BOUNDING RAMIFICATION WITH COHERENT SHEAVES

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ABSTRACT. Given a coherent sheaf $\mathcal{E}$ on a scheme of finite type X over a perfect field, we introduce a category of complexes of étale sheaves on X with logarithmic conductors bounded by $\mathcal{E}$ and study its compatibilities with finite push-forward.

CONTENTS

1. Introduction

The goal of this paper is to use coherent sheaves to bound the wild ramification of complexes of étale sheaves.

Let X be a normal scheme of finite type over a perfect field k of characteristic $p > 0$, let $D \subset X$ be an effective Cartier divisor and put $U := X - D$. Let Λ be a finite local ring of residue characteristic $\ell \neq p$ and let $\mathcal{L}$ be a locally constant constructible sheaf of free Λ-modules of finite type on U. A useful way to bound the wild ramification of $\mathcal{L}$ along D is to look at the Swan conductors of the restrictions of $\mathcal{L}$ to curves C passing through the points of D. Namely, we say following Deligne [EK12] that $\mathcal{L}$ has Swan conductors bounded by D if for every smooth connected curve C over k and every morphism $f : C \to X$ over k with $f(C) \nsubseteq D$, we have for every $x \in C$,

$$\mathrm{Sw}_x\, \mathcal{L}|_{C\cap U} \leq m_x(f^*D) \tag{1.0.1}$$

where $\mathrm{Sw}_x\, \mathcal{L}|_{C\cap U}$ is the Swan conductor of $\mathcal{L}|_{C\cap U}$ at $x \in C$ in the sense of [Ser68] and $m_x(f^*D)$ is the multiplicity of f^*D at x. When X is smooth over k and when D has normal crossings, Abbes and Saito's ramification theory [AS02] implies that the poset of effective Cartier divisors supported on D and satisfying (1.0.1) has a minimal element, called the Swan divisor [Bar16, Hu17].

The interest of this notion comes from the observation that objects with bounded Swan conductors tend to exhibit behaviors similar to their characteristic zero counterparts. For example in the tame case, that is when $D = 0$, a Lefschetz theorem for the tame fundamental group is known [Dri12, EK16]. When k is finite, Hiranouchi proved without assumption on the singularities of X that there are only a finite number of étale coverings on X with bounded degree and wild ramification bounded by a given effective Cartier divisor at infinity [Hir17], which is reminiscent to the finite generation of the fundamental group of a complex algebraic variety. Also, the Betti numbers of locally constant constructible sheaves with bounded rank and wild ramification are bounded [HT25a]. In characteristic zero, this phenomenon was observed for irregular flat bundles on surfaces in [HT22].

Note however that effective Cartier divisors are not sensitive to the wild ramification in codimension ≥ 2 displayed by sheaves which are not locally constant. This confines the characteristic $p > 0$ side of the above analogies to locally constant sheaves on normal varieties, while the most natural set-up is sometimes that of arbitrary complexes of constructible sheaves on arbitrary schemes of finite type. To solve this problem, we extend the above definition by replacing effective Cartier divisors with arbitrary coherent sheaves. Namely, given a scheme of finite type X over k, given $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$ and Λ a finite local ring of residue characteristic $\ell \neq p$, we introduce a full subcategory

$$D^b_{ctf}(X, \mathcal{E}, \Lambda) \subset D^b_{ctf}(X, \Lambda)$$

of tor finite complexes with logarithmic conductors bounded by $\mathcal{E}$. In a nutshell, the right hand side of (1.0.1) gets replaced by the length of the torsion part of $f^*\mathcal{E}$ at x and the Swan conductor gets replaced by the highest ramification slope from [Ser68]. See Definition 5.1 for a precise definition. Replacing the Swan conductor by the highest upper numbering ramification slope, also called the *logarithmic conductor* is justified by the Betti estimates obtained in [HT25a], which are *linear* in the rank. The main result of this paper is the following compatibility of $D^b_{ctf}(X, \mathcal{E}, \Lambda)$ with finite push-forward :

Theorem 1.1 (Theorem 6.12)**.** *Let $f : Y \to X$ be a finite morphism between schemes of finite type over k. Let Σ be a stratification of Y and let $\mathcal{E} \in \mathrm{Coh}(Y)$. Then there exists $\mathcal{E}' \in \mathrm{Coh}(X)$ such that for every Σ-stratified constructible complex $\mathcal{K} \in D^b_{ctf}(Y, \mathcal{E}, \Lambda)$, we have $f_*\mathcal{K} \in D^b_{ctf}(X, \mathcal{E}', \Lambda)$.*

The proof of Theorem 1.1 reduces to the analysis of the compatibility of Abbes and Saito's logarithmic conductor [AS02] with finite push-forward. To state its main output, recall that for a normal scheme of finite type X and for a constructible sheaf of Λ-modules $\mathcal{F}$ of finite tor-dimension on X, we let $LC_X(\mathcal{F})$ be the Weil divisor of X whose multiplicity at Z is the generic logarithmic conductor of $\mathcal{F}$ along Z, in the sense of Abbes and Saito (see Definition 3.1). Then we have the following

Theorem 1.2 (Theorem 6.8,Theorem 6.10)**.** *Let $f : Y \to X$ be a finite surjective morphism of normal schemes of finite type over k. Let D be a reduced effective Cartier divisor on X and put $U := X - D$. Define $E := f^{-1}(D)$ and put $j : V := Y - E \hookrightarrow Y$. Let $\mathcal{L} \in \mathrm{Loc}_{tf}(V, \Lambda)$.*

(1) If the restriction $f_0 : V \to U$ is étale, we have

$$LC_X(f_*j_!\mathcal{L}) \leq LC_X(f_*j_!\Lambda) + f_*LC_Y(j_!\mathcal{L}) \,.$$

(2) If the restriction $f_0 : V \to U$ is radicial, we have

$$LC_X(f_*j_!\mathcal{L}) \leq f_*LC_Y(j_!\mathcal{L}) \,.$$

Note that Theorem 1.2 also holds for the non logarithmic conductor divisor (see Theorem 6.8 and Theorem 6.10).

Theorem 1.2 is one of the crucial ingredients to prove the Betti estimates from [HT25a]. Its proof is local. In a nutshell, it rests upon Hu's semi-continuity results [Hu23] for reducing the proof to a local statement for curves treated in the unpublished note [Tey15]. For a variant of Theorem 1.2-(1) for the Swan conductor obtained by a global argument via the Grothendieck-Ogg-Shafarevich formula, see [ES21].

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Notation 1.3. We introduce the following running notations.

- k denotes a perfect field of characteristic $p > 0$.
- The letter Λ will refer to a finite local ring of residue characteristic $\ell \neq p$.
- For a scheme X of finite type over k, a Weil divisor D of X and an irreducible component Z of D, we denote by $m_Z(D)$ the multiplicity of D along Z and by $m(D)$ the maximal multiplicity of D.
- For a scheme X of finite type over k, we denote by $D^b_{ctf}(X, \Lambda)$ the derived category of complexes of Λ-sheaves of finite tor-dimension with bounded and constructible cohomology sheaves.
- We let $\mathrm{Cons}_{ctf}(X, \Lambda)$ be the category of constructible sheaves of Λ-modules of finite tor-dimension over X and $\mathrm{Loc}_{tf}(X, \Lambda) \subset \mathrm{Cons}_{tf}(X, \Lambda)$ the full subcategory spanned by locally constant constructible sheaves. By [Wei94, Lemma 4.4.14], the germs of any $\mathcal{L} \in \mathrm{Loc}_{tf}(X, \Lambda)$ are automatically free Λ-modules of finite rank.
- For a finite stratification Σ of X, we let $D^b_{\Sigma,tf}(X, \Lambda) \subset D^b_{tf}(X, \Lambda)$ be the full subcategory spanned by Σ-constructible complexes.

2. Conductor and finite direct image: local case

Notation 2.1. Let K be a henselian discrete valuation field over k, $\mathcal{O}_K$ the ring of integer of K, $\mathfrak{m}_K$ the maximal ideal of $\mathcal{O}_K$, F the residue field of $\mathcal{O}_K$, $\overline{K}$ an algebraic closure of K and $K^{sep} \subset K$ the separable closure of K in $\overline{K}$. Let G_K be the Galois group of K^{sep} over K. We denote by $I_K \subset G_K$ the inertia subgroup and by $P_K \subset I_K$ the wild ramification subgroup.

Recollection 2.2. In [AS02], Abbes and Saito defined two decreasing filtrations $\{G^r_K\}_{r\in\mathbb{Q}_{>0}}$ and $\{G^r_{K,\log}\}_{r\in\mathbb{Q}_{\geq 0}}$ on G_K by closed normal subgroups. They are called respectively the *the ramification filtration* and *the logarithmic ramification filtration*. For $r \in \mathbb{Q}_{\geq 0}$, put

$$G^{r+}_K = \overline{\bigcup_{s>r} G^s_K} \quad \text{and} \quad G^{r+}_{K,\log} = \overline{\bigcup_{s>r} G^s_{K,\log}}.$$

Proposition 2.3 ([AS02, AS03, Sai08, Sai17])**.** *The following inclusions hold:*

(1) *For any* $0 < r \leq 1$, *we have*

$$G_K^r = G_{K,\log}^0 = I_K \textit{ and } G_K^{+1} = G_{K,\log}^{0+} = P_K.$$

(2) *For any* $r \in \mathbb{Q}_{\geq 0}$, *we have*

$$G_K^{r+1} \subseteq G_{K,\log}^r \subseteq G_K^r.$$

If F *is perfect, then for any* $r \in \mathbb{Q}_{\geq 0}$, *we have*

$$G_{K,\mathrm{cl}}^r = G_{K,\log}^r = G_K^{r+1}.$$

where $G_{K,\mathrm{cl}}^r$ *is the classical wild ramification subgroup as defined in* [Ser68].

Let M be a finitely generated Λ-module with continuous P_K-action. The module M has decompositions

$$M = \bigoplus_{r\geq 1} M^{(r)} \quad \text{and} \quad M = \bigoplus_{r\geq 0} M_{\log}^{(r)} \tag{2.3.1}$$

into P_K-stable submodules, where $M^{(1)} = M_{\log}^{(0)} = M^{P_K}$, and such that for every $r \in \mathbb{Q}_{>0}$,

$$(M^{(r+1)})^{G_K^{r+1}} = 0 \quad \text{and} \quad (M^{(r+1)})^{G_K^{(r+1)+}} = M^{(r+1)};$$

$$(M_{\log}^{(r)})^{G_{K,\log}^r} = 0 \quad \text{and} \quad (M_{\log}^{(r)})^{G_{K,\log}^{r+}} = M_{\log}^r.$$

The decompositions (2.3.1) are called respectively the *slope decomposition* and the *logarithmic slope decomposition* of M. The values r for which $M^{(r)} \neq 0$ (resp. $M_{\log}^{(r)} \neq 0$) are the *slopes* (resp. the *logarithmic slopes*) of M. We will make repeated use of the following

Lemma 2.4 ([Kat88, Proposition 1.1,Lemma 1.5])**.** *For every* $r \in \mathbb{Q}_{\geq 0}$, *the endofunctors of the category of finitely generated* Λ*-modules with continuous* P_K*-action defined by*

$$M \to M^{(r)} \quad \textit{and} \quad M \to M_{\log}^{(r)}$$

are exact. If furthermore M *is free of finite type over* Λ, *so is* $M_{\log}^{(r)}$.

Lemma 2.5 ([Kat88, Lemma 1.5])**.** *Let* $\Lambda \to \Lambda'$ *be a morphism between finite local rings with residue field of characteristic* $\ell \neq p$. *For every* $r \in \mathbb{Q}_{\geq 0}$ *and every finitely generated free* Λ*-module* M *with continuous* P_K*-action,* r *is a slope (resp. logarithmic slope) of* M *if and only if* r *is a slope (resp. logarithmic slope) of* $\Lambda' \otimes_\Lambda M$.

Let M be a finitely generated Λ-module with continuous P_K-action. We denote by $c_K(M)$ the largest slope of M and refer to $c_K(M)$ as the *conductor of* M. Similarly, we denote by $lc_K(M)$ the largest logarithmic slope of M and refer to $lc_K(M)$ as the *logarithmic conductor of* M. We say that M is *isoclinic* (resp. *logarithmic isoclinic*) if M has only one slope (resp. only one logarithmic slope).

The following is an immediate consequence of Proposition 2.3-(2).

