let us mention [**Gab81**]. We won't use this result, but the following consequence

Theorem 3.1.2. — *The irreducible components of $\mathrm{Char}(\mathcal{M})$ have dimension $\geqslant d_X$.*

Let us mention three reasons which make the characteristic variety of fundamental importance in the theory of $\mathcal{D}$-modules.

3.2. Duality. — The characteristic variety controls the amplitude of the $\mathcal{D}_X^{\mathrm{op}}$-module $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)$, where $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module. This is the following

Theorem 3.2.1. — *We have*

$$\mathcal{H}^i \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \simeq 0$$

unless $\operatorname{codim}_{T^*X} \operatorname{Char}(\mathcal{M}) - d_X \leqslant i \leqslant 0$

For proofs of this theorem, we refer to [**Kas76**][**Bjo93**].

3.3. Holonomy. — The characteristic variety allows to single out a special type of coherent $\mathcal{D}$-modules, that is *holonomic* modules. This is the right "finiteness" notion in the context of $\mathcal{D}$-modules.

Definition 3.3.1. — A coherent $\mathcal{D}_X$-module $\mathcal{M}$ is said to be holonomic if $\mathcal{M} = 0$ or $\dim \operatorname{Char} \mathcal{M} = d_X$.

3.4. Non-characteristic restriction. — The characteristic variety gives rise to the *non-characteristic* condition for morphisms of complex manifolds $f : X \longrightarrow Y$. The slogan here is that

Any reasonable notion for coherent $\mathcal{D}$-modules commutes with inverse image by a non-characteristic morphism

We have for example the following [**Kas95**]

Theorem 3.4.1. — *Let $\mathcal{M}$ be a coherent $\mathcal{D}_Y$-module and suppose that $f : X \longrightarrow Y$ is non-characteristic for $\mathcal{M}$. Then, the canonical comparison morphism*

$$f^{-1} \operatorname{Sol} \mathcal{M} \longrightarrow \operatorname{Sol} f^+ \mathcal{M}$$

is an isomorphism.

The non-characteristic condition is empty for f smooth. Since f can always be factorized as

$$\begin{array}{ccc} Y & \xrightarrow{i_f} & X \times Y \\ & \searrow & \downarrow{\scriptstyle p_2} \\ & & X \end{array}$$

where i_f is the closed immersion given the graph of f, non-charactericity is essentially used for closed immersions in practice. For such a morphism $i : Y \longrightarrow X$ and for a coherent $\mathcal{D}_X$-module $\mathcal{M}$, we say that i *is non-characteristic for* $\mathcal{M}$ if for every $x \in Y$ and every $\varphi \in (\operatorname{Char} \mathcal{M})_x := \operatorname{Char} \mathcal{M} \cap T_x^* X \subset T_x^* X$ such that $\varphi_{|T_x Y} = 0$, we have $\varphi = 0$. This is the same as asking

$$(\operatorname{Char} \mathcal{M})_x \cap (T_Y^* X)_x = \{0\}$$

where $T_Y^* X$ is the conormal bundle of Y in X.

Note finally that the non-characteristic condition is open, in the sense that if it is satisfied at x for a given X and a given $\mathcal{M}$, then it is satisfied in a neighbourhood of

$x^{(5)}$. Along these lines, if Y is non-characteristic for $\mathcal{M}$, any small deformation of Y will be non-characteristic for $\mathcal{M}$.

The bigger $(\operatorname{Char}\mathcal{M})_x$ is, the harder it is to find a subvariety passing through x which is non-characteristic with respect to $\mathcal{M}$. The extreme case is the case where

$$(\operatorname{Char}\mathcal{M})_x = T_x^* X \tag{3.4.2}$$

In that case, a Y passing through x can never be non-characteristic. Hopefully, this does not happen so often in the holonomic case. Since $T_x^* X$ has dimension d_X, the set of points satisfying (3.4.2) is discrete in X if $\mathcal{M}$ is holonomic. For the same dimension reason, the set of points $x \in X$ for which $(\operatorname{Char}\mathcal{M})_x$ has dimension $d_X - 1$ is empty or has dimension at most 1 etc. Finally, the set of points of $x \in X$ for which $(\operatorname{Char}\mathcal{M})_x$ is the zero space is dense open in X.

This property allows the following standard dévissage: suppose that you want to prove a statement like

"If $\mathcal{M}$ holonomic satisfies (condition **C** to be inserted), then $\mathcal{M}$ satisfies **P**"

with **C** and **P** properties of local nature. Argue by recursion on the dimension and pick a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ satisfying **C**. Holonomy implies that away from a discrete set of points $S \subset X$ such that $T_S^* X \subset \operatorname{Char}\mathcal{M}$, one can find for any x a germ of hypersurface passing through x and non characteristic for $\mathcal{M}$. What usually happens is that **C** commutes with non-characteristic restriction. Applying the recursion hypothesis to all these restrictions will (if one is not to unlucky) lead to the fact that $\mathcal{M}$ satisfies **P** away from S. One is thus reduced to analyse what happens around a reluctant point of S. This is a first feature of how nice holonomic $\mathcal{D}$-modules are.

3.5. Functoriality and holonomy. — Let X be a complex manifold. As consequence of 3.1.1 and the discussion 3.4, we have the following

Proposition 3.5.1. — *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. If $\mathcal{M}$ is holonomic, then $\mathcal{M}$ is a flat connection on a complement of a strict closed subset in X.*

From holomorphic Poincaré lemma, we deduce that $\operatorname{DR}\mathcal{M}$ has constructible cohomology on a dense open set in X. This is the starting point of the proof 4.1.1 of the constructibility of DR for holonomic modules.

