ESTIMATES FOR BETTI NUMBERS AND RELATIVE HERMITE-MINKOWSKI THEOREM FOR PERVERSE SHEAVES

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ABSTRACT. We prove estimates for the Betti numbers of constructible sheaves in characteristic $p > 0$ depending only on their rank, stratification and wild ramification. In particular, given a smooth proper variety of dimension n over an algebraically closed field and a divisor D of X, for every $0 \leq i \leq n$, there is a polynomial P_i of degree $\max\{i, 2n - i\}$ such that the i-th Betti number of any rank r local system $\mathcal{L}$ on $X - D$ is smaller than $P_i(\mathrm{lc}_D(\mathcal{L})) \cdot r$ where $\mathrm{lc}_D(\mathcal{L})$ is the highest logarithmic conductor of $\mathcal{L}$ at the generic points of D. As application, we show that the Betti numbers of the inverse and higher direct images of a local system are controlled by the rank and the highest logarithmic conductor only. We also reprove Deligne's finiteness for simple ℓ-adic local systems with bounded rank and ramification on a smooth variety over a finite field and extend it in two different directions. In particular, perverse sheaves over arbitrary singular schemes are allowed and the bounds we obtain are uniform in algebraic families and do not depend on ℓ.

CONTENTS

1. Introduction

Let U be a smooth variety over a finite field $\mathbb{F}$ of characteristic $p > 0$ and let X be a normal compactification of U such that $D := X - U$ is an effective Cartier divisor. Let $\ell \neq p$ be a prime number. A celebrated result of Deligne [Del11, EK12] states that up to geometric isomorphism, there are only finitely many simple $\overline{\mathbb{Q}}_\ell$-local systems pure of weight 0 on U with bounded rank and bounded wild ramification along D. From the existence of companions due to Lafforgue in the curve case [Laf02] and Drinfeld in higher dimension [Dri12], the above number is independent of ℓ. Note that Deligne's finiteness is tight to the finiteness of $\mathbb{F}$ via the use of the Weil conjectures, and breaks down when k is infinite. It is nonetheless a natural question to ask whether some of its consequences survive over more general fields than finite fields.

The goal of this paper is to explore the cohomological aspect of this question. By Deligne's finiteness, we indeed know that there are only a finite number of possible Betti numbers for local systems on U with bounded rank and bounded wild ramification along D. Over an algebraically closed field, explicit bounds for the Betti numbers of affine varieties were obtained by Katz [Ka01]. Boundedness for the Betti numbers of *tame* local systems is due to Orgogozo [Org19]. In wild situations, explicit bounds were obtained for Artin-Schreier sheaves on affine varieties by Bombieri [Bom78], Adolphson-Sperber [AdS88] and Katz [Ka01]. All these bounds were recently sharpened by Zhang [Zh24] and Wan-Zhang [WZ25]. We show the following

Theorem 1.1 (Theorem 7.34)**.** *Let X be a proper smooth scheme of finite type of dimension n over an algebraically closed field k of characteristic $p > 0$. Let D be a reduced effective Cartier divisor of X and put $U := X - D$. Then, there exists $P \in \mathbb{N}[x]^{n+1}$ with P_i of degree i for $i = 0, \dots, n$ such that for every prime $\ell \neq p$, every $j = 0, \dots, 2n$ and every $\mathcal{L} \in \mathrm{Loc}(U, \overline{\mathbb{Q}}_\ell)$ we have*

$$h^j(U, \mathcal{L}) \leq P_{\min(j,2n-j)}(\mathrm{lc}_D(\mathcal{L})) \cdot \mathrm{rk}_{\overline{\mathbb{Q}}_\ell} \mathcal{L} \,. \tag{1.1.1}$$

In the above statement, $h^j(U, \mathcal{L})$ is the j-th Betti number of $\mathcal{L}$ and $\mathrm{lc}_D(\mathcal{L}) \in \mathbb{Q}_{\geq 0}$ is the highest logarithmic conductor of $\mathcal{L}$ at the generic points of D, as defined by Abbes and Saito [AS02, AS03, Sai08, Sai17a]. What makes the inequality (1.1.1) striking is that while the Betti numbers are *global* invariants of $\overline{\mathbb{Q}}_\ell$-sheaves depending on ℓ, the right hand side of (1.1.1) depends only on the *local* behavior of $\mathcal{L}$ at the generic points of D. So if one bounds the rank and the logarithmic conductor, the Betti numbers get bounded independently of ℓ as in the finite field situation. For a characteristic zero analogue for flat bundles, see [HT22].

In the curve case, the polynomials P_i can be made explicit using the Grothendieck-Ogg-Shafarevich formula [Ray68]. For affine spaces, explicit polynomials can also be produced. To this end, let us introduce the following

Definition 1.2. Put $b_0(x) = 1, b_1(x) = x$ and for $n \geq 2$, define inductively $b_n \in \mathbb{N}[x]$ by

$$b_n(x) = \sum_{\substack{i=0 \\ i \neq n \bmod 2}}^{n-1} (x+2) \cdot b_i(x+3) + \sum_{\substack{i=0 \\ i = n \bmod 2}}^{n-1} (x+3) \cdot b_i(x) + \sum_{\substack{i=0 \\ i \neq n \bmod 2}}^{n-1} b_i(x).$$

Then, we have the following

Theorem 1.3 (Theorem 6.1)**.** *Let k be an algebraically closed field of characteristic $p > 0$ and let $\ell \neq p$ be a prime. For every $0 \leq i \leq n$ and every $\mathcal{L} \in \mathrm{Loc}(\mathbb{A}^n_k, \overline{\mathbb{Q}}_\ell)$, we have*

$$h^i(\mathbb{A}^n_k, \mathcal{L}) \leq b_i(\mathrm{lc}_H(\mathcal{L})) \cdot \mathrm{rk}_{\overline{\mathbb{Q}}_\ell} \mathcal{L}$$

where H is the hyperplane at infinity.

In particular for $\mathbb{A}^n_k$, the estimates do not depend on the base field k nor its characteristic. Note that the degrees of the polynomials b_i are optimal as shown in Example 6.5.

It turns out that both the proof of Theorem 1.1 and the applications require more flexibility than allowed by the conditions of Theorem 1.1, which says nothing when singularities are involved or when X varies in a family. When $\mathcal{L}$ is replaced by a constructible sheaf $\mathcal{F}$ and when X is singular, one faces the problem that some wild ramification may hide in codimension > 1, which is typically not captured by the logarithmic conductor. To solve this problem, we generalized the notion of ramification boundedness in [HT25b] by using coherent sheaves instead of effective Cartier divisors. If $\mathbb{Q}[\mathrm{Coh}(X)]$ is the free $\mathbb{Q}$-vector space on the set of isomorphism classes of coherent sheaves on a scheme of finite type X over a field k, we set the following

Definition 1.4. For $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$, we say that a constructible sheaf $\mathcal{F}$ on X has *log conductors bounded by* $\mathcal{E}$ if for every morphism $f : C \to X$ where C is a smooth curve over k and every $x \in C$, the logarithmic conductor of $\mathcal{F}|_C$ at x is smaller than the length of the torsion part of $(f^*\mathcal{E})_x$ viewed as a module over $\mathcal{O}_{C,x}$.

Definition 1.4 captures all constructible sheaves since for every such sheaf $\mathcal{F}$, there is $\mathcal{E} \in \mathrm{Coh}(X)$ such that $\mathcal{F}$ has log conductors bounded by $\mathcal{E}$ (see [HT25b]). Moreover, semi-continuity results for logarithmic conductors [Hu23] provide explicit bounds in many cases, as shown by the following

Example 1.5. In the setting of Theorem 1.1, every $\mathcal{L} \in \mathrm{Loc}(U, \overline{\mathbb{Q}}_\ell)$ has log conductors bounded by $(\mathrm{lc}_D(\mathcal{L}) + 1) \cdot \mathcal{O}_D$.

The main result of this paper is the following

Theorem 1.6 (Theorem 7.15)**.** *Let $X \to S$ be a proper morphism with fibers of dimension $\leq n$ between schemes of finite type over a perfect field of characteristic $p > 0$. Let Σ be a stratification on X. Then, there is a function $\mu : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q}$ and $P \in \mathbb{N}[x]^{n+1}$ with P_i of degree i for every $i = 0, \dots, n$ such that for every algebraic geometric point $\bar{s} \to S$, every prime $\ell \neq p$, every $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$, every $j = 0, \dots, 2n$ and every $\Sigma_{\bar{s}}$-constructible $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on $X_{\bar{s}}$ with log conductors bounded by $\mathcal{E}_{\bar{s}}$, we have*

$$h^j(X_{\bar{s}}, \mathcal{F}) \leq P_{\min(j,2n-j)}(\mu(\mathcal{E})) \cdot \mathrm{Rk}_{\overline{\mathbb{Q}}_\ell} \mathcal{F} \,.$$

In this statement, $\Sigma_{\bar{s}}$-constructibilty means that the restriction of $\mathcal{F}$ to the strata of the stratification induced by Σ on $X_{\bar{s}}$ are locally constant and $\mathrm{Rk}_{\overline{\mathbb{Q}}_\ell} \mathcal{F}$ stands for the maximal rank of the germs of $\mathcal{F}$. When S is the spectrum of an algebraically closed field of characteristic $p > 0$, when X is smooth and $\Sigma = \{U, D\}$ with D an effective Cartier divisor, Theorem 1.6 combined with Example 1.5 immediately implies Theorem 1.1. Observe that to get the particular Theorem 1.1 in its sharpest form where the bounding polynomials have decreasing degrees above n, we need to prove the more general Theorem 1.6 directly. The reason is that allowing arbitrary singular base schemes gives extra flexibility for a recursion on the dimension. In particular, it bypasses the use of De Jong's alterations [dJ96] and the cohomological descent spectral sequence [Co], which makes middle dimension cohomologies on the E_1-page contribute to beyond middle dimension cohomologies of the abutment.

Remark 1.7. Several variations of Theorem 1.6 can be given : one may consider complexes with bounded constructible cohomology or perverse sheaves and one may work with finite coefficients. See Section 7 for precise statements.

Using Sawin's quantitative sheaf theory [SFFK21], Theorem 1.6 gives control on the Betti numbers of inverse images (Theorem 8.5) in terms of the wild ramification. For locally constant constructible sheaves, this specializes to the following

Theorem 1.8 (Corollary 8.6)**.** *Let* $f : Y \to X$ *be a morphism between projective schemes over* k *algebraically closed where* X *is smooth. Let* D *be an effective Cartier divisor of* X *and put* $U := X - D$ *and* $V := Y - f^{-1}(D)$*. Then there is* $P \in \mathbb{N}[x]$ *of degree* $\dim X$ *such that for every prime* $\ell \neq p$ *and every* $\mathcal{L} \in \mathrm{Loc}(U, \overline{\mathbb{Q}}_\ell)$*, we have*

$$\sum_{j \in \mathbb{Z}} h^j_c(V, \mathcal{L}|_V) \leq P(\mathrm{lc}_D(\mathcal{L})) \cdot \mathrm{rk}_{\overline{\mathbb{Q}}_\ell} \mathcal{L} \, .$$

Similarly, Theorem 1.6 implies estimates for the Betti numbers of higher direct images (Theorem 8.7). For locally constant constructible sheaves, this specializes to the following

Theorem 1.9 (Corollary 8.8)**.** *Let* $f : X \to Y$ *be a projective morphism between projective schemes over* k *algebraically closed where* X *is smooth. Let* D *be an effective Cartier divisor of* X *and put* $j : U := X - D \hookrightarrow X$*. Then, there is a function* $C : \mathbb{Q}^+ \times \mathbb{N}^+ \to \mathbb{N}^+$ *such that for every prime* $\ell \neq p$ *and every* $\mathcal{L} \in \mathrm{Loc}(U, \overline{\mathbb{Q}}_\ell)$*, we have*

$$\sum_{i,j \in \mathbb{Z}} h^j(Y, R^i f_* j_! \mathcal{L}) \leq C(\mathrm{lc}_D(\mathcal{L}), \mathrm{rk}_{\overline{\mathbb{Q}}_\ell} \mathcal{L}) \, .$$

Remark 1.10. All continuity results from [SFFK21] translate thanks to Theorem 1.6 into estimates for Betti numbers involving the rank and the wild ramification only.

Our next application of Theorem 1.6 is a relative perverse geometric form of Hermite-Minkowski's theorem in number theory according to which there are only finitely many number fields with bounded discriminant. For more applications to characteristic cycles and to wild Lefschetz type theorems, see [HT25c]. To state the relative perverse Hermite-Minkowski theorem, let us introduce the following

Notation 1.11. Let (X, Σ) be a stratified scheme of finite type over a field k. Let $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$ and $r \geq 0$. We denote by $\mathrm{Perv}^{\leq r}_{\Sigma}(X, \mathcal{E}, \overline{\mathbb{Q}}_\ell)$ the full subcategory of $\mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$ spanned by objects $\mathcal{P}$ such that for every $j \in \mathbb{Z}$, the constructible sheaf $\mathcal{H}^j\mathcal{P}$ is Σ-constructible, has log conductors bounded by $\mathcal{E}$ and satisfies $\mathrm{Rk}_{\overline{\mathbb{Q}}_\ell} \mathcal{H}^j\mathcal{P} \leq r$.

Theorem 1.12 (Theorem 9.7)**.** *Let* $X \to S$ *be a projective morphism between schemes of finite type over a finite field of characteristic* $p > 0$*. Let* Σ *be a stratification on* X*. Then, there is a function* $N : \mathbb{Q}[\mathrm{Coh}(X)] \times \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{N}^+$ *such that for every* $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$*, every* $r \geq 0$*, every prime* $\ell \neq p$ *and every closed point* $s \in S$*, there are up to geometric isomorphism at most* $N(\mathcal{E}, r, \deg s)$ *geometrically simple pure of weight 0 objects in* $\mathrm{Perv}^{\leq r}_{\Sigma_s}(X_s, \mathcal{E}_s, \overline{\mathbb{Q}}_\ell)$*.*

When S is a point, X is normal and $\Sigma = \{U, D\}$ where D is an effective Cartier divisor with U smooth, Theorem 1.12 reproduces Deligne's finiteness theorem [Del11, EK12].

For a variant building on the smooth case where U is normal, see [E17]. See also [Dri81, FKM13, Del15, DF13, Fl15, Yu21, Yu23a, Yu23b] for concrete bounds in the curve case. In a nutshell, Deligne's approach rests upon the identification of simple local systems with bounded rank and ramification with the irreducible components of a scheme of finite type $\mathcal{X}$ over $\mathbb{Q}$. The scheme $\mathcal{X}$ is constructed by putting all the characteristic polynomials of the local Frobenii together. The finite type property of $\mathcal{X}$ is the crucial aspect of the construction. It ultimately relies on the fact that in the bounded rank and bounded ramification situation, only a finite number of Frobenii are needed. Furthermore, the existence of companions [Laf02, Dri12] implies independence in ℓ in Deligne's finiteness. Our approach to Theorem 1.12 is different: it trades off the construction of $\mathcal{X}$ for the use of Beilinson's singular support and Saito's characteristic cycle. Still, the bounds we obtain are independent of ℓ and don't resort to companions. At the cost of using the Langlands correspondence [Dri88, Dri89, Laf02], the purity assumption from Theorem 1.13 can be dropped. This is the following

Theorem 1.13 (Theorem 9.11)**.** *In the setting of Theorem 1.12, there is a function* $N : \mathbb{Q}[\mathrm{Coh}(X)] \times \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{N}^+$ *such that for every* $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$, *every* $r \geq 0$, *every prime* $\ell \neq p$ *and every closed point* $s \in S$, *there are up to geometric isomorphism at most* $N(\mathcal{E}, r, \deg s)$ *geometrically simple objects in* $\mathrm{Perv}^{\leq r}_{\Sigma_s}(X_s, \mathcal{E}_s, \overline{\mathbb{Q}}_\ell)$.