Lemma 2.6. *We have*

$$\mathrm{lc}_K(M) \leq c_K(M) \leq \mathrm{lc}_K(M) + 1 \,.$$

If the residue field F *is perfect, we have*

$$\mathrm{lc}_K(M) + 1 = c_K(M) \,.$$

Lemma 2.7. *Let* $M^\bullet$ *be a complex of finitely generated* Λ*-modules with continuous* G_K*-action and let* $i \in \mathbb{Z}$*. If* $r \in \mathbb{Q}_{\geq 0}$ *is a logarithmic slope of* $\mathcal{H}^i M^\bullet$*, then* r *is a logarithmic slope of* M^i*.*

Proof. Let $Z^i := \mathrm{Ker}(M^i \to M^{i+1}) \subset M_i$. By Lemma 2.4, it is enough to show that r is a logarithmic slope of Z^i. By Lemma 2.4 again , the surjective map $Z^i \to \mathcal{H}^i M^\bullet$ induces a surjective map

$$(Z^i)^{(r)}_{\log} \to (\mathcal{H}^i M^\bullet)^{(r)}_{\log} \,.$$

Since the right hand side is non zero, so is the left hand side and the conclusion follows. □

Lemma 2.8. *Let* $M^\bullet$ *be a bounded finite tor-dimension complex of finitely generated* Λ*-modules with continuous* G_K*-action. Then,* $M^\bullet$ *is quasi-isomorphic to a bounded complex* $N^\bullet$ *of free* Λ*-modules of finite type with continuous* G_K*-action such that* $r \in \mathbb{Q}_{\geq 0}$ *is a logarithmic slope of some* N^i *if and only if it is a logarithmic slope of some* $\mathcal{H}^i N^\bullet$*.*

Proof. By [SP, 03TT], we can assume that $M^\bullet$ is a bounded complex of free Λ-modules of finite type with continuous G_K-action. We have

$$M^\bullet = \bigoplus_{r \in \mathbb{Q}_{\geq 0}} M^{\bullet,(r)}_{\log}$$

where each factor is a bounded complex of free Λ-modules of finite type by Lemma 2.4. If $r \in \mathbb{Q}_{\geq 0}$ is not a logarithmic slope of some $\mathcal{H}^i M^\bullet$, then by Lemma 2.4, we have

$$\mathcal{H}^i(M^{\bullet,(r)}_{\log}) \simeq (\mathcal{H}^i M^\bullet)^{(r)}_{\log} \simeq 0 \,.$$

Thus $M^{\bullet,(r)}_{\log}$ is acyclic. Hence, at the cost of removing the direct factors $M^{\bullet,(r)}_{\log}$ from $M^\bullet$ for all r as above, we can assume that if $r \in \mathbb{Q}_{\geq 0}$ is a logarithmic slope of some M^i, then it is a logarithmic slope of some $\mathcal{H}^i M^\bullet$. Since the converse is true by Lemma 2.7, the proof of Lemma 2.8 is complete. □

Corollary 2.9. *Let* $M^\bullet$ *be a bounded finite tor-dimension complex of finitely generated* Λ*-modules with continuous* G_K*-action. Let* $\Lambda \to \Lambda'$ *be a morphism between finite local rings of characteristic* $\ell \neq p$*. If* $r \in \mathbb{Q}_{\geq 0}$ *is a logarithmic slope of some* $\mathcal{H}^i(M^\bullet \otimes^L_\Lambda \Lambda')$*, then it is a logarithmic slope of some* $\mathcal{H}^i M^\bullet$*.*

Proof. By Lemma 2.8, we can assume that $M^\bullet$ is a bounded complex of free Λ-modules of finite type with continuous G_K-action such that $r \in \mathbb{Q}_{\geq 0}$ is a logarithmic slope of some M^i if and only if it is a logarithmic slope of some $\mathcal{H}^i M^\bullet$. Let $i \in \mathbb{Z}$ and let $r \in \mathbb{Q}_{\geq 0}$ be a logarithmic slope of $\mathcal{H}^i(M^\bullet \otimes^L_\Lambda \Lambda')$. Since,

$$M^\bullet \otimes^L_\Lambda \Lambda' \simeq M^\bullet \otimes_\Lambda \Lambda' \,,$$

we deduce from Lemma 2.7 that r is a logarithmic slope of $M^i \otimes_\Lambda \Lambda'$. By Lemma 2.5, we deduce that is r is a logarithmic slope of M^i. By construction, we get that r is a logarithmic slope of some $\mathcal{H}^j M^\bullet$. □

2.10. Co-induction and ramification.

Recollection 2.11. Let G be a profinite group and let $H \subset G$ be a closed subgroup. Following [Ser73, §2.5], the scalar restriction from Λ-modules with continuous G-action to Λ-modules with continuous H-action admits a right adjoint, the *co-induction* described as

$$\mathrm{CoInd}_G^H(M) := \{f : G \to M \text{ continuous commuting with the left action of } H\}$$

where M is endowed with the discrete topology. By construction $g \in G$ acts on $\mathrm{CoInd}_G^H(M)$ by

$$(g \cdot f)(x) = f(xg)\ .$$

Lemma 2.12. *Let $H'' \subset H' \subset H$ be closed subgroups such that H' and H'' are normal in G. Then, the following holds:*

(1) $\mathrm{CoInd}_G^H(M)^{H'} = \{f \in \mathrm{CoInd}_G^H(M)$ *such that* $f(G) \subset M^{H'}\}$.

(2) If furthermore H is open in G, we have

$$\mathrm{CoInd}_G^H(M)^{H''} = \mathrm{CoInd}_G^H(M)^{H'} \textit{ if and only if } M^{H''} = M^{H'}\ .$$

Proof. We prove (1). Let $f \in \mathrm{CoInd}_G^H(M)^{H'}$. For $h \in H'$ and $g \in G$, we have $g^{-1}hg \in H'$. Thus

$$(g^{-1}hg \cdot f)(g) = f(g) = f(gg^{-1}hg) = h \cdot f(g)$$

Hence, $f(G) \subset M^{H'}$. The reverse inclusion is proved similarly. We now prove (2) and assume that H is open in G. The converse implication is tautological due to (1). Let us prove the direct implication. Since $H'' \subset H'$, we have $M^{H'} \subset M^{H''}$. Let $x \in M^{H''}$ and consider a set of representatives $g_1, \dots, g_n$ for $H\backslash G$. Let $f : G \to M$ be the function defined by

$$f(hg_i) = h \cdot x$$

for every $h \in H$ and $i = 1, \dots, n$. Since $H \subset G$ is open, the continuity of f can be checked on each Hg_i. On the other hand, the restriction of f to Hg_i decomposes as

$$Hg_i \xrightarrow{\cdot g_i^{-1}} H \xrightarrow{\cdot x} M$$

which is indeed continuous as a composition of continuous maps. Thus, $f \in \mathrm{CoInd}_G^H(M)$. Since H'' is normal in H and $x \in M^{H''}$, we have $f(G) \subset M^{H''}$. By (1), we deduce that f lies in $\mathrm{CoInd}_G^H(M)^{H''} = \mathrm{CoInd}_G^H(M)^{H'}$. By (1) again, we deduce $f(g_1) = x \in M^{H'}$. This concludes the proof of Lemma 2.12. □

Lemma 2.13. *In the setting of Recollection 2.2, let K'/K be a finite separable extension in K^{sep}. Let $r \in \mathbb{Q}_{>0}$. Then, the following are equivalent:*

(1) $G^r_{K,\log} \subset G_{K'}$.

(2) $G^r_{K,\log}$ *acts trivially on* $\mathrm{CoInd}_{G_K}^{G_{K'}}(\Lambda)$.

(3) $r > \mathrm{lc}_K(\mathrm{CoInd}_{G_K}^{G_{K'}}(\Lambda))$.

Proof. The equivalence between (2) and (3) is obvious. Suppose that (2) holds. Since $G_{K'}$ is open in G_K, we have

$$\mathrm{CoInd}_{G_K}^{G_{K'}}(\Lambda) \simeq \mathrm{Fun}(G_{K'}\backslash G_K, \Lambda) .$$

Let $g \in G_K$ acting trivially on $\mathrm{CoInd}_{G_K}^{G_{K'}}(\Lambda)$. Let $f : G_{K'}\backslash G_K \to \Lambda$ sending $G_{K'}$ to 1 and sending a class distinct from $G_{K'}$ to 0. Then, $g \cdot f = f$ yields $g \in G_{K'}$. Thus, (1) is true. Suppose that (1) holds and let us prove (2). It is enough to prove that $G^r_{K,\log}$ acts trivially on $G_{K'}\backslash G_K$, which follows immediately from (1) and the fact that $G^r_{K,\log}$ is normal in G.

□

Definition 2.14. In the setting of Recollection 2.2, let L/K be a finite extension in $\overline{K}$ and let K'/K be the separable closure of K in L. We define

$$\mathrm{lc}_{L/K} := \mathrm{lc}_K(\mathrm{CoInd}_{G_K}^{G_{K'}}(\Lambda)) \in \mathbb{Q}_{\geq 0} .$$

Lemma 2.15. *In the setting of Recollection 2.2, assume that the residue field F is perfect. Let* K'/K *be a finite extension in* K^{sep}*. Then for every* $r > \mathrm{lc}_{K'/K}$*, we have*

$$G^r_{K,\log} = G^{\psi_{K'/K}(r)}_{K',\log}$$

in $G_{K'}$*, where* $\psi_{K'/K} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *is Herbrand's function* [Ser68, IV §3].

Proof. The argument below is extracted from the unpublished note [Tey15]. Let L/K be a finite Galois extension in K^{sep} containing K'. In particular, the extension L/K' is finite Galois. Put $G = \mathrm{Gal}(L/K)$ and $H = \mathrm{Gal}(L/K') \subset G$. Since $r > \mathrm{lc}_{K'/K}$, we have $G^r_K \subset G_{K'}$. Hence, $G^r \subset H$. Thus,

$$G^r = H \cap G^r = H \cap G_{\psi_{L/K}(r)} = H_{\psi_{L/K}(r)} = H^{\varphi_{L/K'} \circ \psi_{L/K}(r)}$$

where the third equality comes from the compatibility of the lower-numbering ramification filtration with subgroups [Ser68, IV Proposition 2]. From [Ser68, IV Remark 2], we have $\psi_{L/K} = \psi_{L/K'} \circ \psi_{K'/K}$. Hence,

$$\varphi_{L/K'} \circ \psi_{L/K} = \psi_{K'/K} .$$

Thus,

$$\mathrm{Gal}(L/K)^r = \mathrm{Gal}(L/K')^{\psi_{K'/K}(r)} . \tag{2.15.1}$$

Let $\mathcal{P}$ be the poset of finite Galois extensions L/K in K^{sep} containing K', ordered by the inclusion. Then, $\mathcal{P}$ is cofinal both in the poset of finite Galois extensions of K in K^{sep} and in the poset of finite Galois extensions of K' in K^{sep}. Lemma 2.15 thus follows from (2.15.1) by passing to the limit over $\mathcal{P}^{op}$. □

Lemma 2.16 ([Tey15, Kat18]). *In the setting of Recollection 2.2, assume that the residue field* F *is perfect. Let* L/K *be a finite purely inseparable extension in* $\overline{K}$ *and let* L^{sep} *the separable extensions of* L *in* $\overline{K}$*. Then, the isomorphism* $\iota : G_L \xrightarrow{\sim} G_K$ *induced by restriction from* L^{sep} *to* K^{sep} *is compatible with the logarithmic ramification filtration, that is*

$$\iota(G^r_{L,\log}) = G^r_{K,\log}$$

for every $r \in \mathbb{Q}_{\geq 0}$.

Corollary 2.17. *In the setting of Recollection 2.2, assume that the residue field* F *is perfect. Let* L/K *be a finite extension in* $\overline{K}$ *and let* K'/K *be the separable closure of* K *in* L*. Then for every* $r > lc_{L/K}$*, we have*

$$G^r_{K,\log} = \iota(G^{\psi_{K'/K}(r)}_{L,\log})$$

where $\iota : G_L \xrightarrow{\sim} G_K$ *is the isomorphism induced by restriction from* L^{sep} *to* K^{sep} *in* $\overline{K}$.

Proof. Since $r > lc_{L/K} = lc_{K'/K}$, we have

$$G^r_{K,\log} = G^{\psi_{K'/K}(r)}_{K',\log} = \iota(G^{\psi_{K'/K}(r)}_{L,\log})$$

where the first equality follows from Lemma 2.15 and the second one from Lemma 2.16. □

Proposition 2.18. *In the setting of Recollection 2.2, assume that the residue field* F *is perfect. Let* L/K *be a finite extension in* $\overline{K}$ *and let* K'/K *be the separable closure of* K *in* L*. Let* M *be a finitely generated* Λ*-module with continuous* G_L*-action. For* $r > lc_{L/K}$*, the following are equivalent :*

(1) *r is a logarithmic slope of* $\mathrm{CoInd}^{G_L}_{G_K}(M)$.

(2) *$\psi_{K'/K}(r)$ is a logarithmic slope of* M.

Proof. Note that L/K' is purely inseparable. In the above statement, the coinduction is made along the morphism $G_L \xrightarrow{\sim} G_{K'} \subset G_K$. Since $r > lc_{L/K} = lc_{K'/K}$, we have $G^r_{K,\log} \subset G_{K'}$ in virtue of Lemma 2.13 and the groups

$$G^{r+}_{K,\log} \subset G^r_{K,\log} \subset G_{K'}$$

satisfy the assumptions of Lemma 2.12 with $G = G_K$ and $H = G_{K'}$ open in G_K. Thus,

$$
\begin{aligned}
& r \text{ is not a logarithmic slope of } \mathrm{CoInd}^{G_L}_{G_K}(M) \\
\Longleftrightarrow\ & \mathrm{CoInd}^{G_L}_{G_K}(M)^{G^r_{K,\log}} = \mathrm{CoInd}^{G_L}_{G_K}(M)^{G^{r+}_{K,\log}} \\
\Longleftrightarrow\ & M^{G^r_{K,\log}} = M^{G^{r+}_{K,\log}} && \text{Lemma 2.12} \\
\Longleftrightarrow\ & M^{G^{\psi_{K'/K}(r)}_{L,\log}} = M^{G^{\psi_{K'/K}(r)+}_{L,\log}} && \text{Corollary 2.17} \\
\Longleftrightarrow\ & \psi_{K'/K}(r) \text{ is not a logarithmic slope of } M.
\end{aligned}
$$

□

The following lemma is proved in [Ser68, IV Proposition 12] for finite *Galois* extensions. We however needs the case of an arbitrary finite separable extension.