We denote by $D^b_{\mathrm{hol}}(\mathcal{D}_X)$ the derived category of $\mathcal{D}_X$-module with bounded and holonomic cohomology. Holonomy is preserved by the usual operations on $\mathcal{D}$-modules (inverse image[6], duality, nearby cycles). Of fundamental importance in this notes is the following [**Kas76**]

Theorem 3.5.2. — *Let $f : X \longrightarrow Y$ be a proper morphism between complex manifolds. The functor f_+ sends $D^b_{\mathrm{hol}}(\mathcal{D}_X)$ in $D^b_{\mathrm{hol}}(\mathcal{D}_Y)$.*

---

(5) Take two sub vector spaces of a vector space whose intersections is $\{0\}$ and move them a little. Then there intersection stays $\{0\}$.

(6) Note that this cannot be trivial, since from 2.1.1, we know that inverse image does not preserve $\mathcal{D}$-coherence in general. So holonomic $\mathcal{D}$-modules are very special coherent $\mathcal{D}$-modules.

Note that an extra assumption on the existence of good filtrations locally above Y is needed in [**Kas76**]. Using the existence of canonical lattices for (possibly irregular) meromorphic connections [**Mal96**], Malgrange proved [**Mal04**] that holonomic modules admit globally defined canonical good filtrations.

4. Kashiwara constructibility theorem

4.1. The statement and a lemma. — Following [**LM93**] modulo some slight simplifications, we prove in this section the following

Theorem 4.1.1. — *Let X be a complex manifold and let $\mathcal{M} \in D^b_{\mathrm{hol}}(\mathcal{D}_X)$. Then* $\mathrm{DR}\,\mathcal{M} \in D^b_c(X, \mathbb{C})$.

Note that such a statement cannot be trivial, since the modules defining $\mathrm{DR}\,\mathcal{M}$ are quasi-coherent $\mathcal{O}_X$-modules. So they are far from being objects of $D^b_c(X, \mathbb{C})$.

The main $\mathcal{D}$-module ingredient for this is the commutativity of DR with direct image and the compatibility of holonomy with proper direct image 3.5.2. Note that these properties are also available for coherent $\mathcal{D}$-modules and not only holonomic $\mathcal{D}$-modules. Holonomy will be used through the fact 3.5.1 that a holonomic $\mathcal{D}$-module restricts to a flat connection on a dense open subset. For flat connections, constructibility of DR is a consequence of holomorphic Poincaré lemma.

A topological ingredient for the proof of 4.1.1 is the preservation of constructibility by proper maps. Let us start with the following

Lemma 4.1.2. — *Let (E, ∇) be a flat connection on a punctured polydisc $D^N \times D^*$. Let $j : D^N \times D^* \longrightarrow D^{N+1}$ be the canonical inclusion. Then (E, ∇) extends to a flat meromorphic connection $\mathcal{M}$ on D^{N+1} with poles along $D^N \times \{0\}$ such that $\mathrm{DR}\,\mathcal{M}$ has bounded and constructible cohomology.*

Note that the first part of the statement is an instance of the existence of Deligne extensions. However, we will see that in our particular situation, $\mathcal{M}$ can be found almost by hand using what has been proved so far in these notes.

Proof. — Define $p : D^N \times D^* \longrightarrow D^*$ the canonical projection. We have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Loc}_{\mathbb{C}}(D^*) & \xrightarrow{p^{-1}} & \mathrm{Loc}_{\mathbb{C}}(D^N \times D^*) \\ \downarrow{\scriptstyle \mathcal{H}^0\,\mathrm{Sol}} & & \downarrow{\scriptstyle \mathcal{H}^0\,\mathrm{Sol}} \\ \mathrm{Conn}(D^*) & \xrightarrow{p^{+}} & \mathrm{Conn}(D^N \times D^*) \end{array} \tag{4.1.3}$$

Since p is an equivalence of homotopy, the top horizontal arrow in (4.1.3) is an equivalence of category. From Riemann-Hilbert correspondence for flat connections, the vertical arrows in (4.1.3) are equivalences of categories. Hence, the bottom arrow in (4.1.3) is an equivalence of category. Thus, we have $(E, \nabla) = p^+(E', \nabla')$ for $(E', \nabla') \in \mathrm{Conn}(D^*)$. If we find a $\mathcal{M}'$ for (E', ∇') as in 4.1.2 (case $N = 0$), then

4.1.2 is proved for (E,∇). Indeed $\mathcal{M} := p^{+}\mathcal{M}'$ will be a meromorphic connection extending (E,∇), and since p is smooth, it is non-characteristic for $\mathcal{M}'$, so from the slogan 3.4

$$p^{-1}\operatorname{DR}\mathcal{M}' \simeq \operatorname{DR} p^{+}\mathcal{M}' = \operatorname{DR}\mathcal{M} \tag{4.1.4}$$

And we are done because p^{-1} preserves constructibility.

Thus, we are left to suppose $N = 0$ from now on. Let Γ be the monodromy of the local system associated to (E,∇). Choose a matrix C with complex coefficients such that $e^{2i\pi C} = \Gamma$. Let us define

$$(\mathcal{N},\nabla) := (\mathcal{O}^{n}_{D*}(*0), d + C\frac{dx}{x})$$

where n is the rank of E. On a small disc not containing 0, the columns of $e^{C\log x}$ form a basis for the space of solutions of $(\mathcal{N},\nabla)$. Thus, the local system associated to $(\mathcal{N},\nabla)_{|D*}$ is that associated to (E,∇). By Riemann-Hilbert for flat connections, there is an isomorphism of flat connections

$$(E,\nabla) \simeq (\mathcal{N},\nabla)_{|D*}$$

Taking the preimages of the canonical base of $\mathcal{O}^{n}_{D*}(*0)$ singles out a trivialisation of E on D^{*} in which the matrix of ∇ is Cdx/x. Let us prove that the $\mathcal{O}_D(*0)$-module $\mathcal{M}$ of $j_{*}E$ generated by this basis meets our requirements.