On the proof of Theorem 1.6. The proof of Theorem 1.6 involves a reduction to the case where $f : X \to S$ is projective with geometrically normal fibres (Lemma 7.26). From this point on and up to some noise coming from lower dimension, the function $\mu : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q}$ is concretely constructed in Example 7.10 using the lengths at codimension one points of the restrictions of $\mathcal{E} \in \mathrm{Coh}(X)$ to the fibers of $f : X \to S$. Using a result of Kedlaya [Ke05] combined with some estimates (Theorem 2.15) for the conductor of a finite direct image proved in [HT25b], we reduce to prove Theorem 1.3. To treat the $\mathbb{A}^n_k$ case, we argue by recursion on n. To simplify the explanation, assume that $n = 2$. By a standard weak Lefschetz argument (Proposition 6.6), we are reduced to estimate the Euler-Poincaré characteristic of $\mathcal{L} \in \mathrm{Loc}(\mathbb{A}^2_k, \Lambda)$. The main idea is here to twist $\mathcal{L}$ by a generic enough Artin-Schreier sheaf $\mathcal{N}$ so that the discrepancy between the χ-ies of $\mathcal{L}$ and its twist $\mathcal{L}'$ stays in control (Proposition 6.8). To estimates the χ of $\mathcal{L}'$, a well-chosen pencil (Construction 5.9) reduces us to estimate the Euler-Poincaré characteristic of the higher direct images of $\mathcal{L}'$ along some map $\pi_A : \mathbb{A}^2_k \to \mathbb{A}^1_k$. By choosing the conductor of $\mathcal{N}$ at ∞ to be prime to p and $> \mathrm{lc}_H(\mathcal{L})$, one sees that $R\pi_{A!}\mathcal{L}'$ is locally constant and concentrated in degree 1 with controlled rank (Proposition 5.10). We are thus left to control the conductor of $R^1\pi_{A!}\mathcal{L}'$ at ∞. Note that this question amounts to bound the wild ramification of some nearby cycle complex attached to $\mathcal{L}'$. To achieve this, we invoke the main result of [HT21] to take care of the locus of the special fiber free from any wild horizontal contribution. Inspired by an argument from [Sai21, Corollary 1.5.7] relying on a suitable radicial pullback, we show that at the unique problematic point where [HT21] breaks down, $\mathcal{N}$ can be chosen so that the nearby cycle complex of $\mathcal{L}'$ vanishes (Proposition 5.11).

Linear Overview. Section 2 provides an account of Abbes and Saito theory for the logarithmic conductor (Proposition 2.3) and recall its behaviour under inverse image

(Theorem 2.14) and finite direct image (Theorem 2.15). In section 3, we introduce Beilinson's singular support and Saito's characteristic cycle for étale sheaves. Section 4 provides a control of the wild ramification of the nearby cycle complex generalizing the main result of [HT21] over henselian traits coming from geometric situations (Theorem 4.7). Section 5 provides a crucial conductor estimates (Proposition 5.12) for the higher direct image of some suitably twisted local system. Section 6 is devoted to the proof of Theorem 1.3. In section 7, we recall some basic material from [HT25b] and prove Theorem 1.3 and some immediate consequences. Variations of Theorem 1.3 dealing with inverse and higher direct images are obtained in section 8 by building on the continuity results from [SFFK21]. Section 9 is devoted to the proof of Theorem 1.12. The appendix 10 is a computation making the bounding polynomials explicit in the $\mathbb{A}^n_k$ case.

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Notation 1.14. We introduce the following running notations.

- k denotes a perfect field of characteristic $p > 0$.
- The letter Λ will refer to a finite local ring of residue characteristic $\ell \neq p$.
- For a scheme X of finite type over k, we denote by $D^b_{ctf}(X, \Lambda)$ the derived category of complexes of Λ-sheaves of finite tor-dimension with bounded and constructible cohomology sheaves.
- We let $\mathrm{Cons}_{ctf}(X, \Lambda)$ be the category of constructible sheaves of Λ-modules of finite tor-dimension over X and $\mathrm{Loc}_{tf}(X, \Lambda) \subset \mathrm{Cons}_{tf}(X, \Lambda)$ the full subcategory spanned by locally constant constructible sheaves. By [Wei94, Lemma 4.4.14], the germs of any $\mathcal{L} \in \mathrm{Loc}_{tf}(X, \Lambda)$ are automatically free Λ-modules of finite rank.
- $\mathrm{Perv}_{tf}(X, \Lambda)$ will denote the category of perverse sheaves of Λ-modules of finite tor-dimension over X for the middle perversity function.
- Let X be a scheme of finite type over k and let Λ be a field of characteristic $\neq p$. For $\mathcal{K} \in D^b_{ctf}(X, \Lambda)$, we put

$$\mathrm{Rk}_\Lambda \mathcal{K} := \max\{\mathrm{rk}_\Lambda \mathcal{H}^i \mathcal{K}_{\bar{x}}, \text{ where } i \in \mathbb{Z} \text{ and } \bar{x} \to X \text{ is algebraic geometric}\}.$$

- For $r \geq 0$, we let $D^{\leq r}_{ctf}(X, \Lambda) \subset D^b_{ctf}(X, \Lambda)$ be the full subcategory spanned by objects $\mathcal{K}$ such that $\mathrm{Rk}_\Lambda \mathcal{K} \leq r$, and similarly with perverse complexes.

- For a finite stratification Σ of X, we let $D^b_{\Sigma,\mathrm{tf}}(X,\Lambda) \subset D^b_{\mathrm{tf}}(X,\Lambda)$ be the full subcategory spanned by Σ-constructible complexes, and similarly with perverse complexes.

2. Conductors of étale sheaves

2.1. Ramification filtrations. Let K be a henselian discrete valuation field over k. Let $\mathcal{O}_K$ be the ring of integer of K, let $\mathfrak{m}_K$ be the maximal ideal of $\mathcal{O}_K$ and F the residue field of $\mathcal{O}_K$. Fix $K \subset K^{\mathrm{sep}}$ a separable closure of K and let G_K be the Galois group of K^{sep} over K. Let $I_K \subset G_K$ be the inertia subgroup and let $P_K \subset G_K$ be the wild ramification subgroup.

Recollection 2.2. In [AS02], Abbes and Saito defined two decreasing filtrations $\{G^r_K\}_{r\in\mathbb{Q}_{>0}}$ and $\{G^r_{K,\log}\}_{r\in\mathbb{Q}_{\geq 0}}$ on G_K by closed normal subgroups. They are called the *the ramification filtration* and *the logarithmic ramification filtration* respectively. For $r \in \mathbb{Q}_{\geq 0}$, put

$$G^{r+}_K = \overline{\bigcup_{s>r} G^s_K} \quad \text{and} \quad G^{r+}_{K,\log} = \overline{\bigcup_{s>r} G^s_{K,\log}}.$$

Proposition 2.3 ([AS02, AS03, Sai08, Sai17a])**.** *The following properties hold :*

(1) For any $0 < r \leq 1$, we have

$$G^r_K = G^0_{K,\log} = I_K \text{ and } G^{+1}_K = G^{0+}_{K,\log} = P_K.$$

(2) For any $r \in \mathbb{Q}_{\geq 0}$, we have

$$G^{r+1}_K \subseteq G^r_{K,\log} \subseteq G^r_K.$$

If F is perfect, then for any $r \in \mathbb{Q}_{\geq 0}$, we have

$$G^r_{K,\mathrm{cl}} = G^r_{K,\log} = G^{r+1}_K.$$

where $G^r_{K,\mathrm{cl}}$ is the classical wild ramification subgroup as defined in [Ser68].

(3) For any $r \in \mathbb{Q}_{>0}$, the graded piece $G^r_{K,\log}/G^{r+}_{K,\log}$ is abelian, p-torsion and contained in the center of $P_K/G^{r+}_{K,\log}$.

Let M be a finitely generated Λ-module with a continuous P_K-action. The module M has decompositions

$$M = \bigoplus_{r\geq 1} M^{(r)} \quad \text{and} \quad M = \bigoplus_{r\geq 0} M^{(r)}_{\log} \tag{2.3.1}$$

into P_K-stable Λ-submodules where $M^{(1)} = M^{(0)}_{\log} = M^{P_K}$, and such that for every $r \in \mathbb{Q}_{>0}$,

$$(M^{(r+1)})^{G^{r+1}_K} = 0 \text{ and } (M^{(r+1)})^{G^{(r+1)+}_K} = M^{(r+1)};$$

$$(M^{(r)}_{\log})^{G^r_{K,\log}} = 0 \text{ and } (M^{(r)}_{\log})^{G^{r+}_{K,\log}} = M^{(r)}_{\log}.$$

The decompositions (2.3.1) are respectively called the *slope decomposition* and the *logarithmic slope decomposition* of M. The values r for which $M^{(r)} \neq 0$ (resp. $M^{(r)}_{\log} \neq 0$)

are the *slopes* (resp. the *logarithmic slopes*) of M. We denote by $c_K(M)$ the largest slope of M and refer to $c_K(M)$ as the *conductor of* M. Similarly, we denote by $lc_K(M)$ the largest logarithmic slope of M and refer to $lc_K(M)$ as the *logarithmic conductor of* M. We say that M is *isoclinic* (resp. *logarithmic isoclinic*) if M has only one slope (resp. only one logarithmic slope).

The following is an immediate consequence of Proposition 2.3-(2).

Lemma 2.4. *Let* M *be a finitely generated* Λ*-module with a continuous* P_K*-action. Then,*

$$lc_K(M) \leq c_K(M) \leq lc_K(M) + 1.$$

If M is free as a Λ-module, then so are the $M^{(r)}_{\log}$ and the $M^{(r)}$ in virtue of [Ka88, Lemma 1.5]. In that case, the *total dimension* of M is defined by

$$\mathrm{dimtot}_K(M) := \sum_{r\geq 1} r \cdot \mathrm{rk}_\Lambda M^{(r)}$$

and the *Swan conductor* of M is defined by

$$\mathrm{sw}_K(M) := \sum_{r\geq 0} r \cdot \mathrm{rk}_\Lambda M^{(r)}_{\log} \, .$$

Lemma 2.5 ([AS02])**.** *In the setting of Recollection 2.2, we have*

$$\mathrm{sw}_K(M) \leq \mathrm{dimtot}_K(M) \leq \mathrm{sw}_K(M) + \mathrm{rk}_\Lambda M \, .$$

If the residue field F *is perfect, we have*

$$lc_K(M) + 1 = c_K(M) \, .$$
$$\mathrm{sw}_K(M) + \mathrm{rk}_\Lambda M = \mathrm{dimtot}_K(M) \, .$$

Example 2.6 ([Lau81a, Example 1.1.7])**.** *Let* $\psi : \mathbb{F}_p \to \Lambda^\times$ *be a non trivial character. Let* M *be the free rank one* Λ*-module with continuous* G_K*-action corresponding to the Artin-Schreier cover defined by the equation*

$$t^p - t = \frac{u}{x} \quad u \in K^\times, x \in \mathfrak{m}_K$$

If $m := \nu_K(x)$ *is prime to* p*, then* $lc_K(M) = m$ *and* $c_K(M) = m + 1$*.*

Lemma 2.7 ([HT21, Lemma 1.9])**.** *Let* M *and* N *be free* Λ*-modules of finite type with continuous* G_K*-actions.*

(1) *If* M *and* N *are isoclinic of slope* r *and* s *respectively with* $r > s$*, then* $M \otimes_\Lambda N$ *is isoclinic of slope* r *and*

$$\mathrm{dimtot}_K(M \otimes_\Lambda N) = \mathrm{rk}_\Lambda N \cdot \mathrm{rk}_\Lambda M \cdot r \, .$$

(2) *If* M *and* N *are logarithmic isoclinic of logarithmic slope* r *and* s *respectively with* $r > s$*, then* $M \otimes_\Lambda N$ *is logarithmic isoclinic of logarithmic slope* r *and*

$$\mathrm{sw}_K(M \otimes_\Lambda N) = \mathrm{rk}_\Lambda N \cdot \mathrm{rk}_\Lambda M \cdot r \, .$$

Recollection 2.8. Assume that Λ is a field. Let M be a free Λ-module of finite type with a continuous P_K-action where Λ contains a p-root of unity. Assume that M is logarithmic isoclinic of logarithmic slope r. Let $X(r)$ be the set of isomorphism classes of finite characters

$$\chi : G^r_{K,\log}/G^{r+}_{K,\log} \to \Lambda^\times .$$

As a consequence of Proposition 2.3-(3) (see also [AS11, Lemma 6.7]), the module M has a unique direct sum decomposition

$$M := \bigoplus_{\chi \in X(r)} M_\chi$$

into P_K-stable sub-Λ-modules such that as Λ-module with a continuous $G^r_{K,\log}$-action, M_χ is a direct sum of copies of $\chi : G^r_{K,\log}/G^{r+}_{K,\log} \to \Lambda^\times$.

Lemma 2.9. *Assume that Λ is a field. Let M be a free Λ-module of finite type with a continuous P_K-action where Λ contains a p-root of unity. Let N be a free Λ-module of rank 1 with a continuous P_K-action. Let r be the logarithmic slope of N and let $\chi : G^r_{K,\log}/G^{r+}_{K,\log} \to \Lambda^\times$ be the corresponding character. If χ^{-1} does not contribute to the character decomposition of $M^{(r)}_{\log}$, we have*

$$\mathrm{sw}_K(M) \leq \mathrm{sw}_K(M \otimes_\Lambda N) \leq \mathrm{sw}_K(M) + r \cdot \mathrm{rk}_\Lambda M$$

Proof. We have

$$M \otimes_\Lambda N = \bigoplus_{s \geq 0} M^{(s)}_{\log} \otimes_\Lambda N .$$

By Lemma 2.7, if $s < r$, the tensor product $M^{(s)}_{\log} \otimes_\Lambda N$ is logarithmic isoclinic of logarithmic slope r and if $s > r$, it is logarithmic isoclinic of logarithmic slope s. For every $\nu : G^r_{K,\log}/G^{r+}_{K,\log} \to \Lambda^\times$ contributing to the character decomposition of $M^{(r)}_{\log}$, observe that $G^r_{K,\log}$ acts on $(M^{(r)}_{\log})_\nu \otimes_\Lambda N$ via a direct sum of copies of $\nu \otimes_\Lambda \chi : G^r_{K,\log}/G^{r+}_{K,\log} \to \Lambda^\times$ which is non trivial by assumption. In particular, $G^{r+}_{K,\log}$ acts trivially on $(M^{(r)}_{\log})_\nu \otimes_\Lambda N$ and $G^r_{K,\log}$ acts on $(M^{(r)}_{\log})_\nu \otimes_\Lambda N$ without non zero fixed points. Thus,

$$(M \otimes_\Lambda N)^{(r)}_{\log} = \bigoplus_{\eta} (M^{(r)}_{\log})_\nu \otimes_\Lambda N$$

where ν runs over the set of characters contributing to $M^{(r)}_{\log}$. Thus

$$\begin{aligned} \mathrm{sw}_K(M \otimes_\Lambda N) &= r \cdot \sum_{s<r} \mathrm{rk}_\Lambda M^{(s)}_{\log} + \sum_{s \geq r} s \cdot \mathrm{rk}_\Lambda M^{(s)}_{\log} \\ &= \sum_{s<r} (r-s) \cdot \mathrm{rk}_\Lambda M^{(s)}_{\log} + \mathrm{sw}_K(M) . \end{aligned}$$

The conclusion thus follows. □

2.10. Conductor divisors. Let X be a normal scheme of finite type over k. Let Z be an integral Weil divisor and let $\eta \in Z$ be its generic point. Let K be the fraction field of $\hat{\mathcal{O}}_{X,\eta}$ and fix a separable closure K^{sep} of K. For $\mathcal{F} \in \mathrm{Cons}_{tf}(X, \Lambda)$, the pull-back $\mathcal{F}|_{\operatorname{Spec} K}$ is a Λ-module of finite type with continuous G_K-action. Using the notations from Section 2.1, we put

$$c_Z(\mathcal{F}) := c_K(\mathcal{F}|_{\operatorname{Spec} K}) \text{ and } \operatorname{lc}_Z(\mathcal{F}) := \operatorname{lc}_K(\mathcal{F}|_{\operatorname{Spec} K}) .$$

Definition 2.11. Let X be a normal scheme of finite type over k and let $\mathcal{F} \in \mathrm{Cons}_{tf}(X, \Lambda)$. We define the *conductor divisor of* $\mathcal{F}$ by

$$C_X(\mathcal{F}) := \sum_Z c_Z(\mathcal{F}) \cdot Z$$

and the *logarithmic conductor divisor of* $\mathcal{F}$ by

$$LC_X(\mathcal{F}) := \sum_Z \operatorname{lc}_Z(\mathcal{F}) \cdot Z$$

where the sums run over the set of integral Weil divisors of X.

Remark 2.12. The above divisors are $\mathbb{Q}$-Weil divisors of X. We will sometimes abuse the notations and write $C(\mathcal{F})$ instead of $C_X(\mathcal{F})$ and similarly in the logarithmic case.