Lemma 2.19. *In the setting of Recollection 2.2, assume that the residue field* F *is perfect. Let* K'/K *be a finite separable extension in* K^{sep}*. Then, the following hold :*

1. *The function* $\varphi_{K'/K}$ *is strictly increasing, piecewise linear and concave.*
2. *For every* $r \geq 0$*, we have* $\varphi_{K'/K}(r) \leq r$.

Proof. Let L/K be the Galois closure of K' in K^{sep}. Put $G := \mathrm{Gal}(L/K)$ and $H := \mathrm{Gal}(L/K') \subset G$. By definition, we have

$$\varphi_{K'/K} = \varphi_{L/K} \circ \psi_{L/K'} .$$

Hence, $\varphi_{K'/K}$ is strictly increasing and piecewise linear as composition of strictly increasing and piecewise linear functions. To prove that $\varphi_{K'/K}$ is concave, it is thus enough to prove that the restriction of $\varphi_{K'/K}$ to its affine locus has decreasing derivative. For $r \geq 0$ sufficiently generic, we have

$$\begin{aligned}
\varphi'_{K'/K}(r) &= \psi'_{L/K'}(r)\varphi'_{L/K}(\psi_{L/K'}(r)) \\
&= \frac{|H_0|}{|H_{\psi_{L/K'}(r)}|}\frac{|G_{\psi_{L/K'}(r)}|}{|G_0|} && \text{[Ser68, IV Proposition 12]} \\
&= \frac{|H_0|}{|G_0|}|\operatorname{Im}(G_{\psi_{L/K'}(r)} \to G/H)|
\end{aligned}$$

where the last equality follows from $H_u = H \cap G_u$ for every $u \geq 0$. Since the ramification filtration is decreasing, we deduce that so is $\varphi'_{K'/K}$. This proves (1). We now prove (2). By loc. cit., we have $\psi_{L/K'}(0) = \varphi_{L/K}(0) = 0$, so that $\varphi_{K'/K}(0) = 0$. Since $\varphi_{K'/K}$ is piecewise linear and concave, we are thus left to show that the right derivative $\varphi'_{K'/K,d}(0)$ of $\varphi_{K'/K}$ at 0 is smaller than 1. By the above computation, we have

$$\varphi'_{K'/K,d}(0) = \frac{|H_0|}{|G_0|}\frac{|G_1|}{|H_1|} = \frac{|H_0|/|H_1|}{|G_0|/|G_1|} .$$

On the other hand, the relation $H_0 \cap G_1 = H_1$ implies that the induced map

$$H_0/H_1 \to G_0/G_1$$

is injective. Thus $\varphi'_{K'/K,d}(0) \leq 1$. This concludes the proof of Lemma 2.19. □

Corollary 2.20. *In the setting of Recollection 2.2, assume that the residue field* F *is perfect. Let* L/K *be a finite extension in* $\overline{K}$*. Let* M *be a finitely generated* Λ*-module with continuous* G_L*-action. Then,*

$$\mathrm{lc}_K(\mathrm{CoInd}_{G_K}^{G_L}(M)) \leq \max(\mathrm{lc}_{L/K}, \mathrm{lc}_L(M)) .$$

Proof. Put $r = \mathrm{lc}_K(\mathrm{CoInd}_{G_K}^{G_L}(M))$. If $r > \mathrm{lc}_{L/K}$, Proposition 2.18 implies that $\psi_{K'/K}(r)$ is a logarithmic slope of M. In particular $\psi_{K'/K}(r) \leq \mathrm{lc}_L(M)$. Thus,

$$r \leq \varphi_{K'/K}(\mathrm{lc}_L(M)) \leq \mathrm{lc}_L(M)$$

where the second inequality follows from Lemma 2.19-(2). □

3. Conductor divisors

Let X be a normal scheme of finite type over k. Let Z be an integral Weil divisor and let $\eta \in Z$ be its generic point. Let K be the fraction field of $\hat{\mathcal{O}}_{X,\eta}$ and fix a separable closure K^{sep} of K. For $\mathcal{F} \in \mathrm{Cons}_{tf}(X,\Lambda)$, the pull-back $\mathcal{F}|_{\mathrm{Spec}\, K}$ is a Λ-module of finite type with continuous G_K-action. Using the notations from Notation 2.1, we put

$$c_Z(\mathcal{F}) := c_K(\mathcal{F}|_{\mathrm{Spec}\, K}) \text{ and } lc_Z(\mathcal{F}) := lc_K(\mathcal{F}|_{\mathrm{Spec}\, K}) \ .$$

Definition 3.1. Let X be a normal scheme of finite type over k and let $\mathcal{F} \in \mathrm{Cons}_{tf}(X,\Lambda)$. We define the *conductor divisor of* $\mathcal{F}$ as the Weil divisor with rational coefficients given by

$$C_X(\mathcal{F}) := \sum_Z c_Z(\mathcal{F}) \cdot Z$$

and the *logarithmic conductor divisor of* $\mathcal{F}$ as the Weil divisor with rational coefficients given by

$$LC_X(\mathcal{F}) := \sum_Z lc_Z(\mathcal{F}) \cdot Z$$

where the sums run over the set of integral Weil divisors of X.

Remark 3.2. When there is no ambiguity, we will drop the subscript X in $C_X(\mathcal{F})$ and $LC_X(\mathcal{F})$.

Remark 3.3. Lemma 2.6 implies

$$LC_X(\mathcal{F}) \leq C_X(\mathcal{F}) \leq LC_X(\mathcal{F}) + D \ .$$

where D is the support of $C_X(\mathcal{F})$.

Definition 3.4. In the setting of Section 3, we define the *generic conductor* and the *generic (logarithmic) conductor* of $\mathcal{F}$ along a divisor D respectively by

$$c_D(\mathcal{F}) := \max_Z c_Z(\mathcal{F}) \text{ and } lc_D(\mathcal{F}) := \max_Z lc_Z(\mathcal{F})$$

where Z runs over the set of irreducible components of D.

The conductor and log conductor divisors enjoy the following semi-continuity property :

Theorem 3.5 ([Hu23, Theorem 1.4,1.5])**.** *Let $f : Y \to X$ be a morphism of smooth schemes of finite type over k. Let D be an effective Cartier divisor on X and put $U := X - D$. Assume that $E := Y \times_X D$ is an effective Cartier divisor on Y and that Λ is a finite field. For every $\mathcal{L} \in \mathrm{Loc}_{tf}(U,\Lambda)$, we have*

$$C_Y((j_!\mathcal{L})|_Y) \leq f^* C_X(j_!\mathcal{L}) \ .$$

If furthermore D has normal crossings, we have

$$LC_Y((j_!\mathcal{L})|_Y) \leq f^* LC_X(j_!\mathcal{L}) \ .$$

The conductor divisor can be detected by curves, due to the following :

Proposition 3.6 ([Sai17, Corollary 3.9]). *Let X be a smooth scheme over k of pure dimension n. Let D be an effective Cartier divisor on X and put $j : U := X - D \hookrightarrow X$. Let $\mathcal{L} \in \mathrm{Loc}_{tf}(U, \Lambda)$. Then, there is a closed conical subset $SS \subset \mathbb{T}^*X$ of pure dimension n and a dense open subset $\Omega \subset D$ such that for every immersion $i : S \hookrightarrow X$ over k where S is a smooth curve satisfying*

(1) S meets D transversely at a single smooth point $x \in \Omega$,

(2) every non zero $\omega \in SS_x$ does not vanish on $\mathbb{T}_x S \subset \mathbb{T}_x X$,

we have $C_S((j_!\mathcal{L})|_S) = i^* C_X(j_!\mathcal{L})$.

Theorem 3.7 ([Hu23, Theorem 1.5]). *Let X be a smooth scheme over k. Let D be a smooth irreducible divisor on X and put $j : U := X - D \hookrightarrow X$. For every $\mathcal{L} \in \mathrm{Loc}_{tf}(U, \Lambda)$, we have*

$$\mathrm{lc}_D(j_!\mathcal{L}) = \sup_{\mathcal{I}(X,D)} \frac{\mathrm{lc}_x(j_!\mathcal{L})}{m_x(f^*D)}$$

where $\mathcal{I}(X, D)$ is the set of triples $(C, f : S \to X, x)$ where $f : C \to X$ is an immersion from a smooth curve over k to X such that $x = C \cap D$ is a closed point of X.

Proposition 3.8 ([HT25b, Corollary 5.8]). *Let $f : X \to S$ be a smooth morphism between smooth schemes of finite type over k. Let $D \subset X$ be an effective Cartier divisor relative to S such that $f|_D : D \to S$ is smooth. Put $j : U := X - D \hookrightarrow X$. Then for every $\mathcal{L} \in \mathrm{Loc}_{tf}(U, \Lambda)$ and every algebraic geometric point $\bar{s} \to S$, we have*

$$C((j_!\mathcal{L})|_{X_{\bar{s}}}) \leq i_{\bar{s}}^* C(j_!\mathcal{L}) \text{ and } LC((j_!\mathcal{L})|_{X_{\bar{s}}}) \leq i_{\bar{s}}^* LC(j_!\mathcal{L})$$

where $i_{\bar{s}} : X_{\bar{s}} \to X$ is the canonical morphism.

Remark 3.9. Theorem 3.5,Proposition 3.6, Theorem 3.7 and Proposition 3.8 are proved for a finite field Λ of characteristic $\ell \neq p$ in [Sai17, Hu23, HT25b]. They hold for Λ a finite local ring of residue characteristic $\ell \neq p$ as a consequence of Lemma 2.5.

4. Torsion divisors of coherent sheaves

Construction 4.1. Let X be a normal noetherian scheme and let $\mathcal{E} \in \mathrm{Coh}(X)$. If $X^1 \subset X$ denotes the set of codimension 1 points of X, we define a Weil divisor on X by the formula

$$T(\mathcal{E}) := \sum_{\eta \in X^1} \mathrm{length}_{\mathcal{O}_{X,\eta}}(\mathcal{E}|_{X_\eta}^{\mathrm{tors}}) \cdot \overline{\{\eta\}}$$

where $X_\eta = \mathrm{Spec}\, \mathcal{O}_{X,\eta}$ and where $\mathcal{E}|_{X_\eta}^{\mathrm{tors}}$ is the torsion part of $\mathcal{E}|_{X_\eta}$. We refer to $T(\mathcal{E})$ as the *torsion divisor of* $\mathcal{E}$.

We denote by $\mathbb{Q}[\mathrm{Coh}(X)]$ the free $\mathbb{Q}$-vector space on the set of isomorphism classes of coherent sheaves on X. If $\mathrm{Weil}(X)_\mathbb{Q}$ is the space of $\mathbb{Q}$-Weil divisors on X, the map $T : \mathrm{Coh}(X) \to \mathrm{Weil}(X)_\mathbb{Q}$ induces a map of $\mathbb{Q}$-vector spaces

$$T : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathrm{Weil}(X)_\mathbb{Q} .$$

Example 4.2. If D is an effective Cartier divisor of X, then $T(\mathcal{O}_D) = D$.

Lemma 4.3. *Let X be a normal scheme and let $Z, Z' \subset X$ be effective Cartier divisors. Then*

$$T(\mathcal{O}_{Z+Z'}) = T(\mathcal{O}_Z) + T(\mathcal{O}_{Z'}) .$$

Proof. By definition, we have $\mathcal{I}_{Z+Z'} = \mathcal{I}_Z \cdot \mathcal{I}_{Z'}$. The conclusion thus follows by localizing at a codimension 1 point of X. $\square$

Lemma 4.4. *Let* X *be a normal noetherian scheme and let*

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

be an exact sequence in Coh(X)*. Then, we have*

$$T(\mathcal{E}) \leq T(\mathcal{E}') + T(\mathcal{E}'') .$$

Proof. Obvious from the fact that localization is exact and that the torsion functor is left-exact. $\square$

Recollection 4.5. Let us recall that a morphism $f : X \to S$ between schemes of finite type over k is *normal* if it is flat with geometrically normal fibres. By [GD67, Proposition 11.3.13], the source of a normal morphism is normal if the target is normal. By [GD65, Proposition 2.3.4], a generic point of X is sent to a generic point of S. In particular, if X is irreducible the pull-back of X along any dense open immersion $V \hookrightarrow S$ is irreducible as well.