Notice that $(\mathcal{M},\nabla)$ is a successive extension of rank 1 meromorphic connections. One the other hand, if we have an exact sequence of meromorphic connections

$$0 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M}_2 \longrightarrow \mathcal{M}_3 \longrightarrow 0$$

we have from the description 2.4.4 of DR a distinguished triangle

$$\operatorname{DR}\mathcal{M}_1 \longrightarrow \operatorname{DR}\mathcal{M}_2 \longrightarrow \operatorname{DR}\mathcal{M}_3 \xrightarrow{+1} \tag{4.1.5}$$

So the central term of (4.1.5) lies in $D^{b}_{c}(D,\mathbb{C})$ if this is the case of the other terms. Hence, we can suppose that

$$(\mathcal{M},\nabla) := (\mathcal{O}^{n}_{D*}(*0), d + c\frac{dx}{x})$$

where $c \in \mathbb{C}$. The constructibility in that case is a simple computation. □

4.2. Proof of theorem 4.1.1. — We argue by induction on $d_X = \dim X$. An object in $D^{b}_{\mathrm{hol}}(\mathcal{D}_{\mathrm{pt}})$ is a complex of vector spaces with bounded and finite dimensional cohomology. In the case, DR is the identity functor, so 4.1.1 is true. Let us suppose that $d_X > 0$ and that 4.1.1 is true in dimension $< d_X$. A recursion on the amplitude of $\mathcal{M}$ allows us to reduce to the case where $\mathcal{M}$ is concentrated in degree 0.

From 3.5.1, we know that $\mathcal{M}$ is a flat connection away from a hypersurface Z in X. Let $j : X\backslash Z \longrightarrow X$ and $i : Z \longrightarrow X$ be the canonical inclusions. From 1.4.5, $j^{-1}\operatorname{DR}\mathcal{M}$ is quasi-isomorphic to a local system. Since constructibility is a local property, we are thus left to prove 4.1.1 in a neighbourhood of a point of $0 \in Z$.

We have a distinguished triangle

$$j_{!}j^{-1}\operatorname{DR}\mathcal{M} \longrightarrow \operatorname{DR}\mathcal{M} \longrightarrow i_{*}i^{-1}\operatorname{DR}\mathcal{M} \xrightarrow{+1} \tag{4.2.1}$$

Since $j_!$ preserves constructibility, the left term of (4.2.1) lies in $D^b_c(X,\mathbb{C})$. To prove 4.1.1, we are thus left to prove that $i_*i^{-1}\operatorname{DR}\mathcal{M}$ lies in $D^b_c(X,\mathbb{C})$. Since i_* preserves constructibility, we are left to prove that $i^{-1}\operatorname{DR}\mathcal{M}$ lies in $D^b_c(Z,\mathbb{C})$.

By Weierstrass preparation theorem, one can find a polydisc $D^{d_X-1}\times D$ centred at 0 such that the projection $p: D^{d_X-1}\times D \longrightarrow D^{d_X-1}$ is finite over Z. Define $Y := D^{d_X-1}$. Since

$$(Y\times D)\backslash Z \hookrightarrow (Y\times\mathbb{C})\backslash Z$$

is an equivalence of homotopy, the local system $j^{-1}\operatorname{DR}\mathcal{M}$ extends to a local system $\mathcal{L}$ on $(Y\times\mathbb{C})\backslash Z$. From 4.1.2, the associated connection $E_{\mathcal{L}}$ extends to a meromorphic connection $M_{\mathcal{L}}$ on $(Y\times\mathbb{P}^1_{\mathbb{C}})\backslash Z$. From Riemann-Hilbert correspondence for flat connections, $M_{\mathcal{L}}$ glue with $\mathcal{M}$ along $(Y\times D)\backslash Z$ and define a $\mathcal{D}_{Y\times\mathbb{P}^1_{\mathbb{C}}}$-module denoted by $\overline{\mathcal{M}}$. Since $\mathcal{M}$ and $M_{\mathcal{L}}$ are holonomic(7), $\overline{\mathcal{M}}$ is holonomic. To prove the constructibility of $i^{-1}\operatorname{DR}\mathcal{M}$, we are left to work with $\overline{\mathcal{M}}$ from now on (since nothing has changed in a neighbourhood of Z !).

We define $\overline{p}: Y\times\mathbb{P}^1_{\mathbb{C}} \longrightarrow Y$ and still denote by j and i the inclusions of $(Y\times\mathbb{P}^1_{\mathbb{C}})\backslash Z \hookrightarrow Y\times\mathbb{P}^1_{\mathbb{C}}$ and $Z\hookrightarrow Y\times\mathbb{P}^1_{\mathbb{C}}$. We thus have a distinguished triangle

$$j_!j^{-1}\operatorname{DR}\overline{\mathcal{M}} \longrightarrow \operatorname{DR}\overline{\mathcal{M}} \longrightarrow i_*i^{-1}\operatorname{DR}\overline{\mathcal{M}} \xrightarrow{+1} \tag{4.2.2}$$

Note that by construction of $M_{\mathcal{L}}$, the complex $j^{-1}\operatorname{DR}\overline{\mathcal{M}}$ has constructible and bounded cohomology, so this is also the case of the left hand side of (4.2.2). Let us denote by $\overline{p}_Z$ the restriction of $\overline{p}$ to Z. Since finite and closed direct images are exact, we have

$$\begin{aligned}\mathcal{H}^kR\overline{p}_*i_*i^{-1}\operatorname{DR}\overline{\mathcal{M}} &\simeq \mathcal{H}^k\overline{p}_*i_*i^{-1}\operatorname{DR}\overline{\mathcal{M}}\\ &\simeq \mathcal{H}^k\overline{p}_{Z*}i^{-1}\operatorname{DR}\overline{\mathcal{M}}\\ &\simeq \overline{p}_{Z*}\mathcal{H}^ki^{-1}\operatorname{DR}\overline{\mathcal{M}}\end{aligned}$$