Definition 2.13. In the setting of Section 2.10, we define the *generic conductor* and the *generic logarithmic conductor of* $\mathcal{L}$ *along* D respectively by

$$c_D(\mathcal{L}) := \max_Z c_Z(\mathcal{L}) \text{ and } \operatorname{lc}_D(\mathcal{L}) := \max_Z \operatorname{lc}_Z(\mathcal{L}) .$$

where Z runs over the set of irreducible components of D.

The above divisors enjoy the following semi-continuity property :

Theorem 2.14 ([Hu23, Theorem 1.4,1.5])**.** *Let* $f : Y \to X$ *be a morphism of smooth schemes of finite type over* k*. Let* D *be an effective Cartier divisor on* X *and put* $U := X - D$*. Assume that* $E := Y \times_X D$ *is an effective Cartier divisor on* Y*. For every* $\mathcal{L} \in \mathrm{Loc}_{tf}(U, \Lambda)$*, we have*

$$C_Y((j_!\mathcal{L})|_Y) \leq f^* C_X(j_!\mathcal{L}) .$$

If furthermore D *has normal crossings, we have*

$$LC_Y((j_!\mathcal{L})|_Y) \leq f^* LC_X(j_!\mathcal{L}) .$$

The generic logarithmic conductor divisor satisfies the following compatibility with finite push-forward :

Theorem 2.15 ([HT25b, Theorem 1.3])**.** *Let* $f : Y \to X$ *be a finite surjective morphism of normal schemes of finite type over* k*. Let* D *be an irreducible effective Cartier divisor on* X *and put* $U := X - D$*. Put* $E := D \times_X Y$ *and* $V := Y - E$*. Assume that the restriction* $f_U : V \to U$ *is étale. Then, for every* $\mathcal{L} \in \mathrm{Loc}_{tf}(V, \Lambda)$ *we have*

$$\operatorname{lc}_D(f_{U*}\mathcal{L}) \leq \operatorname{lc}_D(f_{U*}\Lambda) + d \cdot \operatorname{lc}_E(\mathcal{L})$$

where d *is the generic degree of* $f : X \to Y$.

3. Singular support and characteristic cycle of étale sheaves

3.1. The singular support. Let X be a smooth scheme of finite type over k. We denote by $\mathbb{T}^*X$ the cotangent bundle of X. Let $C \subset$ be a closed conical subset. For a point $x \in X$, we put $\mathbb{T}^*_x X = \mathbb{T}^*X \times_X x$ and $C_x = C \times_X x$.

Recollection 3.2. Let $h : U \to X$ be a morphism of smooth schemes of finite type over k. For $u \in U$, we say that $h : U \to X$ is C-*transversal at* u if

$$\ker dh_u \bigcap C_{h(u)} \subseteq \{0\} \subseteq \mathbb{T}^*_{h(u)}X$$

where $dh_u : \mathbb{T}^*_{h(u)}X \to \mathbb{T}^*_u U$ is the cotangent map of h at u. We say that $h : U \to X$ is C-*transversal* if it is C-transversal at every point of U. For a C-transversal morphism $h : U \to X$, we let $h^\circ C$ be the scheme theoretic image of $C \times_X U$ in $\mathbb{T}^*U$ by $dh : \mathbb{T}^*X \times_X U \to \mathbb{T}^*U$.

Let $f : X \to Y$ be a morphism of smooth schemes of finite type over k. For $x \in X$, we say that $f : X \to Y$ is C-*transversal at* x if

$$df_x^{-1}(C_x) \subseteq \{0\} \subseteq \mathbb{T}^*_{f(x)}Y$$

We say that $f : X \to Y$ is C-*transversal* if it is C-transversal at every point of X.

Let $(h, f) : Y \leftarrow U \to X$ be a pair of morphisms of between smooth schemes of finite type over k. We say that (h, f) is C-*transversal* if $h : U \to X$ is C-transversal and if $f : U \to Y$ is $h^\circ C$-transversal.

Definition 3.3. In the setting of Section 3.1, we say that $\mathcal{K} \in D^b_c(X, \Lambda)$ is *micro-supported on* C if for every C-transversal pair $(h, f) : Y \leftarrow U \to X$, the map $f : U \to Y$ is universally locally acyclic with respect to $h^*\mathcal{K}$.

Theorem 3.4 ([Bei16, Theorem 1.3])**.** *For every $\mathcal{K} \in D^b_c(X, \Lambda)$, there is a smallest closed conical subset $SS(\mathcal{K}) \subset \mathbb{T}^*X$ on which $\mathcal{K}$ is micro-supported. Furthermore, if X has pure dimension n, then $SS(\mathcal{K})$ has pure dimension n.*

Definition 3.5. The closed conical subset $SS(\mathcal{K})$ is the *singular support of* $\mathcal{K}$.

The conductor can be detected by curves, due to the following :

Proposition 3.6 ([Sai17a, Corollary 3.9])**.** *Let X be a smooth scheme over k. Let D be an effective Cartier divisor on X and put $j : U := X - D \hookrightarrow X$. Let $\mathcal{L} \in \mathrm{Loc}_{tf}(U, \Lambda)$ and let $i : S \to X$ be an immersion over k where S is a smooth curve. Assume that*

(1) S meets D transversely at a single smooth point $x \in D$.

(2) The map $i : S \to X$ is $SS(j_!\mathcal{L})$-transversal.

(3) The ramification of $j_!\mathcal{L}$ is non-degenerate at x.

Then, $C_S((j_!\mathcal{L})|_S) = i^*C_X(j_!\mathcal{L})$.

Proposition 3.7 ([Sai22, Proposition 1.1.8])**.** *Let*

$$\begin{tikzcd} V \arrow[r, "h'"] \arrow[d, "j'"'] \arrow[dr, phantom, "\square"] & U \arrow[d, "j"] \\ W \arrow[r, "h"'] & X \end{tikzcd}$$

be a cartesian diagram of smooth schemes over k *such that the vertical arrows are open immersions and such that* h *is separated. Let* $\mathcal{F}$ *be an object in* $D^b_c(X, \Lambda)$ *and assume that* $h : W \to X$ *is* $SS(Rj_*\mathcal{F})$*-transversal. Then, the base change morphism*

$$h^* Rj_* \mathcal{F} \to Rj'_* h'^* \mathcal{F}$$

is an isomorphism in $D^b_c(X, \Lambda)$*.*

3.8. The characteristic cycle. Let $f : X \to S$ be a morphism between smooth schemes of finite type over k where S is a curve over k. Let $x \in X$ be a closed point and put $s = f(x)$. Note that any local trivialization of $\mathbb{T}^*S$ in a neighborhood of s gives rise to a local section of $\mathbb{T}^*X$ in a neighborhood of x by applying $df : \mathbb{T}^*S \times_S X \to \mathbb{T}^*X$. We abusively denote by df this section.

We say that x is an *at most* C*-isolated characteristic point* for $f : X \to S$ if $f : X \setminus \{x\} \to S$ is C-transversal. In that case, the intersection of a cycle A supported on C with $[df]$ is supported at most at a single point in $\mathbb{T}^*_x X$. Since C is conical, the intersection number $(A, [df])_{\mathbb{T}^*X,x}$ is independent of the chosen local trivialization for $\mathbb{T}^*S$ in a neighborhood of s.

Theorem 3.9 ([Sai17b, Theorem 5.9])**.** *Let* X *be a smooth scheme of finite type over* k*. For every* $\mathcal{K} \in D^b_{ctf}(X, \Lambda)$*, there is a unique cycle* $CC(\mathcal{K})$ *of* $\mathbb{T}^*X$ *supported on* $SS(\mathcal{K})$ *such that for every étale morphism* $h : U \to X$*, for every morphism* $f : U \to S$ *with* S *a smooth curve over* k*, for every at most* $h^\circ(SS(\mathcal{K}))$*-isolated characteristic point* $u \in U$ *for* $f : U \to S$*, we have the following Milnor type formula*

$$-\operatorname{dimtot}(R\Phi_{\bar{u}}(h^*\mathcal{K}, f)) = (h^*CC(\mathcal{K}), [df])_{\mathbb{T}^*U,u}, \tag{3.9.1}$$

where $R\Phi_{\bar{u}}(h^*\mathcal{K}, f)$ *denotes the stalk of the vanishing cycle of* $h^*\mathcal{K}$ *with respect to* $f : U \to S$ *at a geometric point* $\bar{u} \to U$ *above* u*.*

Definition 3.10. The cycle $CC(\mathcal{K})$ from Theorem 3.9 is the *characteristic cycle of* $\mathcal{K}$.

Example 3.11 ([Sai17b, Lemma 5.11])**.** Assume that X is a smooth connected curve over k. Let $\mathcal{K} \in D^b_{ctf}(X, \Lambda)$ and let $U \subset X$ be a dense open subset where $\mathcal{K}$ is locally constant. Then, we have

$$CC(\mathcal{K}) = -\operatorname{rk}_\Lambda \mathcal{K}_\eta \cdot [\mathbb{T}^*_X X] - \sum_{x \in X - U} (\operatorname{dimtot}_x(\mathcal{K}) - \operatorname{rk}_\Lambda \mathcal{K}_x) \cdot [\mathbb{T}^*_x X]$$

where η is the generic point of X.

3.12. Characteristic cycle and cohomology. The following index formula provides a positive characteristic analogue of Kashiwara-Dubson's formula for $\mathcal{D}$-modules.

Theorem 3.13 ([Sai17b, Theorem 7.13])**.** *Let* X *be a smooth projective variety over an algebraically closed field* k*. For every* $\mathcal{K} \in D^b_{ctf}(X, \Lambda)$*, we have*

$$\chi(X, \mathcal{K}) = (CC(\mathcal{K}), \mathbb{T}^*_X X)_{\mathbb{T}^*X}$$

Remark 3.14. Assume that Λ is a finite field. When $\mathcal{K}$ is of the form $j_!\mathcal{F}$ where $j : U \hookrightarrow X$ is an open immersion and $\mathcal{F} \in D^b_c(U, \Lambda)$, we have $\chi(X, \mathcal{K}) = \chi_c(U, \mathcal{F}) = \chi(U, \mathcal{F})$ in virtue of [Lau81b].

For curves, Theorem 3.13 and Example 3.11 specialize to the Grothendieck-Ogg-Shafarevich formula [Ray68] :

Theorem 3.15. *Let* X *be a smooth proper connected curve of genus* g *over an algebraically closed field* k*. Let* $\mathcal{K} \in D^b_{ctf}(X, \Lambda)$ *and let* $U \subset X$ *be a dense open subset where* $\mathcal{K}$ *is locally constant. Then, we have*

$$\chi(X, \mathcal{K}) = (2-2g) \cdot \mathrm{rk}_\Lambda \mathcal{K}_\eta - \sum_{x \in X-U} (\mathrm{dimtot}_x(\mathcal{K}) - \mathrm{rk}_\Lambda \mathcal{K}_x)$$

where η *is the generic point of* X*.*

The combination of Remark 3.14 and Theorem 3.15 gives the following :

Corollary 3.16. *Let* X *be a smooth proper connected curve of genus* g *over an algebraically closed field* k *and assume that* Λ *is a finite field. Let* $U \subset X$ *be an affine dense open subset and put* $D := X - U$*. For every* $\mathcal{L} \in \mathrm{Loc}(U, \Lambda)$*, we have*

$$\begin{cases} h^0(U, \mathcal{L}) \leq \mathrm{rk}_\Lambda \mathcal{L}, \\ h^1(U, \mathcal{L}) \leq (2g - 1 + |D| + |D| \cdot \mathrm{lc}_D(\mathcal{L})) \cdot \mathrm{rk}_\Lambda \mathcal{L}, \\ h^2(U, \mathcal{L}) = 0 \end{cases}$$

4. Ramification of nearby cycles

4.1. In this section, the notations of Section 2.1 are in use. We further assume that Λ is a finite field of characteristic $\ell \neq p$. We put $\mathcal{S} := \mathrm{Spec}\, \mathcal{O}_K$, denote by s its closed point and by $\bar{s}$ an algebraic geometric point above s. We put $\eta = \mathrm{Spec}\, K$ and let η^t be the maximal tame cover of η dominated by $\bar{\eta} = \mathrm{Spec}\, \overline{K}$. For a morphism of finite type $f : \mathcal{X} \to \mathcal{S}$, consider the following diagram with cartesian squares

$$\begin{tikzcd}
\mathcal{X}_{\bar\eta} \arrow[r] \arrow[d, "f_{\bar\eta}"] \arrow[rrr, bend left=40, "\bar{j}"] & \mathcal{X}_{\eta^t} \arrow[r] \arrow[d] \arrow[rr, bend left=25, "j^t"] & \mathcal{X}_\eta \arrow[r] \arrow[d, "f_\eta"] & \mathcal{X} \arrow[d, "f"] & \mathcal{X}_{\bar s} \arrow[l, "\bar\iota"'] \arrow[d, "f_{\bar s}"] \\
\bar\eta \arrow[r] & \eta^t \arrow[r] & \eta \arrow[r] & \mathcal{S} & \bar{s} \arrow[l]
\end{tikzcd}$$

For $\mathcal{K} \in D^+(\mathcal{X}_\eta, \Lambda)$, we denote by $R\Psi(\mathcal{K}, f) = \bar\iota^* R\bar{j}_* \mathcal{K}|_{\mathcal{X}_{\bar\eta}}$ (resp. $R\Psi^t(\mathcal{K}, f) = \bar\iota^* Rj^t_* \mathcal{K}|_{\mathcal{X}_{\eta^t}}$) the nearby cycle complex (resp. tame nearby cycle complex) of $\mathcal{K}$ with respect to $f : \mathcal{X} \to \mathcal{S}$. The nearby cycle complex is an object of $D^+(\mathcal{X}_{\bar s}, \Lambda)$ with a G_K-action and the tame nearby cycle complex is an object of $D^+(\mathcal{X}_{\bar s}, \Lambda)$ with a G_K/P_K-action.

Definition 4.2. For $\mathcal{K} \in D^b_c(\mathcal{X}_\eta, \Lambda)$, we say that $r \in \mathbb{Q}_{\geq 0}$ is a *logarithmic slope* of $R\Psi(\mathcal{K}, f)$ if there exists a closed point $x \in \mathcal{X}_{\bar s}$ and $i \in \mathbb{Z}$ such that r is a logarithmic slope of $R^i\Psi_x(\mathcal{K}, f)$ with respect to the continuous G_K-action. We say that *the logarithmic ramification of* $R\Psi(\mathcal{K}, f)$ *is bounded by* $c \in \mathbb{Q}_{\geq 0}$ if the action $G^{c+}_{K,\log}$ on each $R^i\Psi(\mathcal{K}, f)$ is trivial.

Lemma 4.3 ([HT21, Lemma 5.3]). *Let $\mathcal{K}$ be an object of $D^b_c(\mathcal{X}_\eta, \Lambda)$ and $\mathcal{N}$ a locally constant constructible sheaf of Λ-modules on η. Then, for any closed point $x \in \mathcal{X}_{\bar{s}}$, we have the following canonical G_K/P_K-equivariant isomorphism:*

$$R^i\Psi^t_x(\mathcal{K} \otimes^L_\Lambda f^*_\eta \mathcal{N}, f) \cong (R^i\Psi_x(\mathcal{K}, f) \otimes_\Lambda \mathcal{N}|_{\bar{\eta}})^{P_K} \tag{4.3.1}$$

The proof follows is the same as [HT21, Lemma 5.3] where the residue field of $\mathcal{O}_K$ was assumed to be perfect. However the proof is still valid without this assumption.