Construction 4.6. Let $f : X \to S$ be a normal morphism between schemes of finite type over k. For $\mathcal{E} \in \mathrm{Coh}(X)$, we define a function $\chi_{\mathcal{E}} : S \to \mathbb{N}$ by

$$\chi_{\mathcal{E}} : S \to \mathbb{N}, \quad s \mapsto m(T(\mathcal{E}|_{X_{\bar{s}}})),$$

where $\bar{s} \to S$ is an algebraic geometric point above $s \in S$. We put

$$\mu_f(\mathcal{E}) := \sup \chi_{\mathcal{E}}(S) \in \mathbb{N} \cup \{\infty\} .$$

The goal of what follows is to prove the following

Proposition 4.7. *Let* $f : X \to S$ *be a normal morphism between schemes of finite type over* k*. For every* $\mathcal{E} \in \mathrm{Coh}(X)$*, there is constructible function* $\psi_{\mathcal{E}} : S \to \mathbb{N}$ *with* $\chi_{\mathcal{E}} \leq \psi_{\mathcal{E}}$*.*

Proposition 4.7 admits the following immediate

Corollary 4.8. *Let* $f : X \to S$ *be a normal morphism between schemes of finite type over* k*. For every* $\mathcal{E} \in \mathrm{Coh}(X)$*, the quantity* $\mu_f(\mathcal{E})$ *is finite.*

Proof of Proposition 4.7. Let $\mathcal{E} \in \mathrm{Coh}(X)$. We run the following dévissage :

(i) Since S is noetherian, it is enough to show that there exists a dense open subset $V \subset S$ and an integer n such that $\chi_{\mathcal{E}}(s) \leq n$ for every $s \in V$.

(ii) At the cost of replacing S by S^{red} we may assume that S is reduced.

(iii) By (i) and (ii), we are reduced to the case where $S = \mathrm{Spec}(A)$ is affine, connected and smooth over k.

(iv) Let $\{U_i\}_{1\leq i\leq m}$ be a finite Zariski open cover of X. Observe that

$$\chi_{\mathcal{E}} = \max_{1\leq i\leq m} \chi_{\mathcal{E}|_{U_i}} .$$

Hence, we are left to show that each $\chi_{\mathcal{E}|_{U_i}}$ is bounded on some dense open subset $V_i \subset S$. Since X is noetherian, we can thus assume that X is affine. By Recollection 4.5, we deduce that X is normal affine. By [SP, 0357], the scheme X is a finite disjoint union of normal affine integral schemes. Hence, at the cost of replacing X by its irreducible components, we can assume that X is integral normal affine. By Recollection 4.5, its pullback stays so after replacing S by an affine dense open subset.

(v) By [SP, 00L0], the sheaf $\mathcal{E}$ admits a finite filtration by coherent subsheaves

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{n-1} \subset \mathcal{E}_n = \mathcal{E}$$

such that for every $i = 0, \dots, n-1$, we have $\mathcal{E}_{i+1}/\mathcal{E}_i \simeq \mathcal{O}_{Z_i}$ for some closed irreducible subscheme $Z_i \subset X$. By generic flatness theorem [GD65, Théorème 6.9.1], we may assume each Z_i is flat over S. Then, for every algebraic geometric point $\overline{s} \to S$, the chain

$$0 = \mathcal{E}_0|_{X_{\overline{s}}} \subset \mathcal{E}_1|_{X_{\overline{s}}} \subset \cdots \subset \mathcal{E}_{n-1}|_{X_{\overline{s}}} \subset \mathcal{E}_n|_{X_{\overline{s}}} = \mathcal{E}|_{X_{\overline{s}}}$$

is a filtration of $\mathcal{E}|_{X_{\overline{s}}}$. By Lemma 4.4, we deduce that

$$\chi_{\mathcal{E}} \leq \sum_{1\leq i\leq n} \chi_{\mathcal{O}_{Z_i}} \,.$$

Hence, we can assume that $\mathcal{E} = \mathcal{O}_Z$ where $Z \subset X$ is a closed irreducible subscheme flat over S.

(vi) By [Vak25, Proposition 11.4.1] applied to $f : X \to S$ and $Z \to S$, at the cost of shrinking S, we can suppose that for every $s \in S$, the fibre X_s has pure dimension $\dim X - \dim S$ and Z_s has pure dimension $\dim Z - \dim S$. Hence, for every $s \in S$ we have

$$\dim Z_s = \dim Z - \dim S = \dim X_s + \dim Z - \dim X \,.$$

If $\dim Z < \dim X - 1$, we have

$$\dim Z_s < \dim X_s - 1$$

where X_s and Z_s have pure dimension. Hence $\chi_{\mathcal{O}_Z} = 0$ in this case. If $\dim Z = \dim X$, we have $X = Z$ so that $\mathcal{O}_Z = 0$ and $\chi_{\mathcal{O}_Z} = 0$. We are thus left to prove Proposition 4.7 for $\mathcal{E} = \mathcal{O}_Z$ where $Z \subset X$ is a closed subscheme of pure codimension 1 flat over S with S irreducible and smooth over k.

(vii) Let $U \subset X$ be a smooth neighbourhood of the generic points of Z. Then $U \cap Z$ is an effective Cartier divisor of U relative to S and U contains the generic points of the generic fibre of Z over S. By [SP, 0573] applied to the open subset $Z \cap U \subset Z$, there is a dense open subset $V \subset S$ such that for every $s \in V$, the open set $Z_s \cap U_s$ is dense in Z_s. In particular, U_s contains all the generic points of $Z_s \subset X_s$. We deduce

$$\chi_{\mathcal{O}_Z}(s) = \chi_{\mathcal{O}_{Z\cap U}}(s)$$

for every $s \in V$. We are thus left to prove Proposition 4.7 in the case where $\mathcal{E} = \mathcal{O}_Z$ where $Z \subset X$ is an effective Cartier divisor relative to S with S irreducible and smooth over k.

(viii) By [SP, 0550] applied to $f|_Z : Z \to S$, there is a commutative square

$$\begin{array}{ccc} Z' & \longrightarrow & Z \\ \downarrow & & \downarrow f|_Z \\ S' & \xrightarrow{g} & S \end{array}$$

where $g : S' \to S$ is open quasi-finite with S' integral affine, where Z'^{red} is flat over S' with geometrically reduced generic fibre. Observe that

$$\chi_{\mathcal{O}_{Z'}}(s') = \chi_{\mathcal{O}_Z}(g(s'))$$

for every $s' \in S'$. At the cost of replacing S by a dense open subset of S', we can assume that $Z \subset X$ is an effective Cartier divisor relative to S, that Z^{red} is flat over S with geometrically reduced generic fibre over S with S irreducible and smooth over k. By [GD66, Theorem 9.7.7] at the cost of shrinking S, we can further assume that Z^{red} has geometrically reduced fibres over S. By shrinking X as in step (vii), we can also assume that Z^{red} is an effective Cartier divisor relative to S. Let $Z_1, \dots, Z_n$ be the irreducible components of Z^{red} and write

$$Z = e_1 \cdot Z_1 + \cdots + e_n \cdot Z_n \, .$$

For every $s \in S$,Lemma 4.3 gives

$$T(\mathcal{O}_{Z_{\bar{s}}}) = e_1 \cdot T(\mathcal{O}_{Z_{1,\bar{s}}}) + \cdots + e_n \cdot T(\mathcal{O}_{Z_{n,\bar{s}}}) \, .$$

Since the $Z_{i,\bar{s}}$ are *reduced* effective Cartier divisors of $X_{\bar{s}}$, the multiplicities of the $T(\mathcal{O}_{Z_{i,\bar{s}}})$ are equal to 1. Thus,

$$\chi_{\mathcal{O}_Z}(s) \leq e_1 + \cdots + e_n \, .$$

□

Definition 4.9. Let $f : Y \to X$ be a morphism between normal noetherian schemes. We say that $D \in \mathrm{Weil}(X)_{\mathbb{Q}}$ *pull-backs along* $f : Y \to X$ if for every integral closed subscheme $Z \subset X$ contributing to D, the pull-back $Y \times_X Z$ has pure codimension 1 in Y.

For every $D \in \mathrm{Weil}(X)_{\mathbb{Q}}$ which pull-backs along $f : Y \to X$, there is a well defined pull-back $f^*D \in \mathrm{Weil}(Y)_{\mathbb{Q}}$. If every $D \in \mathrm{Weil}(X)_{\mathbb{Q}}$ pull-backs along $f : Y \to X$, there is a well-defined $\mathbb{Q}$-linear map

$$f^* : \mathrm{Weil}(X)_{\mathbb{Q}} \to \mathrm{Weil}(Y)_{\mathbb{Q}} \, .$$

We describe two cases of interest.

Example 4.10. Let X be a noetherian geometrically normal scheme over a field k and let K/k be a field extension. Then, every $D \in \mathrm{Weil}(X)_{\mathbb{Q}}$ pull-backs along $X_K \to X$.

Example 4.11. Let $f : Y \to X$ be a dominant morphism between regular noetherian schemes. Then, every Weil divisor on X is Cartier and similarly with Y. On the other hand, Cartier divisors on X pull-back to Y since $f : Y \to X$ is dominant. Hence, every $D \in \mathrm{Weil}(X)_{\mathbb{Q}}$ pull-backs along $f : Y \to X$.

Lemma 4.12. *Let* $f : Y \to X$ *be a morphism between normal noetherian schemes and let* $\mathcal{E} \in \mathrm{Coh}(X)$ *such that* $T(\mathcal{E})$ *pull-backs along* $f : Y \to X$*. Let* $\xi \in Y$ *be a codimension 1 point such that* $\eta = f(\xi) \in X$ *is a codimension 1 point. Then, the multiplicities of* $f^*T(\mathcal{E})$ *and* $T(f^*\mathcal{E})$ *at* ξ *are the same.*

Proof. Let π be a uniformizer of $\mathcal{O}_{X,\eta}$ and write

$$\mathcal{E}|_{X_\eta} \simeq \mathcal{O}_{X_\eta}^r \oplus \mathcal{O}_{X_\eta}/(\pi)^{n_1} \oplus \cdots \oplus \mathcal{O}_{X_\eta}/(\pi)^{n_k} .$$

Then, the multiplicity of $T(\mathcal{E})$ at η is $n_1 + \cdots + n_k$. If e is the valuation of $f^*\pi$ at ξ, the multiplicity $f^*T(\mathcal{E})$ at ξ is thus $(n_1 + \cdots + n_k) \cdot e$. If π' is a uniformizer of $\mathcal{O}_{Y,\xi}$ we have

$$(f^*\mathcal{E})|_{Y_\xi} \simeq \mathcal{O}_{Y_\xi}^r \oplus \mathcal{O}_{Y_\xi}/(\pi')^{e\cdot n_1} \oplus \cdots \oplus \mathcal{O}_{Y_\xi}/(\pi')^{e\cdot n_k} .$$

Hence, the multiplicity $T(f^*\mathcal{E})$ at ξ is $(n_1 + \cdots + n_k) \cdot e$. □

Observe that in the next lemma, no finiteness assumption is required.

Lemma 4.13. *Let* $f : Y \to X$ *be a flat morphism between normal noetherian schemes and let* $\mathcal{E} \in \mathrm{Coh}(X)$ *such that* $T(\mathcal{E})$ *pull-backs along* $f : Y \to X$*. Then,*

$$f^*T(\mathcal{E}) = T(f^*\mathcal{E}) .$$

Proof. We first show that $f^*T(\mathcal{E}) \leq T(f^*\mathcal{E})$. By Lemma 4.12, it is enough to show that for every codimension 1 point $\xi \in Y$ contributing to $f^*T(\mathcal{E})$, the image $\eta = f(\xi)$ is a codimension 1 point of X. Assume that ξ contributes to $f^*T(\mathcal{E})$. Then, ξ is a generic point of $f^*T(\mathcal{E})$. Since $f^*T(\mathcal{E}) \to T(\mathcal{E})$ is flat, we deduce by [GD65, Proposition 2.3.4] that η is a generic point of $T(\mathcal{E})$, and thus a codimension 1 point of X.