From 6.5, we deduce that $\mathcal{H}^ki^{-1}\operatorname{DR}\overline{\mathcal{M}}$ is constructible (resp. 0) if $\overline{p}_{Z*}\mathcal{H}^ki^{-1}\operatorname{DR}\overline{\mathcal{M}}$ is constructible (resp. 0). We are thus left to prove that $R\overline{p}_*i_*i^{-1}\operatorname{DR}\overline{\mathcal{M}}$ lies in $D^b_c(Y,\mathbb{C})$. Applying $R\overline{p}_*$ to (4.2.2) gives the distinguished triangle

$$R\overline{p}_*j_!j^{-1}\operatorname{DR}\overline{\mathcal{M}} \longrightarrow R\overline{p}_*\operatorname{DR}\overline{\mathcal{M}} \longrightarrow R\overline{p}_*i_*i^{-1}\operatorname{DR}\overline{\mathcal{M}} \xrightarrow{+1} \tag{4.2.3}$$

Since DR commutes with direct image 2.4.4, the triangle (4.2.3) is isomorphic to

$$R\overline{p}_*j_!j^{-1}\operatorname{DR}\overline{\mathcal{M}} \longrightarrow \operatorname{DR}\overline{p}_+\overline{\mathcal{M}} \longrightarrow R\overline{p}_*i_*i^{-1}\operatorname{DR}\overline{\mathcal{M}} \xrightarrow{+1} \tag{4.2.4}$$

From 3.5.2, we know that $\overline{p}_+\overline{\mathcal{M}} \in D^b_{\mathrm{hol}}(\mathcal{D}_Y)$. By recursion hypothesis, the middle term of (4.2.4) lies in $D^b_c(Y,\mathbb{C})$. We are left to prove that $R\overline{p}_*j_!j^{-1}\operatorname{DR}\overline{\mathcal{M}}$ is in $D^b_c(Y,\mathbb{C})$, which is a consequence of the properness of $\overline{p}$. □

Since constructibility is preserved by duality, we deduced from 1.4.2 the following

(7)For $M_{\mathcal{L}}$, this is a theorem resulting from the b-function lemma.

Theorem 4.2.5. — *Let X be a complex manifold and let $\mathcal{M} \in D^b_{\mathrm{hol}}(\mathcal{D}_X)$. Then* $\mathrm{Sol}\,\mathcal{M} \in D^b_c(X, \mathbb{C})$.

5. Regularity

Regularity has at least a dozen non-trivially equivalent incarnations, which make it a fascinating subject of study. Let us give a (non-exhaustive) list of them and a reference where they are used as a starting point. Regularity can be defined/characterized

(1) Via logarithmic forms[8] [**Del70**].
(2) Analytically using growth of solutions[9] [**Del70**].
(3) Algebraically I [**KK81**].
(4) Algebraically II [**Ber**].
(5) Sheaf-theoretically [**Meb90**].
(6) Using restriction to curves [**Bor87**].
(7) Via derived endomorphisms $R\mathcal{H}om$ [**Tey14**].
(8) Via nearby cycles [**Tey15**].

Note that (8) can be transposed to contexts where a notion of regular objects is missing but where one has nearby cycles at one's disposal (arithmetic $\mathcal{D}$-modules, ℓ-adic sheaves etc). In this notes, we will put the emphasis on (1) and (5).

5.1. Dimension 1. — In what follows, we define $\mathcal{O} = \mathcal{O}_{\mathbb{C},0}$, $\widehat{\mathcal{O}} = \widehat{\mathcal{O}}_{\mathbb{C},0}$ and

$$\mathcal{D} = D_{\mathbb{C},0} \simeq \bigoplus_{k \in \mathbb{N}} \mathcal{O} \left(\frac{\partial}{\partial x} \right)^k$$

Let $P \in \mathcal{D} \backslash \{0\}$. P acts by usual differentiation on $\mathcal{O}$ and on $\widehat{\mathcal{O}}$. The main slogan here is the following

The irregularity of the $\mathcal{D}$-module $\mathcal{D}/\mathcal{D}P$ is a measure of the difference between the action of P on $\mathcal{O}$ and on $\widehat{\mathcal{O}}$.

Let us give a precise meaning to that. We denote by $\mathrm{Ker}(P, \mathcal{O})$ and $\mathrm{Coker}(P, \mathcal{O})$ (resp. $\mathrm{Ker}(P, \widehat{\mathcal{O}})$ and $\mathrm{Coker}(P, \widehat{\mathcal{O}})$) the kernel and cokernel of the action of P on $\mathcal{O}$ (resp. $\widehat{\mathcal{O}}$). One can show that these $\mathbb{C}$-vector spaces are finite dimensional. Applying $R\mathcal{H}om(\mathcal{D}/\mathcal{D}P, \cdot\,)$ to the exact sequence of $\mathcal{D}$-modules

$$0 \longrightarrow \mathcal{O} \longrightarrow \widehat{\mathcal{O}} \longrightarrow \widehat{\mathcal{O}}/\mathcal{O} \longrightarrow 0$$

gives a distinguished triangle

$$R\mathcal{H}om(\mathcal{D}/\mathcal{D}P, \mathcal{O}) \longrightarrow R\mathcal{H}om(\mathcal{D}/\mathcal{D}P, \widehat{\mathcal{O}}) \longrightarrow R\mathcal{H}om(\mathcal{D}/\mathcal{D}P, \widehat{\mathcal{O}}/\mathcal{O}) \xrightarrow{+1}$$

(8) for meromorphic connections with poles along a normal crossing divisor.
(9) same remark as above.