Proposition 4.4 ([HT21, Lemma 5.3, Corollary 5.4]). *Let $\mathcal{K}$ be an object of $D^b_c(\mathcal{X}_\eta, \Lambda)$. Then, r is a logarithmic numbering slope of $R\Psi(\mathcal{K}, f)$ if and only if there exists a locally constant constructible sheaf $\mathcal{N}$ of Λ-modules on η whose ramification is logarithmic isoclinic at $s \in \mathcal{S}$ with $r = \mathrm{lc}_s(\mathcal{N})$ such that*

$$R\Psi^t(\mathcal{K} \otimes^L_\Lambda f^*_\eta \mathcal{N}, f) \neq 0.$$

In particular, the logarithmic ramification of $R\Psi(\mathcal{K}, f)$ is bounded by c if and only if, for any locally constant constructible sheaf $\mathcal{N}$ of Λ-modules on η whose ramification is logarithmic isoclinic at $s \in \mathcal{S}$ with $\mathrm{lc}_s(\mathcal{N}) > c$, we have

$$R\Psi^t(\mathcal{K} \otimes^L_\Lambda f^*_\eta \mathcal{N}, f) = 0.$$

Proof. Let $\mathcal{N}$ a locally constant constructible sheaf of Λ-modules on η whose ramification is logarithmic isoclinic at $s \in \mathcal{S}$ with $r = \mathrm{lc}_s(\mathcal{N})$ such that

$$R\Psi^t(\mathcal{K} \otimes^L_\Lambda f^*_\eta \mathcal{N}, f) \neq 0.$$

Then, there exists a closed point $x \in \mathcal{X}_{\bar{s}}$ such that $R\Psi^t_x(\mathcal{K} \otimes^L_\Lambda f^*_\eta \mathcal{N}, f) \neq 0$. By Lemma 4.3, we have $(R^i\Psi_x(\mathcal{K}, f) \otimes_\Lambda \mathcal{N}|_{\bar{\eta}})^{P_K} \neq 0$ for some $i \in \mathbb{Z}$. Suppose that $R^i\Psi_x(\mathcal{K}, f)$ does not have a logarithmic slope r, then all logarithmic slopes of $R^i\Psi_x(\mathcal{K}, f) \otimes_\Lambda \mathcal{N}|_{\bar{\eta}}$ are positive rational numbers. Therefore $R^i\Psi_x(\mathcal{K}, f) \otimes_\Lambda \mathcal{N}|_{\bar{\eta}}$ is purely wild, which contradicts to the fact that $(R^i\Psi_x(\mathcal{K}, f) \otimes_\Lambda \mathcal{N}|_{\bar{\eta}})^{P_K} \neq 0$. Hence $R^i\Psi_x(\mathcal{K}, f)$ has a logarithmic slope r.

Conversely, let $r \in \mathbb{Q}_{\geq 0}$ be a logarithmic slope of $R\Psi(\mathcal{K}, f)$. Then, there exists a closed point $x \in \mathcal{X}_{\bar{s}}$ such that the finite generated Λ-module $M = \bigoplus_{i \in \mathbb{Z}} R^i\Psi_x(\mathcal{K}, f)$ has a logarithmic slope r. Let N be the dual of $M^r_{\log}$. Note that N is also a finite generated Λ-module with a continuous G_K-action of isoclinic logarithmic slope r. Let $\mathcal{N}$ be the locally constant constructible sheaf of Λ-modules on η which is associated to N. We have $\mathrm{lc}_s(\mathcal{N}) = r$. By Lemma 4.3, we have

$$\bigoplus_{i \in \mathbb{Z}} R^i\Psi^t_x(\mathcal{K} \otimes^L_\Lambda f^*_\eta \mathcal{N}, f) \cong (M \otimes_\Lambda N)^{P_K}. \tag{4.4.1}$$

Notice that $N^\vee \otimes_\Lambda N$ is a sub G_K-representation of $M \otimes_\Lambda N$ and $N^\vee \otimes_\Lambda N$ has a quotient with the trivial P_K-action. Hence

$$0 \neq (N^\vee \otimes_\Lambda N)^{P_K} \subseteq (M \otimes_\Lambda N)^{P_K}.$$

Therefore, $R\Psi^t_x(\mathcal{K} \otimes^L_\Lambda f^*_\eta \mathcal{N}, f) \neq 0$. □

Definition 4.5. Let $\mathcal{Z}$ be a reduced closed subscheme of $\mathcal{X}$. We say $(\mathcal{X}, \mathcal{Z})$ is a semi-stable pair over $\mathcal{S}$ if, étale locally, $\mathcal{X}$ is étale over an $\mathcal{S}$-scheme

$$\mathrm{Spec} R[t_1, \cdots, t_d]/(t_{r+1} \cdots t_d - \pi)$$

where $r < d$ and π is a uniformizer of R, and if $\mathcal{Z} = \mathcal{Z}_f \bigcup \mathcal{X}_s$ with $\mathcal{Z}_f$ defined by an ideal $(t_1 \cdots t_m) \subset R[t_1, \cdots, t_d]/(t_{r+1} \cdots t_d - \pi)$ $(m \leq r)$.

Definition 4.6. We say that the henselian trait $\mathcal{S}$ is geometric if $\mathcal{S}$ is the henselization of a smooth scheme S over a perfect field of characteristic $p > 0$ at the generic point of a smooth divisor.

Theorem 4.7. *Assume that $\mathcal{S}$ is geometric. Let $(\mathcal{X}, \mathcal{Z})$ be a semi-stable pair and we assume that $\mathcal{X}$ is smooth over $\mathcal{S}$. Let $\mathcal{U}$ be the complement of $\mathcal{Z}$ in $\mathcal{X}$ and let $j : \mathcal{U} \to \mathcal{X}$ be the canonical injection. Let $\mathcal{F} \in \mathrm{Loc}(\mathcal{U}, \Lambda)$ such that its ramification at generic points of $\mathcal{Z}_f$ is tame and let $\mathrm{lc}(\mathcal{F})$ be the maximum of the set of logarithmic conductors of $\mathcal{F}$ at generic points of the special fiber $\mathcal{X}_s$. Then, the logarithmic ramification of $R\Psi(\mathcal{F}, f)$ is bounded by $\mathrm{lc}(\mathcal{F})$.*

Proof. The proof follows a similar strategy as [HT21, Theorem 5.7]. By Proposition 4.4, it is sufficient to show that, for any locally constant constructible sheaf of Λ-modules $\mathcal{N}$ on η whose ramification is logarithmic isoclinic at s with $\mathrm{lc}_s(\mathcal{N}) > \mathrm{lc}(\mathcal{F})$, we have

$$R\Psi^t(j_!\mathcal{F} \otimes^L_\Lambda f^*_\eta \mathcal{N}, f) = 0. \tag{4.7.1}$$

As in step 2 of [HT21, Theorem 5.7], we reduce to the case where $\mathcal{Z}_f = \emptyset$. Since this is an étale local question, we may assume that $\mathcal{X}_s$ has one irreducible component. Let n be an integer co-prime to p and π a uniformizer of $\mathcal{O}_K$. We put $\mathcal{S}_n = \mathrm{Spec}(\mathcal{O}_K[T]/(T^n - \pi))$ and put $\mathcal{X}_n = \mathcal{X} \times_{\mathcal{S}} \mathcal{S}_n$. We have the following diagram

(4.7.2)
$$\begin{tikzcd}
\mathcal{U}_n \arrow[r, "j_n"] \arrow[d, "h_n"'] \arrow[dr, phantom, "\square"] & \mathcal{X}_n \arrow[d, "g_n"] & \mathcal{X}_s \arrow[l, "\iota_n"'] \arrow[d, equal] \\
\mathcal{U} \arrow[r, "j"'] & \mathcal{X} & \mathcal{X}_s \arrow[l, "\iota"]
\end{tikzcd}$$

We have the following isomorphism

$$R\Psi^t(\mathcal{F} \otimes_\Lambda f^*_\eta \mathcal{N}, f) = \varinjlim_{(n,p)=1} \iota_n^* R j_{n*} h_n^*(\mathcal{F} \otimes_\Lambda f^*_\eta \mathcal{N}).$$

Hence, it is sufficient to show that for every positive integer $n > \dfrac{1}{\mathrm{lc}_s(N) - \mathrm{lc}(\mathcal{F})}$ that is co-prime to p, we have

$$\iota_n^* R j_{n*} h_n^*(\mathcal{F} \otimes_\Lambda f^*_\eta \mathcal{N}) = 0. \tag{4.7.3}$$

Since $\mathcal{S}_n/\mathcal{S}$ is tamely ramified cover at s of degree n, we have

$$n \cdot \mathrm{lc}(\mathcal{F}) = \mathrm{lc}_{\mathcal{X}_s}(h_n^*\mathcal{F}), \tag{4.7.4}$$

$$n \cdot \mathrm{lc}_s(\mathcal{N}) = \mathrm{lc}_s(\mathcal{N}|_{\mathcal{S}_n - \{s\}}). \tag{4.7.5}$$

Hence, we have $\mathrm{lc}_s(\mathcal{N}|_{\mathcal{S}_n - \{s\}}) > \mathrm{lc}_{\mathcal{X}_s}(h_n^*\mathcal{F}) + 1 \geq c_{\mathcal{X}_s}(h_n^*\mathcal{F})$. Notice that $\mathcal{S}$ is geometric, $f : X \to S$ is of finite type and $\mathcal{F}$ and $\mathcal{N}$ are constructible. By spreading out, (4.7.3) is due to Proposition 4.8 below. □

Proposition 4.8. *Let S be a smooth scheme over k, let E be an irreducible divisor of S and put $V := S - E$. Let $f : X \to S$ be a smooth morphism of finite type and consider the following commutative diagram*

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[d, "f_V"'] \arrow[dr, phantom, "\Box"] & X \arrow[d, "f"] \arrow[dr, phantom, "\Box"] & D \arrow[l] \arrow[d] \\ V \arrow[r, "g"'] & S & E \arrow[l] \end{tikzcd}$$

with cartesian squares. Assume that D is irreducible. Let $\mathcal{F} \in \mathrm{Loc}(U, \Lambda)$ and $\mathcal{N} \in \mathrm{Loc}(V, \Lambda)$ such that $\mathcal{N}$ has only one logarithmic slope at the generic point of E and $\mathrm{lc}_E(\mathcal{N}) > c_D(\mathcal{F})$. Then there exists an open dense subset E_0 of E such that

$$(Rj_*(\mathcal{F} \otimes_\Lambda f_V^*\mathcal{N}))|_{f^{-1}(E_0)} = 0.$$

Proof. This is an étale local question. Let $c_1 < c_2 < \cdots < c_r$ be all slopes of the ramification of $\mathcal{N}$ at the generic point of E. We have $c_1 \geq \mathrm{lc}_E(\mathcal{N})$. Let E_0 be an open dense subset of E such that the ramification of $\mathcal{N}$ along E_0 is non-degenerate. Since $f : X \to S$ is smooth, the ramification of $f_V^*\mathcal{N}$ along $D_0 = f^{-1}(E_0)$ is also non-degenerate and $c_1 < c_2 < \cdots < c_r$ are all slopes of the ramification of $f_V^*\mathcal{N}$ at the generic point of D. We have

$$\mathrm{dimtot}_D(\mathcal{F} \otimes_\Lambda f_V^*\mathcal{N}) = \mathrm{rk}_\Lambda \mathcal{F} \cdot \mathrm{dimtot}_D(f_V^*\mathcal{N}) \tag{4.8.1}$$

Let C be a smooth curve over k and $h : C \to X$ a quasi-finite morphism such that:

(a) $y = h^{-1}(D)$ is a closed point of C with $x = h(y) \subset D_0$;

(b) $h : C \to X$ is $SS(j_! f_V^*\mathcal{N})$-transversal.

By [Sai17a, Corollary 3.9], we have

$$NP_y(f_V^*\mathcal{N}|_{C_0}) = m_y(h^*D) \cdot NP_D(f_V^*\mathcal{N}). \tag{4.8.2}$$

By Theorem 2.14, we have

$$c_y(\mathcal{F}|_{C_0}) \leq m_y(h^*D) \cdot c_D(\mathcal{F}). \tag{4.8.3}$$

By (4.8.2) and (4.8.3), and the fact that all slopes of the ramification of $f_V^*\mathcal{N}$ at the generic point of D is larger than that of $\mathcal{F}$, we obtain that

$$\begin{aligned} \mathrm{dimtot}_y((\mathcal{F} \otimes_\Lambda f_V^*\mathcal{N})|_{C_0}) &= \mathrm{rk}_\Lambda \mathcal{F} \cdot \mathrm{dimtot}_y(f_V^*\mathcal{N}|_{C_0}) \\ &= \mathrm{rk}_\Lambda \mathcal{F} \cdot m_y(h^*(DT_X(j_! f_V^*\mathcal{N}))) \\ &= m_y(h^*(DT_X(j_!(\mathcal{F} \otimes_\Lambda f_V^*\mathcal{N})))). \end{aligned}$$

By [Hu23, Proposition 5.6], we obtain that, for any closed point $x \in D_0$, we have

$$SS(j_!(\mathcal{F} \otimes_\Lambda f_V^*\mathcal{N}))_{\bar{x}} \subseteq SS(j_! f_V^*\mathcal{N})_{\bar{x}} = (f^\circ SS(g_!\mathcal{N}))_{\bar{x}}. \tag{4.8.4}$$

In particular, $SS(j_!(\mathcal{F} \otimes_\Lambda f_V^*\mathcal{N}))_x$ has dimension 1 for any closed point $x \in D_0$. For any closed point $x \in D_0$, we take a smooth k-curve and an immersion $h : C \to X$ such that

(a) C and D meet transversely at x;

(b) $h : C \to X$ is $SS(j_!(\mathcal{F} \otimes_\Lambda f_V^*\mathcal{N}))$-transversal.

Since $\mathrm{lc}_E(\mathcal{N}) > c_E(\mathcal{F}) \geq 0$, the ramifications of $f_V^*\mathcal{N}$ and $\mathcal{F} \otimes_\Lambda f_V^*\mathcal{N}$ at the generic of D have slopes $c_1, \ldots, c_r > 1$. By [Sai17a, Corollary 3.9], the ramification of $(\mathcal{F} \otimes_\Lambda f_V^*\mathcal{N})|_C$ at x has slopes $c_1, \ldots, c_r > 1$, in particular, is purely wild. By [Hu23, Corollary 3.12], we obtain that $(Rj_*(\mathcal{F} \otimes_\Lambda f_V^*\mathcal{N}))_x = 0$. Hence $(Rj_*(\mathcal{F} \otimes_\Lambda f_V^*\mathcal{N}))|_{D_0} = 0$. □

5. A conductor estimate

Recollection 5.1. In this section, Λ will denote a finite field of characteristic $\ell \neq p$. Let X be a smooth scheme of finite type over k. Let $D \subset X$ be an effective Cartier divisor and put $j : U := X - D \hookrightarrow X$. We denote by $\mathcal{Q}(X, D)$ the set of triples $(S, h : S \to X, x)$ where S is a smooth affine connected curve over k, where $h : S \to X$ is a quasi-finite morphism over k such that $\{x\} = h(S) \cap D$ and such that $h^{-1}(x)$ is a single closed point of S.

Definition 5.2 ([Hu23, Definition 5.7]). Let $C \subset \mathbb{T}^*X$ be a closed conical subset with basis D and with pure 1-dimensional fibers over D. For $\mathcal{L} \in \mathrm{Loc}(U, \Lambda)$, we say that the ramification of $\mathcal{L}$ along D is C-*isoclinic by restricting to curves* if for every $(S, h : S \to X, x) \in \mathcal{Q}(X, D)$ such that $h : S \to X$ is C-transversal at $s := h^{-1}(x)$, the ramification of $\mathcal{L}|_{S-\{s\}}$ at s is isoclinic and we have

$$C_S(\mathcal{L}|_{S-\{s\}}) = h^*C_X(\mathcal{L}) .$$

Definition 5.2 admits the following purely geometric reformulation :

Lemma 5.3 ([Hu23, Proposition 5.17]). *In the setting of Definition 5.2, the following are equivalent :*

1. *the ramification of $\mathcal{L}$ along D is C-isoclinic by restricting to curves.*
2. *The ramification of $\mathcal{L}$ at each generic point of D is isoclinic and $SS(j_!\mathcal{L}) \subseteq \mathbb{T}^*_X X \bigcup C$.*

When C is simple enough, the inclusion in Lemma 5.3 is an equality.

Lemma 5.4 ([Hu23, Lemma 5.9]). *In the setting of Definition 5.2, assume that $\mathcal{L} \neq 0$ and that the map* $\mathrm{IrrCom}(C) \to \mathrm{IrrCom}(D)$ *induced on the sets of irreducible components is bijective. Then,*

$$SS(j_!\mathcal{L}) = \mathbb{T}^*_X X \bigcup C .$$

Lemma 5.5 ([Hu23, Proposition 5.18]). *Let X be a smooth scheme of finite type over k. Let $D \subset X$ be an effective Cartier divisor and put $j : U := X - D \hookrightarrow X$. Let $C \subset \mathbb{T}^*X$ be a closed conical subset with basis D and pure 1-dimensional fibers over D. Let $\mathcal{L}, \mathcal{N} \in \mathrm{Loc}(U, \Lambda)$ where*

1. *the ramification of $\mathcal{N}$ along D is C-isoclinic by restricting to curves.*
2. *we have $c_Z(\mathcal{N}) > c_Z(\mathcal{L})$ for every irreducible component Z of D.*

Then, the ramification of $\mathcal{L} \otimes_\Lambda \mathcal{N}$ along D is C-isoclinic by restricting to curves and

$$SS(j_!(\mathcal{L} \otimes_\Lambda \mathcal{N})) = SS(j_!\mathcal{N}) \subseteq \mathbb{T}^*_X X \bigcup C ,$$
$$CC(j_!(\mathcal{L} \otimes_\Lambda \mathcal{N})) = \mathrm{rk}_\Lambda \mathcal{L} \cdot CC(j_!\mathcal{N}) .$$

5.6. Conductor estimate. The goal of what follows is to give an estimate for the logarithmic conductor of the direct image of some twisted sheaf on $\mathbb{A}^n_k$ (see Proposition 5.12). The letter Λ will denote a finite field of characteristic $\ell \neq p$.