To conclude, it is enough to show that for every codimension 1 point $\xi \in Y$ contributing to $T(f^*\mathcal{E})$, the image $\eta = f(\xi)$ is a codimension 1 point of X. Assume that ξ contributes to $T(f^*\mathcal{E})$. The question is Zariski local around ξ and η, so we can suppose that X is affine. By [SP, Tag 00L0], the sheaf $\mathcal{E}$ admits a finite filtration by coherent subsheaves

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{n-1} \subset \mathcal{E}_n = \mathcal{E}$$

such that for every $i = 0, \dots, n-1$, we have $\mathcal{E}_{i+1}/\mathcal{E}_i \simeq \mathcal{O}_{Z_i}$ for some closed irreducible subscheme $Z_i \subset X$. Put $T_i := Y \times_X Z_i$ for $i = 0, \dots, n-1$. Since $f : Y \to X$ is flat, the chain

$$0 = f^*\mathcal{E}_0 \subset f^*\mathcal{E}_1 \subset \cdots \subset f^*\mathcal{E}_{n-1} \subset f^*\mathcal{E}_n = f^*\mathcal{E}$$

is a finite filtration of $f^*\mathcal{E}$ such that for every $i = 0, \dots, n-1$, we have $f^*\mathcal{E}_{i+1}/f^*\mathcal{E}_i \simeq \mathcal{O}_{T_i}$. Let $\mathfrak{M}$ be the maximal ideal of $\mathcal{O}_{Y_\xi}$. Observe that the ideal sheaves of the $Y_\xi \times_Y T_i$ are of the form $0, \mathcal{O}_{Y_\xi}$ or $\mathfrak{M}^n$ for some $n \geq 1$. If $0, \mathcal{O}_{Y_\xi}$ are the only ideal sheaves that occur, then $f^*\mathcal{E}$ is locally free in a neighbourhood of ξ, which contradicts the assumption that ξ contributes to $T(f^*\mathcal{E})$. Hence, there is $i \in \{0, \dots, n-1\}$ such that the ideal sheaf of $Y_\xi \times_Y T_i$ is $\mathfrak{M}^n$ for some $n \geq 1$. Then, at the cost of shrinking Y around ξ, we can suppose that $T_i^{\mathrm{red}} = \overline{\{\xi\}}$. In particular, T_i has codimension 1 and ξ is a generic point of T_i. Since $T_i \to Z_i$ is flat, we deduce that η is the generic point of Z_i. We are thus left to prove that Z_i has codimension 1 in X. To do this, we adapt the proof of [GD65, Corollaire 6.1.4]. By [GD65, Proposition 5.1.2], we have to show that

$$\dim \mathcal{O}_{X,\eta} = 1 .$$

Let $f_\xi : \mathrm{Spec}(\mathcal{O}_{Y,\xi}) \to \mathrm{Spec}(\mathcal{O}_{X,\eta})$ be the induced morphism. Since ξ is maximal in T_i, we have $\{\xi\} = f_\xi^{-1}(\eta)$. Hence, $\dim f_\xi^{-1}(\eta) = 0$. Since $\mathcal{O}_{Y,\xi}$ is a flat $\mathcal{O}_{X,\eta}$-module, [GD65, Corollaire 6.1.3] gives

$$\dim \mathcal{O}_{X,\eta} = \dim \mathcal{O}_{Y,\xi} = 1$$

and the conclusion follows. □

Corollary 4.14. *Let X be a geometrically normal scheme of finite type over k and let $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$. Then, $T(\mathcal{E})_{\overline{k}} = T(\mathcal{E}_{\overline{k}})$ where the subscript refers to the pullback along $X_{\overline{k}} \to X$.*

Proof. The statement of Corollary 4.14 makes sense by Example 4.10. We can suppose that $\mathcal{E} \in \mathrm{Coh}(X)$. Since X is normal, we can choose a dense open subset $U \subset X$ smooth over k such that $Z := X - U$ has codimension ≥ 2 in X. In particular, $Z_{\overline{k}}$ has codimension ≥ 2 in $X_{\overline{k}}$. Hence, it is enough to prove that the restriction of $T(\mathcal{E})_{\overline{k}}$ and $T(\mathcal{E}_{\overline{k}})$ to $U_{\overline{k}}$ are equal. Thus, at the cost of replacing X by U, we can assume that X is smooth over k. In that case, $T(\mathcal{E})$ is automatically an effective Cartier divisor. Then, Corollary 4.14 follows from Lemma 4.13. □

Corollary 4.15. *Let $f : Y \to X$ be a flat morphism between regular schemes. For every $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$, we have*

$$f^*T(\mathcal{E}) = T(f^*\mathcal{E}) .$$

Proof. The statement of Corollary 4.15 makes sense by Example 4.11. We can reduce to the case where $\mathcal{E} \in \mathrm{Coh}(X)$. This cases follows from Lemma 4.13. □

5. Bounding log conductors with coherent sheaves

Definition 5.1. Let X be a scheme of finite type over k. Let $\mathcal{K} \in D^b_{ctf}(X, \Lambda)$ and $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$. We say that $\mathcal{K}$ *has log conductors bounded by* $\mathcal{E}$ if for every morphism $f : C \to X$ over k where C is a smooth curve over k, we have

$$LC(\mathcal{H}^i\mathcal{K}|_C) \leq T(f^*\mathcal{E})$$

for every $i \in \mathbb{Z}$. We denote by $D^b_{ctf}(X, \mathcal{E}, \Lambda)$ the full subcategory of $D^b_{ctf}(X, \Lambda)$ spanned by objects having log conductors bounded by $\mathcal{E}$.

Remark 5.2. If $Z \subset X$ is closed subscheme, we will note $D^b_{ctf}(X, Z, \Lambda)$ instead of $D^b_{ctf}(X, \mathcal{O}_Z, \Lambda)$ and say that $\mathcal{K}$ has log conductors bounded by Z when $\mathcal{K}$ lies in $D^b_{ctf}(X, Z, \Lambda)$.

Lemma 5.3. *Let X be a scheme of finite type over k, let $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$ and let $\Lambda \to \Lambda'$ be a morphism between finite local rings of characteristic $\ell \neq p$. For every $\mathcal{K} \in D^b_{ctf}(X, \mathcal{E}, \Lambda)$, we have $\mathcal{K} \otimes^L_\Lambda \Lambda' \in D^b_{ctf}(X, \mathcal{E}, \Lambda')$.*

Proof. By [SP, 03TT] every $\mathcal{K} \in D^b_{ctf}(X, \Lambda)$ can be represented by a bounded complex of constructible sheaves of flat Λ-modules. In particular, the germ of $\mathcal{K}$ at any point can be represented by a bounded finite tor-dimension complex of finitely generated Λ-modules. Then, Lemma 5.3 follows from Corollary 2.9. □

Recollection 5.4. Let X be a scheme of finite type over k. Then, for every $\mathcal{K} \in D^b_c(X, \overline{\mathbb{Q}}_\ell)$, there is a finite extension $L/\mathbb{Q}_\ell$ and an integral representative $\mathcal{K}_\bullet = (\mathcal{K}_m)_{m\geq 0}$ for $\mathcal{K}$. If we put $\Lambda_m := \mathcal{O}_L/\mathfrak{m}_L^m$, the sheaf $\mathcal{K}_m$ is an object of $D^b_{ctf}(X, \Lambda_m)$ such that $\Lambda_m \otimes^L_{\Lambda_{m+1}} \mathcal{K}_{m+1} \simeq \mathcal{K}_m$.

The following definition upgrades Definition 5.1 to $\overline{\mathbb{Q}}_\ell$-coefficients.

Definition 5.5. Let X be a scheme of finite type over k. Let $\mathcal{K} \in D^b_c(X, \overline{\mathbb{Q}}_\ell)$ and $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$. We say that $\mathcal{K}$ *has log conductors bounded by* $\mathcal{E}$ if there is a finite extension $L/\mathbb{Q}_\ell$ and an integral representative $\mathcal{K}_\bullet = (\mathcal{K}_m)_{m\geq 0}$ for $\mathcal{K}$ such that for every $m \geq 0$, we have $\mathcal{K}_m \in D^b_{cft}(X, \mathcal{E}, \Lambda_m)$. We denote by $D^b_c(X, \mathcal{E}, \overline{\mathbb{Q}}_\ell)$ the full subcategory of $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ spanned by objects having log conductors bounded by $\mathcal{E}$.

Lemma 5.6. *Let X be a scheme of finite type over k. Let $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$ and $\mathcal{K} \in D^b_{ctf}(X, \Lambda)$.*

(1) *Assume that $\mathcal{K} \in D^b_{ctf}(X, \mathcal{E}, \Lambda)$ and let $f : Y \to X$ be a morphism of schemes of finite type over k. Then $f^*\mathcal{K} \in D^b_{ctf}(Y, f^*\mathcal{E}, \Lambda)$.*

(2) *Assume that $\mathcal{K} \in D^b_{ctf}(X, \mathcal{E}, \Lambda)$. Then for every $\mathcal{E}' \in \mathrm{Coh}(X)$, we have $\mathcal{K} \in D^b_{ctf}(X, \mathcal{E} \oplus \mathcal{E}', \Lambda)$.*

(3) *Let $(f_i : U_i \to X)_{i\in I}$ be an étale cover X. Then $\mathcal{K} \in D^b_{ctf}(X, \mathcal{E}, \Lambda)$ if and only if for every $i \in I$, we have $\mathcal{K}|_{U_i} \in D^b_{ctf}(U_i, f_i^*\mathcal{E}, \Lambda)$.*

(4) *Consider a distinguished triangle*

$$\mathcal{K}_1 \to \mathcal{K}_2 \to \mathcal{K}_3 \to$$

in $D^b_{ctf}(X, \Lambda)$. If 2 out of the 3 above complexes lie in $D^b_{ctf}(X, \mathcal{E}, \Lambda)$, so does the third.

(5) *If the above triangle is an exact sequence of constructible sheaves, then $\mathcal{K}_2$ lies in $D^b_{ctf}(X, \mathcal{E}, \Lambda)$ if and only if so do $\mathcal{K}_1$ and $\mathcal{K}_3$.*

(6) *If $\mathcal{K}_1, \mathcal{K}_2 \in D^b_{ctf}(X, \Lambda)$ are isomorphic away from a finite set of closed points of X, then $\mathcal{K}_1$ has log conductors bounded by $\mathcal{E}$ if and only if so does $\mathcal{K}_2$.*

(7) *Let $(U_i)_{i\in I}$ be a finite Zariski cover of X and let $\mathcal{E}_i \in \mathrm{Coh}(U_i)$, $i \in I$. Then, there exists $\mathcal{E}' \in \mathrm{Coh}(X)$ depending only on the $\mathcal{E}_i$ such that*

$$\mathcal{K}|_{U_i} \in D^b_{ctf}(U_i, \mathcal{E}_i, \Lambda) \text{ for every } i \in I \Rightarrow \mathcal{K} \in D^b_{ctf}(X, \mathcal{E}', \Lambda)\ .$$

(8) *Let $\mathrm{red} : X^{\mathrm{red}} \to X$ be the reduction morphism. Then, $\mathcal{K}$ lies in $D^b_{ctf}(X, \mathcal{E}, \Lambda)$ if and only if $\mathrm{red}^*\mathcal{K}$ lies in $D^b_{ctf}(X, \mathrm{red}^*\mathcal{E}, \Lambda)$.*

(9) *For every closed immersion $i : X \hookrightarrow Y$, we have $\mathcal{K} \in D^b_{ctf}(X, \mathcal{E}, \Lambda)$ if and only if $i_*\mathcal{K} \in D^b_{ctf}(Y, i_*\mathcal{E}, \Lambda)$.*

(10) *Let $j : U \hookrightarrow X$ be an open subset and $i : Z \hookrightarrow X$ its complement. Assume that $\mathcal{E} \in \mathrm{Coh}(X)$, that $j_!(\mathcal{K}|_U) \in D^b_{ctf}(X, \mathcal{E}, \Lambda)$ and $\mathcal{K}|_Z \in D^b_{ctf}(Z, \mathcal{E}_Z, \Lambda)$ for some $\mathcal{E}_Z \in \mathrm{Coh}(Z)$. Then,*

$$\mathcal{K} \in D^b_{ctf}(X, \mathcal{E} \oplus i_*\mathcal{E}_Z, \Lambda)\ .$$

Proof. Items (1)(2)(3)(6) are obvious. Items (4)(5) follow from Lemma 2.4. For (7), we know by [SP, 0G41] that each $\mathcal{E}_i$ extends to a coherent sheaf $\mathcal{E}_i'$ on X. Then, the direct sum of the $\mathcal{E}_i'$ does the job in virtue of (2),(3). Item (8) follows from (1) and the fact that a smooth curve mapping to X canonically factors through X^{red}. The converse implication in (9) follows from (1) and the isomorphism $i^*i_*\mathcal{E} \simeq \mathcal{E}$. For the direct implication, consider the cartesian square of schemes of finite type over k

$$\begin{array}{ccc} X_C & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle i} \\ C & \xrightarrow{f} & Y \end{array}$$

where C is a smooth connected curve over k. We have

$$(i_*\mathcal{F})|_C \simeq i'_*(\mathcal{F}|_{X_C}) .$$

If X_C has dimension 0, the sheaf $(i_*\mathcal{F})|_C$ is generically 0 and there is nothing to prove. Otherwise, the composition $X_C^{\mathrm{red}} \to X_C \to C$ is an isomorphism, and the conclusion follows. Item (11) follows immediately by (2),(4),(9) via the localization triangle. □

The following is our main supply of sheaf with explicit bound on the log conductors.

Proposition 5.7. *Let X be a normal scheme of finite type over k. Let D be an effective Cartier divisor of X and put $j : U := X - D \hookrightarrow X$. Let $\mathcal{L} \in \mathrm{Loc}_{tf}(U, \Lambda)$ and $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$.*

(1) If $j_!\mathcal{L} \in D^b_{ctf}(X, \mathcal{E}, \Lambda)$, then $\mathrm{LC}(j_!\mathcal{L}) \leq T(\mathcal{E})$.

(2) If X is smooth, $j_!\mathcal{L}$ has log conductors bounded by $C(j_!\mathcal{L})$.

(3) If X is smooth and if D has simple normal crossings, $j_!\mathcal{L}$ has log conductors bounded by $\mathrm{LC}(j_!\mathcal{L})$.