whose associated long exact sequence reads
(5.1.1)

$$\begin{array}{l} 0 \longrightarrow \operatorname{Ker}(P,\mathcal{O}) \longrightarrow \operatorname{Ker}(P,\widehat{\mathcal{O}}) \longrightarrow \operatorname{Ker}(P,\widehat{\mathcal{O}}/\mathcal{O}) \longrightarrow \\ \longrightarrow \operatorname{Coker}(P,\mathcal{O}) \longrightarrow \operatorname{Coker}(P,\widehat{\mathcal{O}}) \longrightarrow \operatorname{Coker}(P,\widehat{\mathcal{O}}/\mathcal{O}) \longrightarrow 0 \end{array}$$

From [**Mal71**], we know that $\operatorname{Coker}(P,\widehat{\mathcal{O}}/\mathcal{O}) \simeq 0$. In accordance to the above slogan, we thus define the *irregularity number of* $\mathcal{D}/\mathcal{D}P$ as

$$\operatorname{irr}(\mathcal{D}/\mathcal{D}P) := \dim_{\mathbb{C}} \operatorname{Ker}(P,\widehat{\mathcal{O}}/\mathcal{O})$$

The computation of this number is quite easy for a given P, this is the following [**Mal71**]

Theorem 5.1.2. — *Suppose* $P = \sum_{k=1}^{n} a_k \left(\frac{\partial}{\partial x}\right)^k$ *with* $a_n \neq 0$. *Then*[10]

$$\operatorname{irr}(\mathcal{D}/\mathcal{D}P) = \operatorname{Sup}_k(k - v_x(a_k)) - (n - v_x(a_n))$$

For example, Malgrange formula predicts that $\operatorname{irr}(\mathcal{D}/\mathcal{D}(x^2\frac{\partial}{\partial x} - 1)) = 1$. On the other hand, if we define $f := \sum n! x^{n+1}$ we see that

$$\begin{aligned} \left(x^2\frac{\partial}{\partial x} - 1\right) f &= -x \\ &= 0 \text{ in } \widehat{\mathcal{O}}/\mathcal{O} \end{aligned}$$

So $\operatorname{Ker}(x^2\frac{\partial}{\partial x} - 1, \widehat{\mathcal{O}}/\mathcal{O}) \simeq \mathbb{C}f$.

5.2. General case. — Let X be a complex manifold and let $i : Z \longrightarrow X$ be an analytic subspace of X. Let us define $\mathcal{O}_{X|Z} := i^{-1}\mathcal{O}_X$ and let $\mathcal{O}_{\widehat{X|Z}}$ be the formalization of $\mathcal{O}_X$ along Z. Mimicking the 1-dimensional case, we have the following exact sequence of $\mathcal{D}_X$-modules

$$0 \longrightarrow \mathcal{O}_{X|Z} \longrightarrow \mathcal{O}_{\widehat{X|Z}} \longrightarrow \mathcal{O}_{\widehat{X|Z}}/\mathcal{O}_{X|Z} \longrightarrow 0$$

For $\mathcal{M} \in D^b_{\mathrm{hol}}(X)$, we thus have a distinguished triangle

$$i^{-1}\operatorname{Sol}(\mathcal{M}) \longrightarrow R\mathcal{H}om(\mathcal{M}, \mathcal{O}_{\widehat{X|Z}}) \longrightarrow R\mathcal{H}om(\mathcal{M}, \mathcal{O}_{\widehat{X|Z}}/\mathcal{O}_{X|Z}) \xrightarrow{+1}$$

we define the *irregularity sheaf of* $\mathcal{M}$ *along* Z as

$$\operatorname{Irr}^*_Z \mathcal{M} := R\mathcal{H}om(\mathcal{M}, \mathcal{O}_{\widehat{X|Z}}/\mathcal{O}_{X|Z})[-1]$$

viewed as a complex on X with support in Z.

[10] In this formula, $v_x(f)$ for $f \in \mathcal{O}$ denotes the vanishing order of f at 0.

Definition 5.2.1. — We say that $\mathcal{M} \in D^b_{\mathrm{hol}}(X)$ is regular if $\mathrm{Irr}^*_Z \mathcal{M} \simeq 0$ for every analytic subspace Z of X.

We denote by $\mathrm{Reg}(X)$ the category of regular holonomic $\mathcal{D}_X$-modules on X. With this point of view, the abelianity of $\mathrm{Reg}(X)$ is highly non trivial. Indeed, it amounts to prove that if we have an exact sequence of holonomic $\mathcal{D}_X$-modules

$$0 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M}_2 \longrightarrow \mathcal{M}_3 \longrightarrow 0 \tag{5.2.2}$$

then $\mathcal{M}_1$ and $\mathcal{M}_3$ are regular if $\mathcal{M}_2$ is regular. By applying Irr^*_Z to (5.2.2) for every Z, we get

$$\mathrm{Irr}^*_Z \mathcal{M}_3 \longrightarrow \mathrm{Irr}^*_Z \mathcal{M}_2 \longrightarrow \mathrm{Irr}^*_Z \mathcal{M}_1 \xrightarrow{+1} \tag{5.2.3}$$

The regularity of $\mathcal{M}_2$ gives

$$\mathrm{Irr}^*_Z \mathcal{M}_3 \simeq \mathrm{Irr}^*_Z \mathcal{M}_1[1]$$

and one is stuck a priori. However for every analytic subspaces Z_1, Z_2 in X, there is for every $\mathcal{M} \in D^b_{\mathrm{hol}}(X)$ a distinguished triangle
(5.2.4)

$$(\mathrm{Irr}^*_{Z_1 \cup Z_2} \mathcal{M})_{|Z_{12}} \longrightarrow (\mathrm{Irr}^*_{Z_1} \mathcal{M})_{|Z_{12}} \oplus (\mathrm{Irr}^*_{Z_2} \mathcal{M})_{|Z_{12}} \longrightarrow (\mathrm{Irr}^*_{Z_{12}} \mathcal{M}) \xrightarrow{+1}$$

where $Z_{12} := Z_1 \cap Z_2$. Hence, $\mathcal{M}$ is regular if $\mathrm{Irr}^*_Z \mathcal{M} \simeq 0$ for every Z hypersurface in X.