Lemma 5.7. *Let* $\mathbb{A}^{2,\circ}_k$ *be the complement of the origin in* $\mathbb{A}^2_k = \mathrm{Spec}(k[x,y])$. *Let* D *(resp.* E*) be the divisor of* $\mathbb{A}^{2,\circ}_k$ *defined by* $x = 0$ *(resp.* $y = 1$*). Put* $Z = D \bigcup E$ *and* $j : U := \mathbb{A}^{2,\circ}_k - Z \hookrightarrow \mathbb{A}^{2,\circ}_k$. *Let* $\mathcal{N} \in \mathrm{Loc}(U, \Lambda)$ *be the Artin-Schreier sheaf on* U *defined by*

$$t^p - t = \lambda \left(\frac{y}{x^{p^2}(y-1)} \right)^m \tag{5.7.1}$$

where $\lambda \in k^\times$ *and* $(m,p) = 1$. *Then,*

$$C_{\mathbb{A}^{2,\circ}_k}(j_!\mathcal{N}) = mp^2 \cdot D + (m+1) \cdot E$$

and the ramification of $\mathcal{N}$ *along* Z *is* $\langle dy \rangle \cdot Z$*-isoclinic by restricting to curves.*

Proof. Let $(S, h : S \to \mathbb{A}^{2,\circ}_k, z) \in \mathcal{Q}(\mathbb{A}^2_k, Z)$ such that $h : S \to X$ is $\langle dy \rangle \cdot Z$-transversal at $s := h^{-1}(z)$. By Proposition 3.6, it is enough to show that

$$c_s(\mathcal{N}|_{S-\{s\}}) = mp^2 \cdot m_s(h^*D) + (m+1) \cdot m_s(h^*E) \tag{5.7.2}$$

where $m_s(h^*D)$ (resp. $m_s(h^*E)$) is the multiplicity of h^*D (resp. h^*E) at s. Put $K = \mathrm{Frac}(\widehat{\mathcal{O}}_{S,s})$ and $\mathcal{O}_K = \widehat{\mathcal{O}}_{S,s}$. We first assume that $z = (0,b) \in D - \{(0,1)\}$. Choose a uniformizer T of $\mathcal{O}_{S,s}$ such that $h : S \to \mathbb{A}^{2,\circ}_k$ is given by

$$\begin{aligned} x &\mapsto uT^\alpha, \quad u \in \mathcal{O}^\times_{S,s}, \alpha \geq 1 \\ y &\mapsto b + T. \end{aligned}$$

Then, the restriction $\mathcal{N}|_{\mathrm{Spec}(K)}$ corresponds to an Artin-Schreier cover defined by

$$t^p - t = \frac{\lambda(b+T)^m}{(uT^\alpha)^{mp^2}(b-1+T)^m}.$$

Note that

$$\left(\frac{b+T}{b-1+T}\right)^m = \left(\frac{b}{b-1}\right)^m \left(1 - \frac{m}{b(b-1)}T + \cdots\right) \tag{5.7.3}$$

Since $(m,p) = 1$, the coefficient of T is a unit of k. Hence, $\mathcal{N}|_{\mathrm{Spec}(K)}$ corresponds to an Artin-Schreier cover defined by

$$t^p - t = \frac{\lambda' + u'T}{(uT^\alpha)^{mp^2}} \quad \lambda' \in k^\times, u' \in \widehat{\mathcal{O}}^\times_{S,s}$$

At the cost of taking a αmp^2 root of λ' in k and making a change of variable, we can suppose that $\lambda' = 1$. On the other hand, we have

$$\begin{aligned} t^p - t - \frac{1+u'T}{(uT^\alpha)^{mp^2}} &= t^p - t - \frac{1}{(uT^\alpha)^{mp^2}} - \frac{u'}{u^{mp^2}T^{\alpha mp^2 - 1}} \\ &= t'^p - t' - \frac{u''}{T^{\alpha mp^2 - 1}} \end{aligned}$$

where we put $t' = t - \frac{1}{(uT^{\alpha})^{mp}}$ and where $u'' \in \widehat{\mathcal{O}}_{S,s}^{\times}$. By Example 2.6, we deduce

$$c_s(\mathcal{N}|_{S-\{s\}}) = \mathrm{dimtot}_s(\mathcal{N}|_{S-\{s\}}) = \alpha mp^2 = mp^2 \cdot m_s(h^*E).$$

We now assume that $z = (a, 1) \in E - \{(0, 1)\}$. Choose a uniformizer T of $\mathcal{O}_{S,s}$ such that $h : S \to \mathbb{A}_k^{2,\circ}$ is given by

$$\begin{aligned} x &\mapsto a + uT^{\alpha}, \quad u \in \mathcal{O}_{S,s}^{\times}, \alpha \geq 1, \\ y &\mapsto 1 + T. \end{aligned}$$

The restriction $\mathcal{N}|_{\mathrm{Spec}(K)}$ corresponds to an Artin-Schreier cover defined by

$$t^p - t = \frac{\lambda(1+T)^m}{(a+uT^{\alpha})^{mp^2}T^m} .$$

Since $(m, p) = 1$, there is $v \in \mathcal{O}_K^{\times}$ with $v^m = \dfrac{(a+uT^{\alpha})^{mp^2}}{\lambda(1+T)^m}$. Hence $\mathcal{N}|_{\mathrm{Spec}(K)}$ corresponds to an Artin-Schreier cover defined by

$$t^p - t = \frac{1}{(vT)^m} .$$

From Example 2.6, we deduce

$$c_s(\mathcal{N}|_{S-\{s\}}) = \mathrm{dimtot}_s(\mathcal{N}|_{S-\{s\}}) = m + 1 = (m+1) \cdot m_s(h^*E) .$$

We now assume $z = (0, 1) = D \bigcap E$. Choose a uniformizer T of $\mathcal{O}_{S,s}$ such that $h : S \to \mathbb{A}_k^{2,\circ}$ is given by

$$\begin{aligned} x &\mapsto uT^{\alpha}, \quad u \in \mathcal{O}_{S,s}^{\times}, \alpha \geq 1, \\ y &\mapsto 1 + T. \end{aligned}$$

The restriction $\mathcal{N}|_{\mathrm{Spec}(K)}$ corresponds to an Artin-Schreier cover defined by

$$t^p - t = \frac{\lambda(1+T)^m}{u^{mp^2}T^{\alpha mp^2+m}} .$$

Since $\alpha mp^2 + m$ is coprime to p, there is $v \in \mathcal{O}_K^{\times}$ such that $v^{\alpha mp^2+m} = \dfrac{u^{mp^2}}{\lambda(1+T)^m}$. Hence $\mathcal{N}|_{\mathrm{Spec}(K)}$ corresponds to an Artin-Schreier cover defined by

$$t^p - t = \frac{1}{(vT)^{\alpha mp^2+m}} .$$

By Example 2.6, we deduce

$$c_s(\mathcal{N}|_{S-\{s\}}) = \mathrm{dimtot}_s(\mathcal{N}|_{S-\{s\}}) = \alpha mp^2 + m + 1 = mp^2 \cdot m_s(h^*D) + (m+1) \cdot m_s(h^*E) .$$

$\square$

Corollary 5.8. *In the situation from Lemma 5.7, we have*

$$SS(j_!\mathcal{N}) = \mathbb{T}^*_{\mathbb{A}^{2,\circ}_k}\mathbb{A}^{2,\circ}_k \bigcup \langle dy\rangle \cdot Z,$$
$$CC(j_!\mathcal{N}) = [\mathbb{T}^*_{\mathbb{A}^{2,\circ}_k}\mathbb{A}^{2,\circ}_k] + mp^2 \cdot [D \cdot \langle dy\rangle] + (m+1) \cdot [\langle dy\rangle \cdot E].$$

Proof. The computation of SS is a consequence of Lemma 5.4 and Lemma 5.7. Then, the computation of CC follows from [Sai17b, Theorem 7.6]. ☐

Construction 5.9. Put $\mathbb{P}^n_k = \mathrm{Proj}(k[x_0, \ldots, x_n])$ with $n \geq 2$. Let H be the hyperplane defined by $x_0 = 0$ and H_n the hyperplane defined by $x_n = 0$. Put

$$\mathbb{A}^n_k = \mathrm{Spec}\left(k\left[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right]\right), \quad U_n = \mathrm{Spec}\left(k\left[\frac{x_0}{x_n}, \frac{x_1}{x_n} \ldots, \frac{x_{n-1}}{x_n}\right]\right).$$

Put

$$z_0 = \frac{x_0}{x_n}, z_1 = \frac{x_1}{x_n}, \ldots, z_{n-1} = \frac{x_{n-1}}{x_n}.$$

Hence, $H \cap U_n$ is given by $z_0 = 0$. Let $\mathrm{pr} : \mathbb{A}^n_k \to \mathbb{A}^1_k$ be the projection to the first coordinate. Let Λ be a finite field of characteristic $\ell \neq p$. Let $\mathcal{N}_\lambda$ be the Artin-Schreier sheaf of Λ-modules on $\mathbb{A}^1_k$ defined by the equation $t^p - t = \lambda x^m$ for some $\lambda \in k^\times$ and $(m, p) = 1$. Then, the pull-back $(\mathrm{pr}^* \mathcal{N}_\lambda)|_{U_n \cap \mathbb{A}^n_k}$ is the Artin-Schreier sheaf defined by the equation

$$t^p - t = \lambda \left(\frac{z_1}{z_0}\right)^m.$$

Consider the closed point $x = (0, 1, \ldots, 1) \in H \cap U_n$. Let $X \to \mathbb{P}^n_k$ be the blow-up of $\mathbb{P}^n_k$ at x. Let E be the exceptional divisor. Let $Gr(n, 2)$ be the grassmanniann of projective lines in $\mathbb{P}^n_k$ and let $Q \to Gr(n, 2)$ be the universal projective line. Let $\mathbb{P}^{n-1}_k \subset Gr(n, 2)$ be the subset of projective lines passing through x. Then, there is a pull-back square

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\square"] & Q \arrow[r] \arrow[d] & \mathbb{P}^n_k \\
\mathbb{P}^{n-1}_k \arrow[r] & Gr(n,2) &
\end{tikzcd}$$

exhibiting X as the closed subscheme of $\mathbb{P}^n_k \times \mathbb{P}^{n-1}_k$ formed by couples (y, L) where $x, y \in L$. Furthermore, the restriction $\pi|_E : E \to \mathbb{P}^{n-1}_k$ is an isomorphism. If H' is the strict transform of H and if $D \subset \mathbb{P}^{n-1}_k$ is the hyperplane $\pi(H')$, there is a commutative square

(5.9.1)
$$\begin{tikzcd}
H' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "\pi"] \arrow[dr, phantom, "\square"] & W \arrow[l] \arrow[d, "\pi_W"'] & \mathbb{A}^n_k \arrow[l, "j_W"] \arrow[ll, bend right, "j"'] \arrow[dl, "\pi_A"] \\
D \arrow[r] & \mathbb{P}^{n-1}_k & \mathbb{A}^{n-1}_k \arrow[l] &
\end{tikzcd}.$$

Proposition 5.10. *In the setting of Construction 5.9, let* $\mathcal{L} \in \mathrm{Loc}(\mathbb{A}^n_k, \Lambda)$ *and let* $\mathcal{N}_\lambda$ *be an Artin-Schreier sheaf of* Λ*-modules on* $\mathbb{A}^1_k$ *defined by* $t^p - t = \lambda x^m$ *for some* $\lambda \in k^\times$ *and* $(m,p) = 1$*. Assume that* $m = \mathrm{lc}_\infty(\mathcal{N}_\lambda) > \mathrm{lc}_H(\mathcal{L}) + 1$*. Then the map* $\pi : X \to \mathbb{P}^{n-1}_k$ *is universally locally acyclic relative to* $\jmath_!(\mathrm{pr}^* \mathcal{N}_\lambda \otimes_\Lambda \mathcal{L})$ *at every point of* E *not in* H'*.*

Proof. Since the restriction $\pi|_E : E \to \mathbb{P}^{n-1}_k$ is an isomorphism, it is enough by [Lau81a] to show that for every $L \in \mathbb{P}^{n-1}_k - D$, the number

$$\mathrm{dimtot}_{(x,L)}(\jmath_!(\mathrm{pr}^* \mathcal{N}_\lambda \otimes_\Lambda \mathcal{L})|_{\pi^{-1}(L)})$$

does not depend on L. Notice that for every $L \in \mathbb{P}^{n-1}_k - D$, the composition

$$\pi^{-1}(L) - \{(x,L)\} \to \mathbb{A}^n_k \xrightarrow{\mathrm{pr}} \mathbb{A}^1_k$$

is an isomorphism. Hence, we have

$$\mathrm{lc}_{(x,L)}(\jmath_! \,\mathrm{pr}^* \mathcal{N}_\lambda|_{\pi^{-1}(L)}) = \mathrm{lc}_\infty(\mathcal{N}_\lambda) = m.$$

By Theorem 2.14 applied to the composition $\pi^{-1}(L) \to X \xrightarrow{\pi} \mathbb{P}^n_k$ and to $\mathcal{L}$, we obtain that for $L \in \mathbb{P}^{n-1}_k - D$, we have

$$\mathrm{lc}_{(x,L)}(\jmath_!\mathcal{L}|_{\pi^{-1}(L)}) \leq \mathrm{lc}_E(\mathcal{L}) \leq \mathrm{lc}_H(\mathcal{L}) < m - 1.$$

By Lemma 2.5 and Lemma 2.7, for any $L \in \mathbb{P}^{n-1}_k - D$, we deduce

$$\begin{aligned}\mathrm{dimtot}_{(x,L)}(\jmath_!(\mathrm{pr}^* \mathcal{N}_\lambda \otimes_\Lambda \mathcal{L})|_{\pi^{-1}(L)}) &= \mathrm{sw}_{(x,L)}(\jmath_!(\mathrm{pr}^* \mathcal{N}_\lambda \otimes_\Lambda \mathcal{L})|_{\pi^{-1}(L)}) + \mathrm{rk}_\Lambda \mathcal{L} \\ &= (m+1) \cdot \mathrm{rk}_\Lambda \mathcal{L}.\end{aligned}$$

This concludes the proof of Proposition 5.10. ☐

Proposition 5.11. *In the setting of Construction 5.9, let* $\mathcal{L} \in \mathrm{Loc}(\mathbb{A}^n_k, \Lambda)$ *and let* $\mathcal{N}_\lambda$ *be an Artin-Schreier sheaf of* Λ*-modules on* $\mathbb{A}^1_k$ *defined by* $t^p - t = \lambda x^m$ *for some* $\lambda \in k^\times$ *and* $(m,p) = 1$*. Assume that* $m = \mathrm{lc}_\infty(\mathcal{N}_\lambda) > \mathrm{lc}_H(\mathcal{L}) + 1$*. Then* $\pi : X \to \mathbb{P}^{n-1}_k$ *is universally locally acyclic relative to* $\jmath_!(\mathrm{pr}^* \mathcal{N}_\lambda \otimes_\Lambda \mathcal{L})$ *at a dense open subset of* $H' \bigcap E$*.*

Proof. We are going to achieve universal local acyclicity after a suitable radicial pull-back. We learned this argument in [Sai21, Corollary 1.5.7]. Consider the pull-back square

$$\begin{tikzcd} Y_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "\pi"] \\ U_n \arrow[r] & \mathbb{P}^n_k \end{tikzcd}$$

so that Y_n is the blow-up of U_n at x. Hence, $Y_n = \{(y,L) \in U_n \times \mathbb{P}^{n-1}_k \text{ such that } y \in L\}$. Using the coordinates of U_n from Construction 5.9, the scheme Y_n is the subscheme of $U_n \times \mathbb{P}^{n-1}_k$ defined by the equations

$$z_0 u_i = (z_i - 1)u_0, (z_i - 1)u_j = (z_j - 1)u_i \ \text{ for } \ i \neq j, \ \ i,j \in \{1,2,\ldots,n-1\}.$$