Proof. Item (1) is a local question around the generic points of D. Hence, we can suppose that X and D are smooth connected. We argue for $\mathcal{E} \in \mathrm{Coh}(X)$, the general case being similar by linear combinations. At the cost of shrinking X further, we can suppose that $\mathcal{E}$ is of the form

$$\mathcal{O}_X^r \oplus \mathcal{O}_X/\mathcal{I}_D^{n_1} \oplus \cdots \oplus \mathcal{O}_X/\mathcal{I}_D^{n_k}$$

where $n_i \in \mathbb{N}^*$. In particular $T(\mathcal{E}) = (n_1 + \cdots + n_k) \cdot D$. Let $f : C \to X$ be an immersion over k where C is a smooth connected curve over k such that the generic point of C is sent in U and that $f^{-1}(C)$ is a single point x. Then,

$$f^*\mathcal{E} \simeq \mathcal{O}_C^r \oplus \mathcal{O}_C/\mathcal{I}_x^{m_x(f^*D)\cdot n_1} \oplus \cdots \oplus \mathcal{O}_C/\mathcal{I}_x^{m_x(f^*D)\cdot n_k} .$$

In particular,

$$\mathrm{lc}_x((j_!\mathcal{L})|_C) \leq m_x(f^*D) \cdot m_D(T(\mathcal{E})) .$$

Then (1) follows from Theorem 3.7. Item (2) follows from Theorem 3.5 and Remark 3.3. Item (3) follows from Theorem 3.5. □

Proposition 5.8. *Let X be a scheme of finite type over k. For every $\mathcal{K} \in D^b_{ctf}(X, \Lambda)$, there exists $\mathcal{E} \in \mathrm{Coh}(X)$ such that $\mathcal{K}$ has log conductors bounded by $\mathcal{E}$.*

Proof. If X has dimension 0, there is nothing to do. Let $n \geq 1$, and assume that Proposition 5.8 holds in dimension $\leq n-1$. Let X be a scheme of finite type over k of dimension $\leq n$ and let $\mathcal{K} \in D^b_{ctf}(X,\Lambda)$. To show that Proposition 5.8 holds for $\mathcal{K}$, we can suppose by Lemma 5.6-4 that $\mathcal{K}$ is concentrated in degree 0. By using a finite affine cover of X, we can assume by Lemma 5.6-7 that X is affine. By Noether normalization, there is a finite morphism $f : X \to \mathbb{A}^m_k$ where $m \leq n$. In particular, the counit map $f^*f_*\mathcal{K} \to \mathcal{K}$ is surjective. By Lemma 5.6-5, we are thus left to prove Proposition 5.8 for $f^*f_*\mathcal{K}$. By Lemma 5.6-1, we are left to prove Proposition 5.8 for $f_*\mathcal{K}$. Hence, we can suppose that X is smooth of dimension $\leq n$. Let $Z \subset X$ be an effective Cartier divisor containing the singular locus of $\mathcal{F}$ and put $j : U := X - Z \hookrightarrow X$. By recursion assumption, $\mathcal{K}|_Z \in D^b_{ctf}(Z, \mathcal{E}_Z, \Lambda)$ for some $\mathcal{E}_Z \in \mathrm{Coh}(Z)$. By Lemma 5.6-10, we can thus further suppose that $\mathcal{K} = j_!\mathcal{L}$ where $\mathcal{L} \in \mathrm{Loc}_{tf}(U,\Lambda)$. In that case, $j_!\mathcal{L}$ has log conductors bounded by $C(j_!\mathcal{L})$ in virtue of Proposition 5.7. □

Proposition 5.9. *Let $f : X \to S$ be a morphism between schemes of finite type over k. Let $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$ and $\mathcal{K} \in D^b_{ctf}(X, \mathcal{E}, \Lambda)$. For every algebraic geometric point $\bar{s} \to S$, the complex $\mathcal{K}|_{X_{\bar{s}}}$ has log conductors bounded by $i^*_{\bar{s}}\mathcal{E}$ where $i_{\bar{s}} : X_{\bar{s}} \to X$ is the canonical morphism.*

Proof. We can suppose that $\mathcal{K}$ is concentrated in degree 0. Let $s \in S$ be the point over which $\bar{s}$ is localized and let S' be a smooth connected open subset of $\overline{\{s\}} \subset S$. Consider the commutative diagram with cartesian squares

$$\begin{array}{ccccc} X_{\bar{s}} & \xrightarrow{i'_{\bar{s}}} & X' & \xrightarrow{h} & X \\ \downarrow & & \downarrow f' & & \downarrow f \\ \bar{s} & \longrightarrow & S' & \longrightarrow & S\,. \end{array}$$

By Lemma 5.6-1, the sheaf $\mathcal{K}|_{X'}$ has log conductors bounded by $h^*\mathcal{E}$. Hence, at the cost of replacing $f : X \to S$ by $f' : X' \to S'$ and $\mathcal{K}$ by $\mathcal{K}|_{X'}$, we can suppose that S is smooth connected and that $\bar{s}$ is localized at the generic point of S. Let C be a smooth curve over $\bar{s}$ and let $h : C \to X_{\bar{s}}$ be a morphism over $\bar{s}$. We want to show that

$$LC(\mathcal{K}|_C) \leq T(h^* i^*_{\bar{s}}\mathcal{E})\,.$$

By writing $\bar{s}$ as a filtered limit of smooth connected varieties over k quasi-finite flat over S, there is a spreading out

$$\begin{array}{ccccc} C & \xrightarrow{\iota_{\bar{s}}} & \mathcal{C} & & \\ \downarrow h & & \downarrow & \searrow^{\alpha} & \\ X_{\bar{s}} & \longrightarrow & X_T & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \bar{s} & \longrightarrow & T & \longrightarrow & S \end{array}$$

with cartesian squares where $T \to S$ is quasi-finite flat with T smooth connected over k, and where $\mathcal{C} \to T$ is a smooth relative curve. Since $\mathcal{K}|_{\mathcal{C}}$ has log conductors bounded by $\alpha^*\mathcal{E}$, it is enough to show that if $f : X \to S$ is a smooth relative curve with S smooth connected over k and if $\bar{s} \to S$ is localized at the generic point of S, we have

$$\mathrm{LC}(\mathcal{K}|_{X_{\bar{s}}}) \leq \mathrm{T}(i_{\bar{s}}^{*}\mathcal{E}) .$$

Observe that X is smooth over k. Choose a reduced effective Cartier divisor $D \subset X$ containing the singular locus of $\mathcal{K}$. At the cost of shrinking S, we can suppose that D is flat over S. Put $U := X - D$ and let $j : U \hookrightarrow X$ be the inclusion. Note that $(j_!(\mathcal{K}|_U))|_{X_{\bar{s}}}$ and $\mathcal{K}|_{X_{\bar{s}}}$ are isomorphic away from a finite number of points of the smooth curve $X_{\bar{s}}$. Hence, at the cost of replacing $\mathcal{K}$ by $j_!(\mathcal{K}|_U)$, we can suppose that $\mathcal{K}$ is of the form $j_!\mathcal{L}$ where $\mathcal{L} \in \mathrm{Loc}_{tf}(U, \Lambda)$. By [SP, 0550] there is a commutative diagram with cartesian squares

$$\begin{array}{ccccc} X_{\bar{s}} & \longrightarrow & X_T & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \bar{s} & \longrightarrow & T & \longrightarrow & S \end{array}$$

where $T \to S$ is quasi-finite flat with T smooth connected over k and where the generic fibre of $(T \times_S D)^{\mathrm{red}} \to T$ is geometrically reduced. At the cost of pulling back the situation to T and replacing D by $(T \times_S D)^{\mathrm{red}}$, we can assume that $D_{\bar{s}}$ is reduced. Since $D_{\bar{s}}$ is a finite set of points over $\bar{s}$, we deduce that $D_{\bar{s}}$ is étale over $\bar{s}$. At the cost of shrinking S we can thus suppose that the induced map $D \to S$ is étale. Hence, we deduce

$$\mathrm{LC}((j_!\mathcal{L})|_{X_{\bar{s}}}) \leq i_{\bar{s}}^{*}\mathrm{LC}(j_!\mathcal{L}) \leq i_{\bar{s}}^{*}\mathrm{T}(\mathcal{E}) = \mathrm{T}(i_{\bar{s}}^{*}\mathcal{E})$$

where the first inequality follows from Proposition 3.8 and the second inequality follows from Proposition 5.7. Since $\bar{s}$ is localized at the generic point of S, the map $i_{\bar{s}} : X_{\bar{s}} \to X$ is flat. On the other hand, X and $X_{\bar{s}}$ are regular. Thus the last equality follows from Corollary 4.15. □

6. Conductor and finite direct image: global case

Lemma 6.1. *Let $f : Y \to X$ be a finite surjective morphism between smooth curves over k. Let D be a divisor on X and let $x \in D$. Put $E = f^{-1}(D)$ and $j : V := Y - E \hookrightarrow Y$. Then, for every $\mathcal{L} \in \mathrm{Loc}_{tf}(V, \Lambda)$, we have*

$$\mathrm{lc}_x(f_*j_!\mathcal{L}) \leq \max\{\mathrm{lc}_x(f_*j_!\Lambda), \mathrm{lc}_y(j_!\mathcal{L}), y \in f^{-1}(x)\} .$$

Proof. This is a geometric rephrasing of Corollary 2.20. □

Definition 6.2. Let $f : Y \to X$ be a finite surjective morphism of normal schemes of finite type over k. Let D be an irreducible divisor on X with generic point η and E an irreducible component of $(D \times_X Y)^{\mathrm{red}}$ with generic point ξ. We denote by $f(E/D)$ the degree of $k(\xi)/k(\eta)$, by $e(E/D)$ the ramification index of the extension of discrete valuation rings $\mathcal{O}_{Y,\xi}/\mathcal{O}_{X,\eta}$, by $f(E/D)^s$ the separable degree of $k(\xi)/k(\eta)$ and by $f(E/D)^{\mathrm{ins}}$ its purely inseparable degree.

Lemma 6.3. *Let $f : Y \to X$ be a finite surjective morphism of smooth schemes over k. Let D be a smooth irreducible effective Cartier divisor on X and put $U := X - D$. Assume that $E := (D \times_X Y)^{\mathrm{red}}$ is smooth irreducible and put $V := Y - E$. Let $\iota : C \to X$ be an immersion* *from a smooth connected curve over* k *meeting* D *at a unique closed point* x*. Let* $\iota' : C' \to Y$ *be the normalization of* $C \times_X Y$*. Then, for every closed point* $x' \in C'$ *lying over* x*, we have*

$$m_{x'}(\iota'^* E) \leq (D, C)_x \cdot f(E/D) \, .$$

Proof. At the cost of throwing away the components of Y not containing E, we can suppose that Y is connected. Let η be the generic point of D and let ξ be the generic point of E. By Zariski's main theorem, the normalization of $\mathcal{O}_{X,\eta}$ in $K(Y)$ is $\mathcal{O}_{Y,\xi}$. In particular, [Ser68, Proposition I.10] gives

$$[K(Y) : K(X)] = f(E/D) \cdot e(E/D) \, .$$

Let $S \subset C'$ be the irreducible component containing x'. Then,

$$e(x'/x) \leq [K(S) : K(C)] \leq [K(Y) : K(X)]$$

where the first inequality comes from [Ser68, Proposition I.10] and the second from the fact that $V \to U$ is finite *flat* as finite surjective morphism with smooth base and target. Consider the following commutative diagram

$$\begin{array}{ccccc} E & \longrightarrow & Y & \xleftarrow{\iota'} & C' \\ \downarrow & & \downarrow f & & \downarrow f_C \\ D & \longrightarrow & X & \xleftarrow[\iota]{} & C \, . \end{array}$$

Then,

$$\begin{aligned} m_{x'}(\iota'^* E) &= \frac{1}{e(E/D)} \cdot m_{x'}\left(\iota'^* f^* D\right) = \frac{1}{e(E/D)} \cdot m_{x'}\left(f_C^* \iota^* D\right) = \frac{(D,C)_x}{e(E/D)} \cdot m_{x'}\left(f_C^* x\right) \\ &= (D,C)_x \cdot \frac{e(x'/x)}{e(E/D)} \leq (D,C)_x \cdot \frac{[K(Y):K(X)]}{e(E/D)} = (D,C)_x \cdot f(E/D). \end{aligned}$$

□

Lemma 6.4. *Let* $f : Y \to X$ *be a finite surjective morphism of smooth schemes over* k*. Let* D *be a smooth effective Cartier irreducible divisor on* X*. Assume that* $E := (D \times_X Y)^{\mathrm{red}}$ *is smooth irreducible. Put* $U := X - D$ *and* $j : V := Y - E \hookrightarrow Y$*. Assume that the restriction* $f_0 : V \to U$ *is étale. For* $\mathcal{L} \in \mathrm{Loc}_{tf}(V, \Lambda)$*, we have*

$$\mathrm{lc}_D(f_* j_! \mathcal{L}) \leq \max\{\mathrm{lc}_D(f_* j_! \Lambda), f(E/D) \cdot \mathrm{lc}_E(j_! \mathcal{L})\}.$$

Proof. Let $\iota : C \to X$ be an immersion from a smooth curve over k meeting D at only one point x. Put $C_0 := C - \{x\}$ and let C' be the normalization of $C \times_X Y$. Put $C'_0 := C' - (x \times_C C')$. We have the following commutative diagrams

$$\begin{array}{ccccc} E & \longrightarrow & Y & \xleftarrow{\iota'} & C' \\ f_D \downarrow & & f \downarrow & & \downarrow f_C \\ D & \longrightarrow & X & \xleftarrow[\iota]{} & C \end{array} \qquad \begin{array}{ccc} V & \xleftarrow{\iota'_0} & C'_0 \\ f_0 \downarrow & & \downarrow f_{C_0} \\ U & \xleftarrow[\iota_0]{} & C_0 \end{array}$$