Going back to the abelianity of $\mathrm{Reg}(X)$, we now have to deduce the vanishing of $\mathrm{Irr}^*_Z \mathcal{M}_i, i = 1, 3$ from that of $\mathrm{Irr}^*_Z \mathcal{M}_2$ for Z hypersurface in X. This would not be much of an improvement if we had not the following [**Meb90**]

Theorem 5.2.5. — *For every holonomic $\mathcal{D}_X$-module $\mathcal{M}$ and for every hypersurface Z in X, the complex $\mathrm{Irr}^*_Z \mathcal{M}$ is perverse.*

Hence, for Z hypersurface, the triangle (5.2.3) is not only a distinguished triangle in $D^b_c(X, \mathbb{C})$. It is an exact sequence in $\mathrm{Perv}(X)$. Thus, the vanishing of the middle-term implies the vanishing of the other terms.

Let us explain why 5.2.5 cannot be obvious. For an hypersurface $i : Z \longrightarrow X$, the sheaf $\mathcal{O}(*Z)$ has a canonical $\mathcal{D}_X$-module structure(11). For $\mathcal{M} \in D^b_{\mathrm{hol}}(X)$, we denote by $\mathcal{M}(*Z) := \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}(*Z)$ endowed with its usual $\mathcal{D}_X$-module structure. Note that as a consequence of the b-function lemma, $\mathcal{M}(*Z) \in D^b_{\mathrm{hol}}(X)$. From [**Meb04**], one has the following

$$\mathrm{Irr}^*_Z(\mathcal{M}) \simeq i_* i^{-1} \mathrm{Sol}(\mathcal{M}(*Z))$$

This identity(12) tells us that theorem 5.2.5 cannot be trivial. For $\mathcal{M}$ holonomic, $\mathcal{M}(*Z)$ is holonomic, so $\mathrm{Sol}(\mathcal{M}(*Z))$ is perverse. But in general, the restriction of a perverse sheaf to a subspace has no reason to stay perverse !

Example 5.2.6. — Flat connections are regular $\mathcal{D}$-modules.

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(11) by usual differentiation !
(12) actually it stays true for any Z analytic but one has to be careful with the definition of the localization functor in that case.

Note that in Mebkhout's approach, this is a non trivial statement since when applied to $\mathcal{O}_X$, it implies almost immediately Grothendieck comparison theorem [**Gro66**] between algebraic and analytic de Rham cohomology on smooth varieties. See [**Meb04**, 5] for more details.

5.3. The case of meromorphic connections. —

Definition 5.3.1. — Let X be a complex manifold and let D be a divisor in X. A *flat meromorphic connection on X with poles along D* is a $\mathcal{D}_X$-module whose underlying sheaf $\mathcal{M}$ is a locally free sheaf of finite rank over $\mathcal{O}_X(*D)$.

A meromorphic connection is usually denoted by $(\mathcal{M}, \nabla)$, where ∇ is the first differential in the de Rham complex. Note in particular that $\mathcal{M} \simeq \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(*D)$, and that ∇ is a $\mathbb{C}$-linear map

$$\nabla : \mathcal{M} \longrightarrow \Omega^1_X(*D) \otimes_{\mathcal{O}_X} \mathcal{M}$$

By definition, $\xi \in \Theta_X$ acts through $\nabla_\xi := (\iota_\xi \otimes \mathrm{id}) \circ \nabla$. Thus, condition (2) in 1.2 translates into what is called *Leibniz rule*

$$\nabla(fm) = df \otimes m + f\nabla m$$

for every $f \in \mathcal{O}_X$ and every $m \in \mathcal{M}$. Condition (3) is the *flatness* property of ∇, that is

$$\nabla_{[\xi,\xi']} = \nabla_{\xi'}\nabla_\xi - \nabla_\xi\nabla_{\xi'}$$

for every $\xi, \xi' \in \Theta_X$.

Let us bridge the gap between Deligne's approach to regularity for meromorphic connections [**Del70**] and Mebkhout's approach. Let D be a normal crossing divisor. In a local system of coordinates $(x_1, \dots, x_n)$ centred at $x \in D$ such that D is given by $x_1 \cdots x_k = 0$ on an open U, define

$$\Omega^1_U(\log D) = \mathcal{O}_U \frac{dx_1}{x_1} \oplus \cdots \oplus \mathcal{O}_U \frac{dx_k}{x_k} \oplus \mathcal{O}_U dx_{k+1} \oplus \cdots \oplus \mathcal{O}_U dx_n$$

As subsheaves of the $\Omega^1_U(*D)$, the sheaves $\Omega^1_U(\log D)$ do not depend on choices of coordinates and glue into a sheaf $\Omega^1_X(\log D)$ on X called the *sheaf of logarithmic differential forms on X*.

Theorem 5.3.2. — *Let $(\mathcal{M}, \nabla)$ be a meromorphic connection on a complex manifold X with poles along a normal crossing divisor D. The following conditions are equivalents*

(1) $(\mathcal{M}, \nabla)$ is regular.

(2) $(\mathcal{M}, \nabla)$ is regular along D, that is

$$\mathrm{Irr}^*_D(\mathcal{M}, \nabla) = \mathrm{Sol}(\mathcal{M}, \nabla)_{|D} \simeq 0$$

(3) $(\mathcal{M}, \nabla)$ is generically regular along D, that is there exists a closed subset Z in D with complement in D dense in D such that $\mathrm{Irr}^*_D(\mathcal{M}, \nabla)_{|D\setminus Z} \simeq 0$.