Let $W_1 \subset \mathbb{P}^{n-1}_k$ be the complement of the hyperplane defined by $u_1 = 0$. Consider the pull-back square

$$\begin{tikzcd}
V_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y_n \arrow[d] \\
W_1 \arrow[r] & \mathbb{P}^{n-1}_k
\end{tikzcd}$$

In particular V_1 is a locally closed subset of $U_n \times W_1$. If we put $s_i = u_i/u_1$ for $i \neq 1$, then $V_1 \simeq \operatorname{Spec}(k[s_0, s_2, \dots, s_{n-1}, s])$ such that $V_1 \to U_n$ is given by

$$\begin{aligned}
z_0 &\mapsto s_0(s-1), \\
z_1 &\mapsto s, \\
z_i &\mapsto s_i(s-1)+1, \quad (2 \leq i \leq n-1).
\end{aligned}$$

and $V_1 \to W_1$ is the projection on the first $(n-1)$-coordinates. Put $E_1 = V_1 \cap E$ and $H_1' = V_1 \cap H'$. Then, $V_1 \cap \pi^{-1}(H) = H_1' \cup E_1$ is defined by $s_0(s-1) = 0$ and if we denote by Ω its complement in V_1, the pull-back $(\mathrm{pr}^* \mathcal{N}_\lambda)|_\Omega$ is the Artin-Schreier sheaf defined by the equation

$$t^p - t = \lambda \left(\frac{s}{s_0(s-1)} \right)^m .$$

Consider the cube with pull-back faces

$$\begin{tikzcd}
V_1 \arrow[rr] \arrow[dr, "\mathrm{Fr}"] \arrow[dd] & & \mathbb{A}^2_k \arrow[dr] \arrow[dd] & \\
& V_1 \arrow[rr] \arrow[dd] & & \mathbb{A}^2_k \arrow[dd] \\
W_1 \arrow[rr] \arrow[dr] & & \mathbb{A}^1_k \arrow[dr, "\mathrm{Fr}_1"] & \\
& W_1 \arrow[rr] & & \mathbb{A}^1_k
\end{tikzcd}$$

where the top horizontal arrows are $(s_0, s_2, \dots, s_{n-1}, s) \to (s_0, s)$, the lower horizontal arrows are $(s_0, s) \to s_0$ and where $\mathrm{Fr}_1 : s_0 \to s_0^{p^2}$. Let $Z \subset H_1'$ defined by the ideal (s, s_0) and let $j : \Omega \to V_1 - Z$ be the canonical inclusion. Then, $\mathrm{Fr}^* j_!(\mathrm{pr}^* \mathcal{N}_\lambda)|_\Omega$ is the Artin-Schreier sheaf defined by the equation

$$t^p - t = \lambda \left(\frac{s}{s_0^{p^2}(s-1)} \right)^m .$$

Since $V_1 \to \mathbb{A}^2_k$ is smooth, Corollary 5.8 and [Sai17b, Theorem 7.6] give

$$SS(\mathrm{Fr}^* j_!(\mathrm{pr}^* \mathcal{N}_\lambda)|_\Omega) = \mathbb{T}^*_{(V_1 - Z)}(V_1 - Z) \bigcup ((H_1' - Z) \cup E_1) \cdot \langle ds \rangle$$

and

$$CC(\mathrm{Fr}^* j_!(\mathrm{pr}^*\mathcal{N}_\lambda)|_\Omega) = (-1)^n[\mathbb{T}^*_{(V_1-Z)}(V_1-Z)] + (-1)^n mp^2\cdot[(H'_1-Z)\cdot\langle du\rangle] \\ + (-1)^n(m+1)\cdot[E_1\cdot\langle du\rangle].$$

By Theorem 2.14, we have

$$c_{H'}(\mathrm{Fr}^* j_!(\mathcal{L}|_\Omega)) \leq p^2\cdot c_H(\mathcal{L}) \leq p^2\cdot(\mathrm{lc}_H(\mathcal{L})+1) < mp^2 = c_{H'}(\mathrm{Fr}^* j_!(\mathrm{pr}^*\mathcal{N}_\lambda|_\Omega))$$

and

$$c_E(\mathrm{Fr}^* j_!(\mathcal{L}|_\Omega)) \leq c_H(\mathcal{L}) \leq \mathrm{lc}_H(\mathcal{L})+1 < m+1 = c_E(\mathrm{Fr}^* j_!(\mathrm{pr}^*\mathcal{N}_\lambda|_\Omega)).$$

By Lemma 5.7 and Lemma 5.5, we have

$$SS(\mathrm{Fr}^* j_!((\mathcal{L}\otimes_\Lambda \mathrm{pr}^*\mathcal{N}_\lambda)|_\Omega)) = SS(\mathrm{Fr}^* j_!(\mathrm{pr}^*\mathcal{N}_\lambda)|_\Omega))$$

and

$$CC(\mathrm{Fr}^* j_!((\mathcal{L}\otimes_\Lambda \mathrm{pr}^*\mathcal{N}_\lambda)|_\Omega)) = \mathrm{rk}_\Lambda\,\mathcal{L}\cdot CC(\mathrm{Fr}^* j_!(\mathrm{pr}^*\mathcal{N}_\lambda)|_\Omega).$$

Hence $V_1 \to W_1$ is $SS(\mathrm{Fr}^* j_!((\mathcal{L}\otimes_\Lambda \mathrm{pr}^*\mathcal{N}_\lambda)|_\Omega))$-transversal. Thus, $V_1 \to W_1$ is universally locally acyclic with respect to $\mathrm{Fr}^* j_!((\mathcal{L}\otimes_\Lambda \mathrm{pr}^*\mathcal{N}_\lambda)|_\Omega)$. Since Fr is a universal homeomorphism, we deduce that $V_1 \to W_1$ is universally locally acyclic with respect to $j_!((\mathcal{L}\otimes_\Lambda \mathrm{pr}^*\mathcal{N}_\lambda)|_\Omega)$. Since Z is disjoint from $H'\cap E$, we deduce that $\pi : X \to \mathbb{P}^{n-1}_k$ is universally locally acyclic with respect to $\jmath_!(\mathrm{pr}^*\mathcal{N}_\lambda\otimes_\Lambda\mathcal{L})$ at every point of $V_1\cap(H'\bigcap E)$. This concludes the proof of Proposition 5.11. □

Proposition 5.12. *Let $n \geq 2$ be an integer. In the setting of Construction 5.9, let $\mathcal{L} \in \mathrm{Loc}(\mathbb{A}^n_k,\Lambda)$ and let $\mathcal{N}_\lambda$ be an Artin-Schreier sheaf of Λ-modules on $\mathbb{A}^1_k$ defined by $t^p - t = \lambda x^m$ for some $\lambda \in k^\times$ and $(m,p) = 1$. Assume that $m = \mathrm{lc}_\infty(\mathcal{N}_\lambda) > \mathrm{lc}_H(\mathcal{L}) + 1$. Then, the following holds :*

1. *the complex $R\pi_{A!}(\mathrm{pr}^*\mathcal{N}_\lambda\otimes_\Lambda\mathcal{L})$ is concentrated in degree 1, has locally constant constructible cohomology sheaves and its formation commutes with base change.*
2. *We have*

$$\mathrm{lc}_D(R^1\pi_{A!}(\mathrm{pr}^*\mathcal{N}_\lambda\otimes_\Lambda\mathcal{L})) \leq m.$$

Proof. With the notations from the diagram (5.9.1), we have

$$R\pi_{A!}(\mathrm{pr}^*\mathcal{N}_\lambda\otimes_\Lambda\mathcal{L}) = R\pi_{W*}\jmath_{W!}(\mathrm{pr}^*\mathcal{N}_\lambda\otimes_\Lambda\mathcal{L}).$$

Hence, (1) follows by Proposition 5.10. We now bound the logarithmic conductor of $R\pi_{A!}(\mathrm{pr}^*\mathcal{N}_\lambda\otimes_\Lambda\mathcal{L})$ at the generc point ξ of D. To do this, observe that the proper base change gives

$$(R\pi_*\jmath_!(\mathrm{pr}^*\mathcal{N}_\lambda\otimes_\Lambda\mathcal{L}))|_{\mathbb{A}^{n-1}_k} \simeq R\pi_{A!}(\mathrm{pr}^*\mathcal{N}_\lambda\otimes_\Lambda\mathcal{L})\ .$$

Hence, we have to bound the logarithmic conductor of $R\pi_{*}\jmath_{!}(\mathrm{pr}^{*}\mathcal{N}_{\lambda}\otimes_{\Lambda}\mathcal{L})$ at ξ. Put $\mathcal{S}:=\mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{\mathbb{P}^{n-1}_{k},\bar{\xi}})$ and consider the following pull-back squares

$$\begin{tikzcd}
\mathcal{X}_{\bar{\xi}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[r] \arrow[d, "\pi_{\mathcal{S}}"] \arrow[dr, phantom, "\square"] & X \arrow[d, "\pi"] \\
\bar{\xi} \arrow[r] & \mathcal{S} \arrow[r] & \mathbb{P}^{n-1}_{k} .
\end{tikzcd}$$

Since the logarithmic conductor of $R\pi_{*}\jmath_{!}(\mathrm{pr}^{*}\mathcal{N}_{\lambda}\otimes_{\Lambda}\mathcal{L})$ at ξ is smaller than the logarithmic conductors of the

$$(R^{*}\Psi_{\pi_{S}}(\jmath_{!}(\mathrm{pr}^{*}\mathcal{N}_{\lambda}\otimes_{\Lambda}\mathcal{L})|_{\mathcal{S}}))_{y}, y\in\mathcal{X}_{\bar{\xi}}$$

we are left to bound the logarithmic conductors of the above Galois modules. Put $\mathcal{E}:=\mathcal{S}\times_{\mathbb{P}^{n-1}_{k}}E$. Since $\pi|_{E}:E\to\mathbb{P}^{n-1}_{k}$ is an isomorphism, so is the pullback $\pi_{\mathcal{S}}|_{\mathcal{E}}:\mathcal{E}\to\mathcal{S}$. Hence, the closed subscheme $\mathcal{E}\subset\mathcal{X}$ meets the special fibre of $\pi_{\mathcal{S}}:\mathcal{X}\to\mathcal{S}$ at a unique point $\bar{\eta}$ lying over the generic point of $H'\cap E$. By Proposition 5.11, the map $\pi:X\to\mathbb{P}^{n-1}_{k}$ is universally locally acyclic with respect to $\jmath_{!}(\mathrm{pr}^{*}\mathcal{N}_{\lambda}\otimes_{\Lambda}\mathcal{L})$ in an open neighborhood of the generic point of $H'\bigcap E$. Hence, the pull-back $\pi_{\mathcal{S}}:\mathcal{X}\to\mathcal{S}$ is universally locally acyclic at $\bar{\eta}$ with respect to $\jmath_{!}(\mathrm{pr}^{*}\mathcal{N}_{\lambda}\otimes_{\Lambda}\mathcal{L})|_{\mathcal{S}}$. In particular,

$$(R^{*}\Psi_{\pi_{S}}(\jmath_{!}(\mathrm{pr}^{*}\mathcal{N}_{\lambda}\otimes_{\Lambda}\mathcal{L})|_{\mathcal{S}}))_{\bar{\eta}}\simeq 0.$$

By Theorem 4.7, the logarithmic conductor of $(R^{*}\Psi_{\pi_{S}}(j_{!}(\mathrm{pr}^{*}\mathcal{N}_{\lambda}\otimes_{\Lambda}\mathcal{L})))_{y}$ is smaller than m for every $y\in X_{\bar{\xi}}$ with $y\neq\bar{\eta}$. This concludes the proof of Proposition 5.12. □

6. Estimates for Betti numbers on affine spaces

In this section, we assume that k is algebraically closed of characteristic $p>0$ and that Λ is a finite field of characteristic $\ell\neq p$. Let $\mathbb{A}^{n}_{k}=\mathrm{Spec}(k[x_{1},\dots,x_{n}])$ and let $\mathbb{P}^{n}_{k}=\mathrm{Proj}(k[x_{0},\dots,x_{n}])$ be the canonical completion of $\mathbb{A}^{n}_{k}$ obtained by adding a homogeneous coordinate x_{0}. Let $j:\mathbb{A}^{n}_{k}\hookrightarrow\mathbb{P}^{n}_{k}$ be the inclusion and let $H:=\mathbb{P}^{n}_{k}-\mathbb{A}^{n}_{k}$ be the hyperplane at infinity. The goal of this subsection is to prove the following

Theorem 6.1. *Let k be an algebraically closed field of characteristic $p>0$ and let Λ be a finite field of characteristic $\ell\neq p$. For every $0\leq i\leq n$ and every $\mathcal{L}\in\mathrm{Loc}(\mathbb{A}^{n}_{k},\Lambda)$, we have*

$$h^{i}(\mathbb{A}^{n}_{k},\mathcal{L})\leq b_{i}(\mathrm{lc}_{H}(\mathcal{L}))\cdot\mathrm{rk}_{\Lambda}\mathcal{L}$$

where the b_{i} are defined in Definition 1.2.

Remark 6.2. The estimates from Theorem 6.1 do not depend on the base field k nor the coefficients.

Theorem 6.1 will be proved in subsection 6.9. Before this, let us draw some immediate consequences.

Corollary 6.3. *Let* k *be an algebraically closed field of characteristic* $p > 0$ *and let* Λ *be a finite field of characteristic* $\ell \neq p$*. For every* $\mathcal{L} \in \mathrm{Loc}(\mathbb{A}^n_k, \Lambda)$*, we have*

$$\begin{aligned}
&h^0(\mathbb{A}^n_k, \mathcal{L}) \leq \mathrm{rk}_\Lambda \mathcal{L}, \\
&h^1(\mathbb{A}^n_k, \mathcal{L}) \leq \mathrm{lc}_H(\mathcal{L}) \cdot \mathrm{rk}_\Lambda \mathcal{L} \\
&h^2(\mathbb{A}^n_k, \mathcal{L}) \leq (\mathrm{lc}_H(\mathcal{L})^2 + 7\,\mathrm{lc}_H(\mathcal{L}) + 9) \cdot \mathrm{rk}_\Lambda \mathcal{L} \\
&h^i(\mathbb{A}^n_k, \mathcal{L}) \leq (\mathrm{lc}_H(\mathcal{L}) + 3i - 3) \prod_{j=1}^{i-1} (\mathrm{lc}_H(\mathcal{L}) + 3j + 1) \cdot \mathrm{rk}_\Lambda \mathcal{L} \text{ for every } 3 \leq i \leq n.
\end{aligned}$$

Proof. It is a direct consequence of Lemma 10.1 and Theorem 6.1. ☐

Corollary 6.4. *Let* k *be an algebraically closed field of characteristic* $p > 0$ *and let* Λ *be a finite field of characteristic* $\ell \neq p$*. For every* $0 \leq i \leq n$ *and every* $\mathcal{L} \in \mathrm{Loc}(\mathbb{A}^n_k, \Lambda)$*, we have*

$$h_c^{2n-i}(\mathbb{A}^n_k, \mathcal{L}) \leq b_i(\mathrm{lc}_H(\mathcal{L})) \cdot \mathrm{rk}_\Lambda \mathcal{L}\,.$$

Proof. This follows from Theorem 6.1 by Poincaré duality and the fact that a Λ-module and its dual have the same logarithmic slopes. ☐

Example 6.5. The degrees of the polynomials appearing in Theorem 6.1 are optimal. Indeed, if $m \geq 1$ is an integer coprime to p and if $\mathcal{N}$ is the Artin-Schreier sheaf on $\mathbb{A}^1_k$ corresponding to $t^p - t = x^m$, then $h^i(\mathbb{A}^1_k, \mathcal{N}) = 0$ for $i \neq 1$ and $h^1(\mathbb{A}^1_k, \mathcal{N}) = m - 1$ by Theorem 3.15. For $1 \leq i \leq n$, let $\mathrm{pr}_i : \mathbb{A}^n_k \to \mathbb{A}^1_k$ be the projection on the i-th coordinate and put

$$\mathcal{M} := \mathrm{pr}_1^* \mathcal{N} \otimes_\Lambda \cdots \otimes_\Lambda \mathrm{pr}_i^* \mathcal{N}\,.$$