Since $f_0 : V \to U$ is étale and since smoothness descends along étale morphisms, the scheme $V \times_U C_0$ is smooth over k. Hence, the right square above is cartesian. By proper base change, we deduce

$$\iota_0^* f_{0*}(\mathcal{L}) \simeq f_{C_0 *}\iota_0'^*(\mathcal{L}) .$$

Hence, we deduce that

$$\iota^* f_*(j_!\mathcal{L}) \simeq f_{C*}\iota'^*(j_!\mathcal{L}) .$$

On the other hand,

$$\begin{aligned}
\mathrm{lc}_x(f_{C*}\iota'^*(j_!\mathcal{L})) &\leq \max\left\{\mathrm{lc}_x(f_{C*}\iota'^*(j_!\Lambda)), \mathrm{lc}_{x'}(\iota'^* j_!\mathcal{L}), x' \in f_C^{-1}(x)\right\} && \text{Lemma 6.1} \\
&\leq \max\left\{\mathrm{lc}_x(f_{C*}\iota'^*(j_!\Lambda)), m_{x'}(\iota'^* E)\cdot \mathrm{lc}_E(j_!\mathcal{L}), x' \in f_C^{-1}(x)\right\} && \text{Theorem 3.5} \\
&\leq \max\{\mathrm{lc}_x(\iota^* f_*(j_!\Lambda)), (D, C)_x \cdot f(E/D)\cdot \mathrm{lc}_E(j_!\mathcal{L})\} && \text{Lemma 6.3}
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{\mathrm{lc}_x(\iota^* f_*(j_!\mathcal{L}))}{(D, C)_x} &\leq \max\left\{\frac{\mathrm{lc}_x(\iota^* f_*(j_!\Lambda))}{(D, C)_x}, f(E/D)\cdot \mathrm{lc}_E(j_!\mathcal{L})\right\} \\
&\leq \max\{\mathrm{lc}_D(f_*(j_!\Lambda)), f(E/D)\cdot \mathrm{lc}_E(j_!\mathcal{L})\} && \text{Theorem 3.5}
\end{aligned}$$

Applying Theorem 3.7 concludes the proof of Lemma 6.4. □

Lemma 6.5. *Let $f : Y \to X$ be a finite surjective morphism of smooth schemes over k. Let D be a smooth irreducible effective Cartier divisor on X and put $U := X - D$. Assume that $E := (D \times_X Y)^{\mathrm{red}}$ is smooth irreducible and put $j : V := Y - E \hookrightarrow Y$. Assume that the restriction $f_0 : V \to U$ is étale. For every $\mathcal{L} \in \mathrm{Loc}_{tf}(V, \Lambda)$, we have*

$$c_D(f_* j_!\mathcal{L}) \leq \max\{c_D(f_* j_!\Lambda), f(E/D)\cdot c_E(j_!\mathcal{L})\} .$$

Proof. By Proposition 3.6 applied to $f_* j_!\mathcal{L}$ and $f_* j_!\Lambda$, there is an immersion $\iota : C \hookrightarrow X$ from a smooth curve over k and a point $x \in C \cap D$ such that

$$c_D(f_* j_!\mathcal{L}) = c_x(\iota^* f_* j_!\mathcal{L}) \text{ and } c_D(f_* j_!\Lambda) = c_x(\iota^* f_* j_!\Lambda) .$$

Let C' be the normalization of $C \times_X Y$. We have the following commutative diagrams

$$\begin{array}{ccccc}
E & \longrightarrow & Y & \xleftarrow{\ \iota'\ } & C' \\
{\scriptstyle f_D}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_C} \\
D & \longrightarrow & X & \xleftarrow[\ \iota\ ]{} & C
\end{array}$$

As in the proof of Lemma 6.4, we have

$$\iota^* f_*(j_!\mathcal{L}) \simeq f_{C*}\iota'^*(j_!\mathcal{L}) .$$

Hence, we deduce

$$c_D(f_* j_!\mathcal{L}) = c_x(f_{C*}\iota'^*(j_!\mathcal{L})) \text{ and } c_D(f_* j_!\Lambda) = c_x(f_{C*}\iota'^*(j_!\Lambda)) .$$

On the other hands, we have

$$
\begin{aligned}
c_x(f_{C*}\iota'^*(j_!\mathcal{L})) &= lc_x(f_{C*}\iota'^*(j_!\mathcal{L}))+1 && \text{Lemma 2.6}\\
&\leq \max\left\{lc_x(f_{C*}\iota'^*(j_!\Lambda)), lc_{x'}(\iota'^*(j_!\mathcal{L})), x'\in f_C^{-1}(x)\right\}+1 && \text{Lemma 6.1}\\
&\leq \max\left\{c_x(f_{C*}\iota'^*(j_!\Lambda)), c_{x'}(\iota'^*(j_!\mathcal{L})), x'\in f_C^{-1}(x)\right\} && \text{Lemma 2.6}\\
&\leq \max\left\{c_x(f_{C*}\iota'^*(j_!\Lambda)), m_{x'}(\iota'^*E)\cdot c_E(j_!\mathcal{L}), x'\in f_C^{-1}(x)\right\} && \text{Theorem 3.5}\\
&\leq \max\left\{c_x(f_{C*}\iota'^*(j_!\Lambda)), f(E/D)\cdot c_E(j_!\mathcal{L})\right\} && \text{Lemma 6.3}
\end{aligned}
$$

Lemma 6.5 thus follows.

□

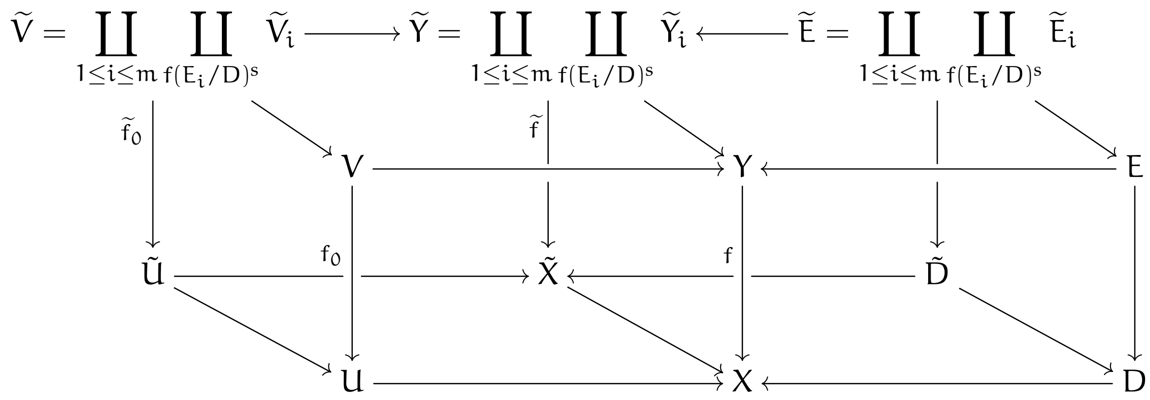
Proposition 6.6. *Let $f : Y \to X$ be a finite surjective morphism of normal schemes of finite type over k. Let D be an irreducible effective Cartier divisor on X and put $U := X - D$. Define $E := (D \times_X Y)^{\mathrm{red}}$ and put $j : V := Y - E \hookrightarrow Y$. Let $\{E_i\}_{1\leq i\leq m}$ be the irreducible components of E. Assume that the restriction $f_0 : V \to U$ is étale. For every $\mathcal{L} \in \mathrm{Loc}_{tf}(V,\Lambda)$, we have*

$$c_D(f_*j_!\mathcal{L}) \leq \max_{1\leq i\leq m}\{c_D(f_*j_!\Lambda), f(E_i/D)^{\mathrm{ins}}\cdot c_{E_i}(j_!\mathcal{L})\}$$

and

$$lc_D(f_*j_!\mathcal{L}) \leq \max_{1\leq i\leq m}\{lc_D(f_*j_!\Lambda), f(E_i/D)^{\mathrm{ins}}\cdot lc_{E_i}(j_!\mathcal{L})\}.$$

Proof. To lighten the notations, we will omit the lower shrieks in what follows. Let η be the generic point of D and $\overline{\eta} \to X$ an algebraic geometric point above η. Let ξ_i be the generic point of E_i $(1 \leq i \leq m)$, and $\overline{\xi}_i \to Y$ an algebraic geometric point above ξ_i such that the composition $\overline{\xi}_i \to \xi_i \to \eta$ factors through $\overline{\eta} \to \eta$. By pulling-back above the strict henselianization of X at $\overline{\eta}$ and then spreading out, there exists a commutative diagram



where every square but the right front and back squares are cartesian, and where $\widetilde{X}$ is an affine smooth connected étale neighborhood of $\overline{\eta} \to X$ such that $\widetilde{D}$ is irreducible and smooth, $\widetilde{Y}_i$ are affine smooth connected étale neighborhood of $\overline{\xi}_i \to Y$ with $\widetilde{E}_i = \widetilde{Y}_i \times_Y E_i$ irreducible and smooth and $\widetilde{E}_i \to \widetilde{D}$ finite, surjective and radiciel of degree $f(E_i/D)^{\mathrm{ins}}$ for $1 \leq i \leq m$. Since $\widetilde{X} \to X$ is étale, the proper base change yields

$$\begin{aligned} c_D(f_*\Lambda) &= c_{\widetilde{D}}(\widetilde{f}_*\Lambda) = \max_{1\leq i\leq m}\{c_{\widetilde{D}}(\widetilde{f}_{i*}\Lambda)\}; \\ c_D(f_*\mathcal{L}) &= c_{\widetilde{D}}(\widetilde{f}_*(\mathcal{L}|_{\widetilde{V}})) = \max_{1\leq i\leq m}\{c_{\widetilde{D}}(\widetilde{f}_{,i*}(\mathcal{L}|_{\widetilde{V}_i}))\}; \\ c_{E_i}(\mathcal{L}) &= c_{\widetilde{E}_i}(\mathcal{L}|_{\widetilde{V}_i}); \end{aligned}$$

and similarly with the logarithmic conductors, where $\widetilde{f}_i : \widetilde{Y}_i \to \widetilde{X}$ is induced by $f : Y \to X$. Hence, it is enough to prove

$$c_{\widetilde{D}}(\widetilde{f}_{i*}(\mathcal{L}|_{\widetilde{V}_i})) \leq \max\{c_{\widetilde{D}}(\widetilde{f}_{i*}\Lambda), f(\widetilde{E}_i/\widetilde{D}) \cdot c_{\widetilde{E}_i}(\mathcal{L}|_{\widetilde{V}_i})\}$$

and similarly with the logarithmic conductors. Thus, we are left to prove Proposition 6.6 in the case where $D \subset X$ and $E = (D \times_X Y)_{\mathrm{red}} \subset Y$ are irreducible and smooth. This case follows from Lemma 6.4 and Lemma 6.5.

□

Corollary 6.7. *In the situation from Proposition 6.6, we have*

$$c_D(f_* j_! \mathcal{L}) \leq c_D(f_* j_! \Lambda) + d \cdot c_E(j_! \mathcal{L})$$

and

$$lc_D(f_* j_! \mathcal{L}) \leq lc_D(f_* j_! \Lambda) + d \cdot lc_E(j_! \mathcal{L})$$

where d is the degree of $f : X \to Y$. Notice that d, $c_D(f_* j_! \Lambda)$ and $lc_D(f_* j_! \Lambda)$ only depend on $f : X \to Y$.