(4) For every $x \in D$, one can find a local trivialization $\mathbf{e}$ of $\mathcal{M}$ around x such that the entries of the matrix of ∇ in $\mathbf{e}$ lies in $\Omega^1_X(\log D)$.

Note that as a consequence of a theorem of Mebkhout [**Meb04**, 4.3-16], the equivalence between (1), (2), (3) holds for an arbitrary divisor D. Description (3) is quite concrete but regularity is not easy to recognize in this way since one requires that *there exists* trivializations such that etc. For a randomly given trivialization, the poles of the matrix of ∇ may be arbitrary even though the connection is regular.

We denote by $\mathrm{Mer}(X, D)$ the category of meromorphic connections with poles along D and $\mathrm{Mer}_{\mathrm{reg}}(X, D)$ the subcategory of regular objects in $\mathrm{Mer}(X, D)$.

5.4. A warning. — The statement

"The connection $(\mathcal{O}_{\mathbb{C}^*}, d)$ is regular at 0"

does not make sense, since 5.3.2 (4) says regularity is a notion attached to a meromorphic connection in a neighbourhood of 0 whereas $(\mathcal{O}_{\mathbb{C}^*}, d)$ lives on $\mathbb{C}^*$. Talking about regularity at 0 thus requires to extend $\mathcal{O}_{\mathbb{C}^*}$ to a rank one $\mathcal{O}_{\mathbb{C}}(*0)$-bundle around 0. If such an extension $(\mathcal{M}, \nabla^{\mathcal{M}})$ is given, a trivialization e of $\mathcal{M}$ around 0 restricts to a function $f_e := e_{|D_r^*} \in \mathcal{O}_{\mathbb{C}^*}(D_r^*)$ for r small enough, where D_r^* is the punctured disc of radius r. Another trivialization e' is of the form fe where f is a germ of meromorphic function at 0. So

$$f_e = f_{e'} \text{ in } (j_*\mathcal{O}_{\mathbb{C}^*})_0^\times / \mathcal{O}_{\mathbb{C},0}(*0)^\times$$

Conversely any class of g in $(j_*\mathcal{O}_{\mathbb{C}^*})_0^\times / \mathcal{O}_{\mathbb{C},0}(*0)^\times$ such that g'/g is meromorphic at 0 will give rise to an extension of $(\mathcal{O}_{\mathbb{C}^*}, d)$. Define $\mathcal{M} := \mathcal{O}_{\mathbb{C},0}(*0)g$ and $\nabla^{\mathcal{M}} g = (g'/g)g$. For every $f \in \mathcal{O}_{\mathbb{C},0}(*0)$ we have

$$\nabla^{\mathcal{M}}(fg) = f'g + f\nabla^{\mathcal{M}} g = (f' + \frac{g'}{g} f)g \tag{5.4.1}$$

Thus there are plenty of choices to extend $(\mathcal{O}_{\mathbb{C}^*}, d)$. Such a choice does not affect the local structure of the $\mathcal{O}_{\mathbb{C},0}(*0)$-bundle we end up with, **but it affects the local structure of the extended connection** $\nabla^{\mathcal{M}}$. As formula (5.4.1) shows, the extension associated with the constant function 1 is regular but the extension associated with $e^{1/z}$ is not.

5.5. Riemann-Hilbert correspondence (algebraic version). — Let U be a smooth complex algebraic variety and let $\mathrm{Conn}(U)$ be the category of algebraic flat connections on U. In general, there are more algebraic connections on U than on U^{an}. Intuitively, this comes from the fact that for $(E_1, \nabla_1), (E_2, \nabla_2) \in \mathrm{Conn}(U)$, we have

$$\mathrm{Hom}_{\mathrm{Conn}(U)}((E_1, \nabla_1), (E_2, \nabla_2)) \simeq \mathcal{H}om((E_1, \nabla_1), (E_2, \nabla_2))^{\nabla}$$

and the same after analytification. That is, a morphism between (E_1, ∇_1) and (E_2, ∇_2) (resp. $(E_1, \nabla_1)^{\mathrm{an}}$ and $(E_2, \nabla_2)^{\mathrm{an}}$) can be interpreted as an algebraic (resp. analytic) solution of an algebraic linear differential equation. There are more analytic solutions to this equation than algebraic solutions.

Example 5.5.1. — $(\mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}, d)$ and $(\mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}, d + dx)$ are non isomorphic as algebraic connections. The exponential function induces an isomorphism between their analytifications.

Let $(E, \nabla) \in \mathrm{Conn}(U)$. Let $j : U \longrightarrow X$ be a smooth compactification such that $D := X \backslash U$ has normal crossing divisor. Then $j_* E$ is an algebraic $\mathcal{O}_X(*D)$-module and by applying j_* to $\nabla : E \longrightarrow \Omega^1_U \otimes E$, we obtain a $\mathbb{C}$-linear morphism

$$j_*\nabla : j_*E \longrightarrow \Omega^1_X(*D) \otimes j_*E$$

satisfying the Leibniz rule. Since $(\mathcal{O}_X(*D))^{\mathrm{an}} = \mathcal{O}_{X^{\mathrm{an}}}(*D^{\mathrm{an}})$, the analytification

$$((j_*E)^{\mathrm{an}}, (j_*\nabla)^{\mathrm{an}})$$

is a meromorphic connection.

Definition 5.5.2. — We say that (E, ∇) is *regular* if one can find a smooth compactification as above such that $((j_*E)^{\mathrm{an}}, (j_*\nabla)^{\mathrm{an}})$ is regular in the sense of 5.3.2.

One can prove that if regularity is achieved for a given compactification, then it is achieved for *every* compactification.