Then, $\mathcal{M}$ is the Artin-Schreier sheaf on $\mathbb{A}^n_k$ corresponding to $t^p - t = x_1^m + \cdots + x_i^m$. By Example 2.6, we have $\mathrm{lc}_H(\mathcal{M}) = m$. On the other hand, the Künneth formula for the étale cohomology [SP23, 0F1P] yields

$$h^i(\mathbb{A}^n_k, \mathcal{M}) = h^1(\mathbb{A}^1_k, \mathcal{N})^i = (m-1)^i\,.$$

Proposition 6.6. *Let* $n \geq 2$ *be an integer. Assume that Theorem 6.1 is valid for* $\mathbb{A}^{n-1}_k$*. For every* $\mathcal{L} \in \mathrm{Loc}(\mathbb{A}^n_k, \Lambda)$ *and every* $i = 0, 1, \dots, n-1$*, we have*

$$h^i(\mathbb{A}^n_k, \mathcal{L}) \leq b_i(\mathrm{lc}_H(\mathcal{L})) \cdot \mathrm{rk}_\Lambda \mathcal{L}\,.$$

Proof. By [Sai21, Lemma 1.3.7], there is a hyperplane $D \neq H$ such that the map $\iota : D \hookrightarrow \mathbb{P}^n_k$ is properly $SS(Rj_*\mathcal{L})$-transversal. Put $U := \mathbb{P}^n_k - D$. The cartesian squares

$$\begin{tikzcd}
D_0 \arrow[r, "\iota_0"] \arrow[d, "j_D"'] \arrow[dr, phantom, "\square"] & \mathbb{A}^n_k \arrow[d, "j"] \arrow[dr, phantom, "\square"] & U_0 \arrow[l, "\jmath_0"'] \arrow[d, "j_U"] \\
D \arrow[r, "\iota"'] & \mathbb{P}^n_k & U \arrow[l, "\jmath"]
\end{tikzcd}$$

give rise to the distinguished triangle

$$\jmath_!\jmath^* Rj_*\mathcal{L} \to Rj_*\mathcal{L} \to \iota_*\iota^* Rj_*\mathcal{L} \xrightarrow{+1}$$

Since $\jmath$ is an open immersion, we have

$$\jmath^* Rj_* \mathcal{L} \simeq Rj_{U*}\jmath_0^* \mathcal{L} \,.$$

Since $\iota : D \to \mathbb{P}^n_k$ is properly $SS(Rj_*\mathcal{L})$-transversal, Proposition 3.7 gives

$$\iota^* Rj_* \mathcal{L} \simeq Rj_{D*}\iota_0^* \mathcal{L} \,.$$

Hence, applying $R\Gamma(\mathbb{P}^n_k, -)$ to the above triangle yields a distinguished triangle

$$R\Gamma_c(U, Rj_{U*}\jmath_0^*\mathcal{L}) \to R\Gamma(\mathbb{A}^n_k, \mathcal{L}) \to R\Gamma(D_0, \iota_0^*\mathcal{L}) \xrightarrow{+1}$$

Note that $Rj_{U*}\jmath_0^*\mathcal{L}[n]$ is a perverse sheaf on $U \simeq \mathbb{A}^n_k$. By Artin's vanishing theorem, we deduce

$$H^i_c(U, Rj_{U*}\jmath_0^*\mathcal{L}) = 0$$

for $i = 0, 1, \dots, n-1$. Hence,

$$h^i(\mathbb{A}^n_k, \mathcal{L}) \leq h^i(D_0, \iota_0^*\mathcal{L})$$

for $i = 0, 1, \dots, n-1$. Note that $D \simeq \mathbb{P}^{n-1}_k$, that $D \bigcap H$ is a hyperplane of D with complement $D_0 \simeq \mathbb{A}^{n-1}_k$ in D. By Theorem 2.14, we have

$$\mathrm{lc}_{D\cap H}(\iota_0^*\mathcal{L}) \leq \mathrm{lc}_H(\mathcal{L}) \,.$$

Since Theorem 6.1 is valid on $\mathbb{A}^{n-1}_k$ by assumption, we get for every $i = 0, 1, \dots, n-1$,

$$h^i(D_0, \iota_0^*\mathcal{L}) \leq b_i(\mathrm{lc}_{D\cap H}(\iota_0^*\mathcal{L})) \cdot \mathrm{rk}_\Lambda(\iota_0^*\mathcal{L}) \leq b_i(\mathrm{lc}_H(\mathcal{L})) \cdot \mathrm{rk}_\Lambda \mathcal{L}.$$

This concludes the proof of Proposition 6.6. □

Proposition 6.7. *Let $n \geq 2$ be an integer. Assume that Theorem 6.1 is valid for $\mathbb{A}^{n-1}_k$. Let $\mathcal{L} \in \mathrm{Loc}(\mathbb{A}^n_k, \Lambda)$. Let $\mathrm{pr} : \mathbb{A}^n_k \to \mathbb{A}^1_k$ be the first projection. Let $\mathcal{N}_\lambda$ be an Artin-Schreier sheaf of Λ-modules on $\mathbb{A}^1_k$ defined by $t^p - t = \lambda x^m$ for some $\lambda \in k^\times$ and $(m, p) = 1$. Assume that $m = \mathrm{lc}_\infty(\mathcal{N}_\lambda) > \mathrm{lc}_H(\mathcal{L}) + 1$. Then, we have*

$$\chi(\mathbb{A}^n_k, \mathrm{pr}^* \mathcal{N}_\lambda \otimes_\Lambda \mathcal{L}) \geq -\left(\sum_{\substack{i=0 \\ i \text{ even}}}^{n-1} b_i(m)\right)(m-1) \cdot \mathrm{rk}_\Lambda \mathcal{L}$$

and

$$\chi(\mathbb{A}^n_k, \mathrm{pr}^* \mathcal{N}_\lambda \otimes_\Lambda \mathcal{L}) \leq \left(\sum_{\substack{i=0 \\ i \text{ odd}}}^{n-1} b_i(m)\right)(m-1) \cdot \mathrm{rk}_\Lambda \mathcal{L} \,.$$

Proof. We use the setup from Construction 5.9. Let $X \to \mathbb{P}^n_k$ be the blow-up of $\mathbb{P}^n_k$ at $x := [0 : 1 : \cdots : 1]$. Let E be the exceptional divisor and let $\pi : X \to \mathbb{P}^{n-1}_k$ be the pencil of lines passing through x. Let H' be the strict transform of H and put $D := \pi(H')$. We have the following commutative diagram:

$$\begin{tikzcd}
H' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "\pi"] \arrow[dr, phantom, "\square"] & W \arrow[l] \arrow[d, "\pi_W"'] & \mathbb{A}^n_k \arrow[l, "\jmath_W"] \arrow[dl, "\pi_A"] \arrow[ll, bend right, "\jmath"'] \\
D \arrow[r] & \mathbb{P}^{n-1}_k & \mathbb{A}^{n-1}_k \arrow[l] &
\end{tikzcd}$$

By Proposition 5.12, the complex $R\pi_{A!}(\mathrm{pr}^* \mathcal{N}_\lambda \otimes_\Lambda \mathcal{L})$ is concentrated in degree 1, has locally constant constructible cohomology sheaves, its formation commutes with base change and

$$\mathrm{lc}_D(R^1\pi_{A!}(\mathrm{pr}^* \mathcal{N}_\lambda \otimes_\Lambda \mathcal{L})) \leq m\,. \tag{6.7.1}$$

By proper base change combined with Theorem 3.15, we deduce

$$\mathrm{rk}_\Lambda(R^1\pi_{A!}(\mathrm{pr}^* \mathcal{N}_\lambda \otimes_\Lambda \mathcal{L})) = (m-1)\cdot \mathrm{rk}_\Lambda \mathcal{L}\,. \tag{6.7.2}$$

On the other hand, we have

$$\begin{aligned}
\chi(\mathbb{A}^n_k, \mathrm{pr}^* \mathcal{N}_\lambda \otimes_\Lambda \mathcal{L}) &= \chi_c(\mathbb{A}^n_k, \mathrm{pr}^* \mathcal{N}_\lambda \otimes_\Lambda \mathcal{L}) && \text{Remark 3.14}\\
&= \chi_c(\mathbb{A}^{n-1}_k, R\pi_{A!}(\mathrm{pr}^* \mathcal{N}_\lambda \otimes_\Lambda \mathcal{L}))\\
&= -\chi_c(\mathbb{A}^{n-1}_k, R^1\pi_{A!}(\mathrm{pr}^* \mathcal{N}_\lambda \otimes_\Lambda \mathcal{L}))\\
&= -\chi(\mathbb{A}^{n-1}_k, R^1\pi_{A!}(\mathrm{pr}^* \mathcal{N}_\lambda \otimes_\Lambda \mathcal{L})) && \text{Remark 3.14}\\
&= \sum_{i=0}^{n-1} (-1)^{i+1} h^i(\mathbb{A}^{n-1}_k, R^1\pi_{A!}(\mathrm{pr}^* \mathcal{N}_\lambda \otimes_\Lambda \mathcal{L})).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
-\sum_{\substack{i=0\\ i \text{ even}}}^{n-1} h^i(\mathbb{A}^{n-1}_k, R^1\pi_{A!}(\mathrm{pr}^* \mathcal{N}_\lambda \otimes_\Lambda \mathcal{L})) &\leq \chi(\mathbb{A}^n_k, \mathrm{pr}^* \mathcal{N}_\lambda \otimes_\Lambda \mathcal{L})\\
&\leq \sum_{\substack{i=0\\ i \text{ odd}}}^{n-1} h^i(\mathbb{A}^{n-1}_k, R^1\pi_{A!}(\mathrm{pr}^* \mathcal{N}_\lambda \otimes_\Lambda \mathcal{L})).
\end{aligned}$$

Since Theorem 6.1 is valid for $\mathbb{A}^{n-1}_k$, we have for every $i = 0, 1, \ldots, n-1$,

$$h^i(\mathbb{A}^{n-1}_k, R^1\pi_{A!}(\mathrm{pr}^* \mathcal{N}_\lambda \otimes_\Lambda \mathcal{L})) \leq b_i(\mathrm{lc}_D(R^1\pi_{A!}(\mathrm{pr}^* \mathcal{N}_\lambda \otimes_\Lambda \mathcal{L})) \cdot \mathrm{rk}_\Lambda(R^1\pi_{A!}(\mathrm{pr}^* \mathcal{N}_\lambda \otimes_\Lambda \mathcal{L})).$$

From (6.7.1) and (6.7.2), we deduce

$$h^i(\mathbb{A}^{n-1}_k, R^1\pi_{A!}(\mathrm{pr}^* \mathcal{N}_\lambda \otimes_\Lambda \mathcal{L})) \leq b_i(m)\cdot(m-1)\cdot \mathrm{rk}_\Lambda \mathcal{L}\,.$$

This concludes the proof of Proposition 6.7. ☐

Proposition 6.8. *Let $n \geq 2$ be an integer. Assume that Theorem 6.1 is valid for $\mathbb{A}^{n-1}_k$. Let $\mathrm{pr} : \mathbb{A}^n_k \to \mathbb{A}^1_k$ be the first projection. Then, for every $\mathcal{L} \in \mathrm{Loc}(\mathbb{A}^n_k, \Lambda)$, the Euler-Poincaré characteristic $\chi(\mathbb{A}^n_k, \mathcal{L})$ is smaller than*

$$\left(\sum_{\substack{i=0\\ i \text{ odd}}}^{n-1} (\mathrm{lc}_H(\mathcal{L})+2)\cdot b_i(\mathrm{lc}_H(\mathcal{L})+3) + \sum_{\substack{i=0\\ i \text{ even}}}^{n-1} (\mathrm{lc}_H(\mathcal{L})+3)\cdot b_i(\mathrm{lc}_H(\mathcal{L}))\right)\cdot \mathrm{rk}_\Lambda \mathcal{L}$$

and bigger than

$$-\left(\sum_{\substack{i=0\\ i \text{ even}}}^{n-1} (\mathrm{lc}_H(\mathcal{L})+2)\cdot b_i(\mathrm{lc}_H(\mathcal{L})+3) + \sum_{\substack{i=0\\ i \text{ odd}}}^{n-1} (\mathrm{lc}_H(\mathcal{L})+3)\cdot b_i(\mathrm{lc}_H(\mathcal{L}))\right)\cdot \mathrm{rk}_\Lambda \mathcal{L}\,.$$

Proof. Note that enlarging the finite field Λ does not change the Betti numbers. Hence, we can suppose that Λ contains a p-root of unity. Let $\mathcal{L} \in \mathrm{Loc}(\mathbb{A}^n_k, \Lambda)$. Observe that at least one of the consecutive integers $[\mathrm{lc}_H(\mathcal{L})]+2$, $[\mathrm{lc}_H(\mathcal{L})]+3$ is prime to p. Pick one of them and denote it by m. In particular,

$$\mathrm{lc}_H(\mathcal{L})+1 < m \leq \mathrm{lc}_H(\mathcal{L})+3\,.$$

Let $\mathcal{N}_\lambda$ be an Artin-Schreier sheaf of Λ-modules on $\mathbb{A}^1_k$ defined by the equation $t^p - t = \lambda x^m$ for some $\lambda \in k^\times$. By the projection formula and Theorem 3.15, we have

$$\chi(\mathbb{A}^n_k, \mathrm{pr}^*\mathcal{N}_\lambda \otimes_\Lambda \mathcal{L}) = \chi(\mathbb{A}^1_k, R\,\mathrm{pr}_*(\mathrm{pr}^*\mathcal{N}_\lambda \otimes_\Lambda \mathcal{L})) = \chi(\mathbb{A}^1_k, \mathcal{N}_\lambda \otimes_\Lambda R\,\mathrm{pr}_*\mathcal{L}))$$
$$\chi(\mathbb{A}^n_k, \mathcal{L}) = \chi(\mathbb{A}^1_k, R\,\mathrm{pr}_*\mathcal{L})\,.$$

Since $\mathcal{N}_\lambda$ is a rank 1 locally constant sheaf on $\mathbb{A}^1_k$, Theorem 3.15 gives

$$\chi(\mathbb{A}^n_k, \mathcal{L}) - \chi(\mathbb{A}^n_k, \mathrm{pr}^*\mathcal{N}_\lambda \otimes_\Lambda \mathcal{L}) = \mathrm{sw}_\infty(\mathcal{N}_\lambda \otimes_\Lambda R\,\mathrm{pr}_*\mathcal{L}) - \mathrm{sw}_\infty(R\,\mathrm{pr}_*\mathcal{L})\,.$$

By generic universally locally acyclicity, there is a dense open subset $U \subset \mathbb{A}^1_k$ such that

1. For every geometric point $\bar{y} \to U$, we have $R\Gamma((\mathbb{A}^n_k)_{\bar{y}}, \mathcal{L}|_{(\mathbb{A}^n_k)_{\bar{y}}}) \cong (R\,\mathrm{pr}_*\mathcal{L})|_{\bar{y}}$
2. For every $i \in \mathbb{Z}$, the sheaf $R^i\mathrm{pr}_*\mathcal{L}$ is locally constant and constructible in U.