Theorem 6.8. *Let $f : Y \to X$ be a finite surjective morphism of normal schemes of finite type over k. Let D be a reduced effective Cartier divisor on X and put $U := X - D$. Define $E := (D \times_X Y)^{\mathrm{red}}$ and put $j : V := Y - E \hookrightarrow Y$. Assume that the restriction $f_0 : V \to U$ is étale. For $\mathcal{L} \in \mathrm{Loc}_{tf}(V, \Lambda)$, we have*

$$C(f_* j_! \mathcal{L}) \leq C(f_* j_! \Lambda) + f_* C(j_! \mathcal{L})$$

and

$$LC(f_* j_! \mathcal{L}) \leq LC(f_* j_! \Lambda) + f_* LC(j_! \mathcal{L}) .$$

Proof. We argue for the non logarithmic conductor divisor as the proof is the same for the logarithmic conductor divisor. The question is local on X. Hence, we may assume that D is irreducible. We denote by E_i's ($1 \leq i \leq m$) the irreducible components of E. By Proposition 6.6, we have

$$\begin{aligned} c_D(f_* j_! \mathcal{L}) &\leq \max_{1\leq i\leq m}\{c_D(f_* j_! \Lambda), f(E_i/D)^{\mathrm{ins}} \cdot c_{E_i}(j_! \mathcal{L})\} \\ &\leq c_D(f_* j_! \Lambda) + \max_{1\leq i\leq m}\{f(E_i/D)^{\mathrm{ins}} \cdot c_{E_i}(j_! \mathcal{L})\} \\ &\leq c_D(f_* j_! \Lambda) + \max_{1\leq i\leq m}\{f(E_i/D) \cdot c_{E_i}(j_! \mathcal{L})\} \\ &\leq c_D(f_* j_! \Lambda) + \sum_{1\leq i\leq m} f(E_i/D) \cdot c_{E_i}(j_! \mathcal{L}). \end{aligned}$$

Theorem 6.8 thus follows. □

Corollary 6.9. *In the situation from Theorem 6.8, assume that* X *is smooth. Let* $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$. *Then, for every* $\mathcal{L} \in \mathrm{Loc}_{tf}(V, \mathcal{E}, \Lambda)$, *the direct image* $f_*(j_!\mathcal{L})$ *has log conductor bounded by*

$$C(f_*j_!\Lambda) + f_*(T(\mathcal{E}) + E)\ .$$

If furthermore D *has normal crossings,* $f_*(j_!\mathcal{L})$ *has logarithmic ramification bounded by*

$$LC(f_*j_!\Lambda) + f_*T(\mathcal{E})\ .$$

Proof. Since X is smooth, $f_*(j_!\mathcal{L})$ has log conductor bounded by $C(f_*(j_!\mathcal{L}))$ in virtue of Proposition 5.7-(2). By Theorem 6.8, we have

$$C(f_*j_!\mathcal{L}) \leq C(f_*j_!\Lambda) + f_*C(j_!\mathcal{L})\ .$$

By Remark 3.3 and Proposition 5.7-(1), we have

$$C(j_!\mathcal{L}) \leq LC(j_!\mathcal{L}) + E \leq T(\mathcal{E}) + E\ .$$

This proves the first claim. The second claim follows similarly via Proposition 5.7-(3). □

Theorem 6.10. *Let* $f : Y \to X$ *be a finite surjective morphism between normal schemes of finite type over* k. *Let* D *be a reduced effective Cartier divisor on* X *and put* $U: = X - D$. *Define* $E: = (D \times_X Y)^{\mathrm{red}}$ *and put* $j : V := Y - E \hookrightarrow Y$. *Assume that the restriction* $f_0 : V \to U$ *is radicial. Then, for* $\mathcal{L} \in \mathrm{Loc}_{tf}(V, \Lambda)$, *we have*

$$C(f_*j_!\mathcal{L}) \leq f_*C(j_!\mathcal{L})$$

and

$$LC(f_*j_!\mathcal{L}) \leq f_*LC(j_!\mathcal{L})\ .$$

Proof. This is a local question in a neighbourhood of the generic points of D. Hence we can assume that X is smooth over k and that D is integral. Let η be the generic point of D. Since $f : Y \to X$ is finite, for every $y \in Y$, we have $f(\overline{\{y\}}) = \overline{\{f(y)\}}$, so that $\dim \overline{\{y\}} = \dim \overline{\{f(y)\}}$. Hence, $f^{-1}(\eta)$ is the set of generic points of E. Since Y is normal, we have $f^{-1}(\eta) \cap Y^{\mathrm{sing}} = \emptyset$. After replacing X by $X - f(Y^{\mathrm{sing}})$, we can further assume that Y is also smooth over k. In this case, $f : Y \to X$ is flat.

Let $\xi_1, \ldots, \xi_m$ be the generic points of E. Let R be the henselization of X at η, let K be the fraction field of R, let S_i be the henselization of Y at ξ_i $(1 \leq i \leq m)$ and let L_i be the fraction field of S_i. Note that R and the S_i's are henselian discrete valuation rings. Since $f : X \to Y$ is finite and flat, we have

$$\mathrm{Spec}(R) \times_X Y \cong \coprod_{1 \leq i \leq m} \mathrm{Spec}(S_i),$$

and $R \to S_i$ are finite and flat $(1 \leq i \leq m)$. Hence, we have

$$\mathrm{Spec}(K) \times_X Y = \mathrm{Spec}(K) \times_U V = \coprod_{1 \leq i \leq m} \mathrm{Spec}(L_i).$$

Since $f_0 : V \to U$ is radicial, we get $m = 1$. Hence, E has only one irreducible component with generic point ξ. Let S be the henselization of Y at ξ and let L be the fraction field of S. Since $\mathrm{Spec}(K) \times_X Y = \mathrm{Spec}(L)$, we have $[K(Y) : K(X)] = [L : K]$ and L/K is purely inseparable. Let κ_K and κ_L be the residue fields of R and S respectively.

Since κ_K is separably closed, the extension κ_L/κ_K is purely inseparable. Let d be its degree and let e be the ramification index of L/K. By [Ser68, Proposition I.10] we have

$$[L:K] = d \cdot e\,.$$

Let $\overline{K}$ be an algebraic closure of K containing L. Since L/K is purely inseparable, the canonical inclusion $\gamma : G_L \to G_K$ is a bijection. Let M be the Λ-module with continuous G_L-action associated to $\mathcal{L}|_{\mathrm{Spec}(L)}$ and N the Λ-module with continuous G_K-action associated to $(f_*j_!\mathcal{L})|_{\mathrm{Spec}(K)}$. By [Hu21, Theorem 1.1], we have

$$G_K^{dr} \subseteq \gamma(G_L^r) \quad \text{for } r \geq 1\,,$$
$$G_{K,\log}^{dr} \subseteq \gamma(G_{L,\log}^r) \quad \text{for } r \geq 0\,.$$

Thus, we have

$$c_K(N) \leq d \cdot c_L(M) \quad \text{and} \quad lc_K(N) \leq d \cdot lc_L(M)\,.$$

Thus we deduce

$$C(f_*j_!\mathcal{L}) = c_K(N)\cdot D \leq d\cdot c_L(M)\cdot D = c_L(M)\cdot f_*E = f_*C(j_!\mathcal{L})$$

and

$$LC(f_*j_!\mathcal{L}) = lc_K(N)\cdot D \leq d\cdot lc_L(M)\cdot D = cl_L(M)\cdot f_*E = f_*LC_Y(j_!\mathcal{L})\,.$$

□

Lemma 6.11. *Let $f : Y \to X$ be a finite surjective morphism between integral schemes of finite type over k. Then, $f : Y \to X$ factors through a finite surjective map $g : Y \to T$ where T is a normal scheme over k followed by a finite surjective map $h : T \to X$ such that there is a dense open subset $U \subset X$ satisfying :*

(1) the induced map $h_0 : h^{-1}(U) \to U$ is étale.

(2) the induced map $g_0 : f^{-1}(U) \to h^{-1}(U)$ is radicial.

Proof. Let ξ be the generic point of Y and let η be the generic point of X. Since $f : Y \to X$ is finite surjective, we have $f^{-1}(\eta) = \xi$. Consider the finite field extension $k(Y)/k(X)$. There exists a unique intermediate extension

$$k(X) \subset k(Y)_{sep} \subset k(Y)$$

such that $k(Y)_{sep}/k(X)$ is separable and $k(Y)/k(Y)_{sep}$ is purely inseparable. Let T be the normalization of X in $k(Y)_{sep}$. Then, $f : Y \to X$ factors through a finite surjective map $g : Y \to T$ with $k(Y)/k(T)$ purely inseparable followed by a finite surjective map $h : T \to X$ with $k(T)/k(X)$ separable. By affine base change, the coherent sheaf $h_*\Omega_{T/X}$ thus vanishes at η. Hence, it vanishes on a dense open neighbourhood $U \subset X$. Since the formation of $\Omega_{T/X}$ commutes with pull-back, we deduce that $\Omega_{T/X}$ vanishes on $h^{-1}(U)$. That is the pull-back $h_0 : h^{-1}(U) \to U$ is unramified. At the cost of shrinking U, we can suppose that $h_0 : h^{-1}(U) \to U$ is étale. By [GD66, Théorème 8.10.5], there is a dense open subset $V \subset h^{-1}(U)$ such that the induced map $g^{-1}(V) \to V$ is radicial. Observe that $E := h^{-1}(U) - V$ is a strict closed subset of $h^{-1}(U)$. Since $h_0 : h^{-1}(U) \to U$ is finite, $h(E)$ is a strict closed subset of U. Furthermore $h^{-1}(U - h(E)) \subset V$. Hence, the open subset $U - h(E)$ satisfies (1)(2). □

Theorem 6.12. *Let* $f : Y \to X$ *be a finite morphism between schemes of finite type over* k*. Let* Σ *be a stratification of* Y *and let* $\mathcal{E} \in \mathrm{Coh}(Y)$*. Then there exists* $\mathcal{E}' \in \mathrm{Coh}(X)$ *such that for every* $\mathcal{K} \in D^b_{\Sigma,tf}(Y, \mathcal{E}, \Lambda)$*, we have* $f_*\mathcal{K} \in D^b_{ctf}(X, \mathcal{E}', \Lambda)$*.*

Proof. We endow the closed subset $p(Y) \subset X$ with its canonical reduced structure. Consider the factorization $Y \to p(Y) \hookrightarrow X$ where the first map is finite and where the second map is a closed immersion. By Lemma 5.6-9, it is enough to prove Theorem 6.12 for $Y \to p(Y)$. Hence, we can assume that $f : Y \to X$ is finite surjective.

We argue by recursion on the dimension n of Y. If $n = 0$, there is nothing to do. Assume that $n > 0$ and that Theorem 6.12 holds in dimension $< n$. Since the étale topos is insensitive to nilpotents, we can by Lemma 5.6-8 assume that X, Y are reduced. By Lemma 5.6-7, we can suppose that X, Y are affine. Let $\mathcal{K} \in D^b_{\Sigma,tf}(Y, \mathcal{E}, \Lambda)$. Since finite direct images are exact, we can suppose that $\mathcal{K}$ is concentrated in degree 0. Let $\nu : Y^\nu \to Y$ be the normalization map. Since the unit map $\mathcal{K} \to \nu_*\nu^*\mathcal{K}$ is injective, the induced map

$$f_*\mathcal{K} \to f_*\nu_*\nu^*\mathcal{K}$$

is an injective map of constructible sheaves. By Lemma 5.6-5, at the cost of replacing Y by Y^ν and f by $f \circ \nu$, we can suppose that X, Y are affine and that Y is normal. By Noether normalization lemma, there is a finite morphism $\pi : X \to \mathbb{A}^d_k$. Observe that the counit map

$$\pi^*\pi_*f_*\mathcal{K} \to f_*\mathcal{K}$$

is surjective. By Lemma 5.6-5, at the cost of replacing X by $\mathbb{A}^d_k$ and f by $\pi \circ f$, we can thus suppose that X is affine smooth over k and Y is affine normal. Since a normal scheme is a disjoint union of its irreducible components, we can further suppose that X, Y are irreducible.

Let $Z \subset Y$ be a strict closed subset containing the strata of Σ of dimension $< \dim Y$. By Lemma 6.11, $f : Y \to X$ factors through a finite surjective map $g : Y \to T$ of normal schemes over k followed by a finite surjective map $h : T \to X$ such that there is a dense open subset $U \subset X$ with $h_0 : h^{-1}(U) \to U$ étale and $g_0 : f^{-1}(U) \to h^{-1}(U)$ radicial. Since $Z \subset Y$ is a strict closed subset, $f(Z) \subset X$ is a strict closed subset of X. At the cost of shrinking U, we can by smoothness of X assume that $D := X - U$ is a reduced effective Cartier divisor containing $f(Z)$. Put $E := (Y \times_X D)^{red}$. Consider the following commutative diagram

$$\begin{array}{ccccccc}
V & \xrightarrow{j} & Y & \xleftarrow[i]{} & E & \longleftarrow & Z \\
\downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_D} & \swarrow & \\
U & \longrightarrow & X & \longleftarrow & D\,. & &
\end{array}$$

By proper base change and recursion applied to the finite morphism $f_D : E \to D$, to $\Sigma_E := \Sigma \cap E$ and $i^*\mathcal{E} \in \mathrm{Coh}(E)$, there is $\mathcal{E}_D \in \mathrm{Coh}(D)$ such that for every $\mathcal{K} \in \mathrm{Cons}_{\Sigma,tf}(Y, \mathcal{E}, \Lambda)$, we have

$$(f_*\mathcal{K})|_D \simeq f_{D*}(\mathcal{K}|_E) \in \mathrm{Cons}_{tf}(D, \mathcal{E}_D, \Lambda)\,.$$

By Lemma 5.6-10, we can thus suppose that $\mathcal{K} = j_!\mathcal{L}$ where $\mathcal{L} \in \mathrm{Loc}_{tf}(V, \Lambda)$.

Note that h^*D is a well-defined effective Cartier divisor of T. By Theorem 6.10

applied to the morphism of normal schemes $g : Y \to T$ and to $h^*D \subset T$, we know that

$$C(g_* j_! \mathcal{L}) \leq g_* C(j_! \mathcal{L}) .$$

By Theorem 6.8 applied to the morphism of normal schemes $h : T \to X$ and to $D \subset X$, we know that

$$C(h_* g_* j_! \mathcal{L}) \leq C(f_* j_! \Lambda) + h_* C(g_* j_! \mathcal{L})$$

Putting everything together gives

$$C(f_* j_! \mathcal{L}) \leq C(f_* j_! \Lambda) + f_* C(j_! \mathcal{L}) .$$

By Remark 3.3 and Proposition 5.7, we have

$$C(j_! \mathcal{L}) \leq LC(j_! \mathcal{L}) + E \leq T(\mathcal{E}) + E .$$

Hence, we deduce

$$C(f_* j_! \mathcal{L}) \leq C(f_* j_! \Lambda) + f_* T(\mathcal{E}) + f_* E := C(f, \mathcal{E}) .$$

By Proposition 5.7, the sheaf $f_* j_! \mathcal{L}$ has log conductors bounded by $C(f, \mathcal{E})$ and the conclusion follows. □

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