The main difference between the algebraic and analytic notions of regularity is the following. As emphasized in 5.4, regularity of a flat connection at infinity does not make sense in the analytic setting since we need a meromorphic structure at infinity. In the algebraic context, such a structure is given by nature.

Example 5.5.3. — $(\mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}, d)$ is regular but $(\mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}, d + dx)$ is not.

Let $\mathrm{Conn}_{\mathrm{reg}}(U)$ be subcategory of $\mathrm{Conn}(U)$ of flat regular connections on U. The algebraic version of the "connection version" of Riemann-Hilbert correspondence is the following [**Del70**]

Theorem 5.5.4. — *The functor*

$$\begin{array}{ccc} \mathrm{Conn}_{\mathrm{reg}}(U) & \longrightarrow & \mathrm{Conn}(U^{\mathrm{an}}) \\ (E, \nabla) & \longrightarrow & (E^{\mathrm{an}}, \nabla^{\mathrm{an}}) \end{array}$$

is an equivalence of category.

Using GAGA and Hironaka desingularization, the essential surjectivity in theorem 5.5.4 is a consequence of the following

Theorem 5.5.5. — *Let X be a complex manifold and let D be a divisor with normal crossing divisor in X. Then the functor*

$$\begin{array}{ccc} \mathrm{Mer}_{\mathrm{reg}}(X, D) & \longrightarrow & \mathrm{Conn}(X \backslash D) \\ (\mathcal{M}, \nabla) & \longrightarrow & (\mathcal{M}, \nabla)_{|X \backslash D} \end{array}$$

is an equivalence of category.

Essential surjectivity in 5.5.5 is a direct consequence of the following more precise

Theorem 5.5.6. — *Let X be a complex manifold and let D be a normal crossing divisor in X. Define $U := X \backslash D$, $j : U \longrightarrow X$ the canonical inclusion and consider $(E, \nabla) \in \mathrm{Conn}(U)$. Then, for every section τ of $\mathbb{C} \longrightarrow \mathbb{C}/\mathbb{Z}$, there is a unique locally free sheaf $\widetilde{E}$ of rank $\mathrm{rg}\, E$ in j_*E*[13] *such that*

[13] Note that since we are in the analytic category, the sheaf j_*E is huge !

(1) The restriction $\widetilde{\nabla}$ *of* $j_*\nabla$ *to* E *as logarithmic poles along* D.
(2) The residues of $\widetilde{\nabla}$ *along the components of* D *have eigenvalues in* $\tau(\mathbb{C}/\mathbb{Z})$.

An extension as in 5.5.6 is called a *Deligne lattice*. In particular, the meromorphic connection along D obtained from $(\widetilde{E}, \widetilde{\nabla})$ by tensoring with $\mathcal{O}_X(*D)$ is regular.

Note that without a connection around, the fact that E extends over D is not granted, even if D is a normal crossing divisor. For example, let D be the divisor of $\mathbb{C}^2$ given by $x_1x_2 = 0$ and define $U := \mathbb{C}^2\backslash D$. Since $\mathbb{C}^2$ is a Stein manifold, the exponential exact sequence gives

$$\mathrm{Pic}(\mathbb{C}^2) \simeq H^2(\mathbb{C}^2, \mathbb{Z}) \simeq 0$$

For the same reason, we have

$$\mathrm{Pic}(U) \simeq H^2(U, \mathbb{Z}) \simeq H^2(S^1 \times S^1, \mathbb{Z}) \simeq \mathbb{Z}.$$

So one can find a non trivial line bundle $\mathcal{L}$ on U. Such a bundle cannot be extended to $\mathbb{C}^2$. As a by-product of 5.5.6, the sheaf $\mathcal{L}$ cannot underlie a flat connection.

6. A few exercises

In the following exercises, the symbol $F^\bullet$ will denote the order filtration on $\mathcal{D}_X$ as defined in 1.1.

6.1. Let X be a complex manifold. Show that $F^1\mathcal{D}_X = \mathcal{O}_X \oplus \Theta_X$.

6.2. Let X be a complex manifold. Let $\mathcal{M}$ be a $\mathcal{D}_X$-module whose underlying sheaf is $\mathcal{O}_X$-coherent. Show that the underlying sheaf of $\mathcal{M}$ is a locally free sheaf.

6.3. Prove that for every $k \geqslant 0$, the morphism

$$\begin{array}{rcl} \mathrm{gr}_k^F \mathcal{D}_X & \longrightarrow & \mathcal{H}om_{\mathbb{C}}(\otimes_{\mathbb{C}}^k \mathcal{O}_X, \mathcal{O}_X) \\ P & \longrightarrow & f_1 \otimes \cdots \otimes f_k \longrightarrow [\cdots[[P, f_1], f_2] \ldots,]f_k] \end{array}$$

identifies $\mathrm{gr}_k^F \mathcal{D}_X$ with $\mathrm{Sym}^k \Theta_X$.

6.4. For $c \in \mathbb{C}$, define

$$(\mathcal{M}, \nabla) := (\mathcal{O}_{\mathbb{C}}(*0), d + c\frac{dx}{x})$$

Prove[14] that $(\mathrm{Sol}\,\mathcal{M})_0 \simeq 0$ in $D(0, \mathbb{C})$.

6.5. Let $p : X \longrightarrow Y$ be a finite map between complex manifolds. Let $\mathcal{F}$ be a sheaf of complex vector spaces on X such that $p_*\mathcal{F}$ is constructible. Using proper base change theorem, prove that $\mathcal{F}$ is constructible.

[14] Note that $i^{-1}\,\mathrm{Sol}\,\mathcal{M}$ is the irregularity sheaf of $\mathcal{M}$ at 0, so6.4 exactly says that $\mathcal{M}$ is regular at 0.

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