Let $\eta_\infty = \mathrm{Spec}\,(\mathrm{Frac}(\mathcal{O}^{\mathrm{sh}}_{\mathbb{P}^1_k,\infty}))$ and let $\bar{\eta}_\infty$ be a geometric point over η_∞. Put $G_\infty = \mathrm{Gal}(\bar{\eta}_\infty/\eta_\infty)$ and let $P_\infty \subset G_\infty$ be the wild inertia subgroup. For $s < 0$ and $s > n-1$, we have

$$(R^s\,\mathrm{pr}_*\mathcal{L})|_{\bar{\eta}_\infty} = 0\,.$$

Let $y \in U$. Since Theorem 6.1 is valid for $\mathbb{A}^{n-1}_k$, we have

$$\mathrm{rk}_\Lambda((R^s\,\mathrm{pr}_*\mathcal{L})|_{\bar{\eta}_\infty}) = \mathrm{rk}_\Lambda((R^s\,\mathrm{pr}_*\mathcal{L})|_y) \leq b_s(\mathrm{lc}_H(\mathcal{L}))\cdot \mathrm{rk}_\Lambda \mathcal{L}$$

for every $0 \leq s \leq n-1$. For $\lambda_1, \lambda_2 \in k^\times$ with $\lambda_1 - \lambda_2 \notin \mathbb{F}_p$, the P_∞-representations $\mathcal{N}_{\lambda_1}|_{\eta_\infty}$ and $\mathcal{N}_{\lambda_2}|_{\eta_\infty}$ are not isomorphic. Hence, their induced characters (see Recollection 2.8)

$$\chi_{\lambda_1}, \chi_{\lambda_2} : G^{(r)}_{\infty,\log}/G^{(r+)}_{\infty,\log} \to \Lambda^\times$$

are not isomorphic either. Since k is infinite, we can thus find $\lambda \in k^\times$ such that χ_λ^{-1} does not contribute to

$$\bigoplus_{i=0}^{n-1} (R^i \operatorname{pr}_* \mathcal{L})|_{\eta_\infty}.$$

By Lemma 2.9, for every $i = 0, 1, \dots, n-1$, we thus have

$$\mathrm{sw}_\infty(R^i \operatorname{pr}_* \mathcal{L}) \leq \mathrm{sw}_\infty(\mathcal{N}_\lambda \otimes_\Lambda R^i \operatorname{pr}_* \mathcal{L}) \leq \mathrm{sw}_\infty(R^i \operatorname{pr}_* \mathcal{L}) + m \cdot \mathrm{rk}_\Lambda((R^i \operatorname{pr}_* \mathcal{L})|_{\overline{\eta}_\infty}).$$

In particular for $i = 0, 1, \dots, n-1$, we have

$$0 \leq \mathrm{sw}_\infty(\mathcal{N}_\lambda \otimes_\Lambda R^i \operatorname{pr}_* \mathcal{L}) - \mathrm{sw}_\infty(R^i \operatorname{pr}_* \mathcal{L}) \leq m \cdot \mathrm{rk}_\Lambda((R^i \operatorname{pr}_* \mathcal{L})|_{\overline{\eta}_\infty}) \, .$$

We obtain

$$\begin{aligned}
\chi(\mathbb{A}^n_k, \mathcal{L}) - \chi(\mathbb{A}^n_k, \operatorname{pr}^* \mathcal{N}_\lambda \otimes_\Lambda \mathcal{L}) &= \mathrm{sw}_\infty(\mathcal{N}_\lambda \otimes_\Lambda R \operatorname{pr}_* \mathcal{L}) - \mathrm{sw}_\infty(R \operatorname{pr}_* \mathcal{L}) \\
&= \sum_{i=0}^{n-1} (-1)^i (\mathrm{sw}_\infty(\mathcal{N}_\lambda \otimes_\Lambda R^i \operatorname{pr}_* \mathcal{L}) - \mathrm{sw}_\infty(R^i \operatorname{pr}_* \mathcal{L})) \\
&\leq \sum_{\substack{i=0 \\ i \text{ even}}}^{n-1} (\mathrm{sw}_\infty(\mathcal{N}_\lambda \otimes_\Lambda R^i \operatorname{pr}_* \mathcal{L}) - \mathrm{sw}_\infty(R^i \operatorname{pr}_* \mathcal{L})) \\
&\leq \sum_{\substack{i=0 \\ i \text{ even}}}^{n-1} m \cdot \mathrm{rk}_\Lambda(R^i \operatorname{pr}_* \mathcal{L}) \\
&\leq \left(\sum_{\substack{i=0 \\ i \text{ even}}}^{n-1} b_i(\mathrm{lc}_H(\mathcal{L})) \right) \cdot m \cdot \mathrm{rk}_\Lambda \mathcal{L} \, .
\end{aligned}$$

and

$$\begin{aligned}
\chi(\mathbb{A}^n_k, \mathcal{L}) - \chi(\mathbb{A}^n_k, \operatorname{pr}^* \mathcal{N}_\lambda \otimes_\Lambda \mathcal{L}) &= \mathrm{sw}_\infty(\mathcal{N}_\lambda \otimes_\Lambda R \operatorname{pr}_* \mathcal{L}) - \mathrm{sw}_\infty(R \operatorname{pr}_* \mathcal{L}) \\
&= \sum_{i=0}^{n-1} (-1)^i (\mathrm{sw}_\infty(\mathcal{N}_\lambda \otimes_\Lambda R^i \operatorname{pr}_* \mathcal{L}) - \mathrm{sw}_\infty(R^i \operatorname{pr}_* \mathcal{L})) \\
&\geq - \sum_{\substack{i=0 \\ i \text{ odd}}}^{n-1} (\mathrm{sw}_\infty(\mathcal{N}_\lambda \otimes_\Lambda R^i \operatorname{pr}_* \mathcal{L}) - \mathrm{sw}_\infty(R^i \operatorname{pr}_* \mathcal{L})) \\
&\geq - \sum_{\substack{i=0 \\ i \text{ odd}}}^{n-1} m \cdot \mathrm{rk}_\Lambda(R^i \operatorname{pr}_* \mathcal{L}) \\
&\geq - \left(\sum_{\substack{i=0 \\ i \text{ odd}}}^{n-1} b_i(\mathrm{lc}_H(\mathcal{L})) \right) \cdot m \cdot \mathrm{rk}_\Lambda \mathcal{L} \, .
\end{aligned}$$

Together with Proposition 6.7, we have

$$\begin{aligned}
\chi(\mathbb{A}^n_k,\mathcal{L}) &\geq \chi(\mathbb{A}^n_k,\mathrm{pr}^*\mathcal{N}_\lambda\otimes_\Lambda\mathcal{L}) - \left(\sum_{\substack{i=0\\ i \text{ odd}}}^{n-1} b_i(\mathrm{lc}_H(\mathcal{L}))\right)\cdot m\cdot \mathrm{rk}_\Lambda\,\mathcal{L}\\
&\geq -\left(\sum_{\substack{i=0\\ i \text{ even}}}^{n-1}(m-1)\cdot b_i(m) + \sum_{\substack{i=0\\ i \text{ odd}}}^{n-1} m\cdot b_i(\mathrm{lc}_H(\mathcal{L}))\right)\cdot \mathrm{rk}_\Lambda\,\mathcal{L}\\
&\geq -\left(\sum_{\substack{i=0\\ i \text{ even}}}^{n-1}(\mathrm{lc}_H(\mathcal{L})+2)\cdot b_i(\mathrm{lc}_H(\mathcal{L})+3) + \sum_{\substack{i=0\\ i \text{ odd}}}^{n-1} (\mathrm{lc}_H(\mathcal{L})+3)\cdot b_i(\mathrm{lc}_H(\mathcal{L}))\right)\cdot \mathrm{rk}_\Lambda\,\mathcal{L}\,,
\end{aligned}$$

and

$$\begin{aligned}
\chi(\mathbb{A}^n_k,\mathcal{L}) &\leq \chi(\mathbb{A}^n_k,\mathrm{pr}^*\mathcal{N}_\lambda\otimes_\Lambda\mathcal{L}) + \left(\sum_{\substack{i=0\\ i \text{ even}}}^{n-1} b_i(\mathrm{lc}_H(\mathcal{L}))\right)\cdot m\cdot \mathrm{rk}_\Lambda\,\mathcal{L}\\
&\leq \left(\sum_{\substack{i=0\\ i \text{ odd}}}^{n-1}(m-1)\cdot b_i(m) + \sum_{\substack{i=0\\ i \text{ even}}}^{n-1} m\cdot b_i(\mathrm{lc}_H(\mathcal{L}))\right)\cdot \mathrm{rk}_\Lambda\,\mathcal{L}\\
&\leq \left(\sum_{\substack{i=0\\ i \text{ odd}}}^{n-1}(\mathrm{lc}_H(\mathcal{L})+2)\cdot b_i(\mathrm{lc}_H(\mathcal{L})+3) + \sum_{\substack{i=0\\ i \text{ even}}}^{n-1} (\mathrm{lc}_H(\mathcal{L})+3)\cdot b_i(\mathrm{lc}_H(\mathcal{L}))\right)\cdot \mathrm{rk}_\Lambda\,\mathcal{L}\,.
\end{aligned}$$

This concludes the proof of Proposition 6.8. ☐

6.9. Proof of Theorem 6.1. We proceed by induction on n. When $n = 1$, Theorem 6.1 follows from Corollary 3.16. Let $n \geq 2$ and assume that Theorem 6.1 is valid for $\mathbb{A}^{n-1}_k$. In particular, Proposition 6.6 gives

$$h^i(\mathbb{A}^n_k,\mathcal{L}) \leq b_i(\mathrm{lc}_H(\mathcal{L}))\cdot \mathrm{rk}_\Lambda(\mathcal{L})\,,$$

for every $0 \leq i \leq n-1$. Hence, we are left to show that

$$h^n(\mathbb{A}^n_k,\mathcal{L}) \leq b_n(\mathrm{lc}_H(\mathcal{L}))\cdot \mathrm{rk}_\Lambda(\mathcal{L}).$$

Let $\mathrm{pr} : \mathbb{A}^n_k \to \mathbb{A}^1_k$ be the projection on the first coordinate. When n is odd, we deduce

$$\begin{aligned}
h^n(\mathbb{A}^n_k, \mathcal{L}) &= -\chi(\mathbb{A}^n_k, \mathcal{L}) + \sum_{i=0}^{n-1} (-1)^i h^i(\mathbb{A}^n_k, \mathcal{L}), \\
&\leq -\chi(\mathbb{A}^n_k, \mathcal{L}) + \sum_{\substack{i=0 \\ i \text{ even}}}^{n-1} h^i(\mathbb{A}^n_k, \mathcal{L}), \\
&\leq \left(\sum_{\substack{i=0 \\ i \text{ even}}}^{n-1} (\mathrm{lc}_H(\mathcal{L}) + 2) \cdot b_i(\mathrm{lc}_H(\mathcal{L}) + 3) + \sum_{\substack{i=0 \\ i \text{ odd}}}^{n-1} (\mathrm{lc}_H(\mathcal{L}) + 3) \cdot b_i(\mathrm{lc}_H(\mathcal{L})) \right) \cdot \mathrm{rk}_\Lambda \mathcal{L} \\
&\quad + \left(\sum_{\substack{i=0 \\ i \text{ even}}}^{n-1} b_i(\mathrm{lc}_H(\mathcal{L})) \right) \cdot \mathrm{rk}_\Lambda \mathcal{L} \\
&= b_n(\mathrm{lc}_H(\mathcal{L})) \cdot \mathrm{rk}_\Lambda \mathcal{L} \, .
\end{aligned}$$

If n is even, we have

$$\begin{aligned}
h^n(\mathbb{A}^n_k, \mathcal{L}) &= \chi(\mathbb{A}^n_k, \mathcal{L}) - \sum_{i=0}^{n-1} (-1)^i h^i(\mathbb{A}^n_k, \mathcal{L}) \\
&\leq \chi(\mathbb{A}^n_k, \mathcal{L}) + \sum_{\substack{i=0 \\ i \text{ odd}}}^{n-1} h^i(\mathbb{A}^n_k, \mathcal{L}) \\
&\leq \left(\sum_{\substack{i=0 \\ i \text{ odd}}}^{n-1} (\mathrm{lc}_H(\mathcal{L}) + 2) \cdot b_i(\mathrm{lc}_H(\mathcal{L}) + 3) + \sum_{\substack{i=0 \\ i \text{ even}}}^{n-1} (\mathrm{lc}_H(\mathcal{L}) + 3) \cdot b_i(\mathrm{lc}_H(\mathcal{L})) \right) \cdot \mathrm{rk}_\Lambda \mathcal{L} \\
&\quad + \left(\sum_{\substack{i=0 \\ i \text{ odd}}}^{n-1} b_i(\mathrm{lc}_H(\mathcal{L})) \right) \cdot \mathrm{rk}_\Lambda \mathcal{L} \\
&\leq b_n(\mathrm{lc}_H(\mathcal{L})) \cdot \mathrm{rk}_\Lambda \mathcal{L} \, .
\end{aligned}$$

In any case we have $h^n(\mathbb{A}^n_k, \mathcal{L}) \leq b_n(\mathrm{lc}_H(\mathcal{L})) \cdot \mathrm{rk}_\Lambda \mathcal{L}$. This concludes the proof of Theorem 6.1.

7. Estimates for Betti numbers of étale sheaves

7.1. Bounding the ramification with coherent sheaves. Let X be a scheme of finite type over k. We denote by $\mathbb{Q}[\mathrm{Coh}(X)]$ the free $\mathbb{Q}$-vector space on the set of isomorphism classes of coherent sheaves on X. Observe that the pullback along every morphism $f : Y \to X$ of schemes of finite type over k induces a morphism of $\mathbb{Q}$-vector spaces

$$f^* : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q}[\mathrm{Coh}(Y)] \, .$$

Assume now that X is normal and let $\mathcal{E} \in \mathrm{Coh}(X)$. If $X^1 \subset X$ denotes the set of codimension 1 points of X, we define a Weil divisor on X by the formula

$$T(\mathcal{E}) := \sum_{\eta \in X^1} \mathrm{length}_{\mathcal{O}_{X,\eta}}(\mathcal{E}|^{\mathrm{tors}}_{X_\eta}) \cdot \overline{\{\eta\}}$$

where $X_\eta = \mathrm{Spec}\, \mathcal{O}_{X,\eta}$ and where $\mathcal{E}|^{\mathrm{tors}}_{X_\eta}$ is the torsion part of $\mathcal{E}|_{X_\eta}$.

Example 7.2. If R is an effective Cartier divisor of X with ideal sheaf $\mathcal{I}_R$ and if $\mathcal{E} = \mathcal{O}_X/\mathcal{I}_R$, then $T(\mathcal{E}) = R$.

If $\mathrm{Weil}(X)_{\mathbb{Q}}$ is the space of $\mathbb{Q}$-Weil divisors on X, the map $T : \mathrm{Coh}(X) \to \mathrm{Weil}(X)_{\mathbb{Q}}$ induces a map of $\mathbb{Q}$-vector spaces

$$T : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathrm{Weil}(X)_{\mathbb{Q}} .$$

Definition 7.3. Let X be a scheme of finite type over k. Let $\mathcal{K} \in D^b_{ctf}(X, \Lambda)$ and $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$. We say that $\mathcal{K}$ *has log conductors bounded by* $\mathcal{E}$ if for every morphism $f : C \to X$ over k where C is a smooth curve over k, we have

$$\mathrm{LC}(\mathcal{H}^i\mathcal{K}|_C) \leq T(f^*\mathcal{E})$$

for every $i \in \mathbb{Z}$. We denote by $D^b_{ctf}(X, \mathcal{E}, \Lambda)$ the full subcategory of $D^b_{ctf}(X, \Lambda)$ spanned by objects having log conductors bounded by $\mathcal{E}$.

The following is our main example of sheaf with explicit bound on the log conductors.

Proposition 7.4 ([HT25b, Proposition 5.7])**.** *Let X be a normal scheme of finite type over k. Let D be an effective Cartier divisor of X and put $j : U := X - D \hookrightarrow X$. Let $\mathcal{L} \in \mathrm{Loc}_{tf}(U, \Lambda)$ and $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$. Then*

(1) If $j_!\mathcal{L}$ has log conductors bounded by $\mathcal{E}$, then $\mathrm{LC}_X(j_!\mathcal{L}) \leq T(\mathcal{E})$.

(2) If X is smooth over k, then $j_!\mathcal{L}$ has log conductors bounded by $(\mathrm{lc}_D(\mathcal{L}) + 1) \cdot \mathcal{O}_D$.

Lemma 7.5 ([HT25b, Lemma 5.6])**.** *Let X be a scheme of finite type over k and let $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$.*

(1) For every $\mathcal{K} \in D^b_{ctf}(X, \mathcal{E}, \Lambda)$ and every $f : Y \to X$ morphism of schemes of finite type over k, we have $f^*\mathcal{K} \in D^b_{ctf}(Y, f^*\mathcal{E}, \Lambda)$.

(2) Consider an exact sequence in $\mathrm{Cons}_{tf}(X, \Lambda)$

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0 .$$

Then $\mathcal{F}_2$ lies has log conductors bounded by $\mathcal{E}$ if and only if so do $\mathcal{F}_1$ and $\mathcal{F}_3$.

Lemma 7.6 ([HT25b, Proposition 5.9])**.** *Let $f : X \to S$ be a morphism between schemes of finite type over k. Let $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$ and $\mathcal{K} \in D^b_{ctf}(X, \mathcal{E}, \Lambda)$. For every algebraic geometric point $\bar{s} \to S$, the complex $\mathcal{K}|_{X_{\bar{s}}}$ has log conductors bounded by $i^*_{\bar{s}}\mathcal{E}$ where $i_{\bar{s}} : X_{\bar{s}} \to X$ is the canonical morphism.*

7.7. Betti bound. In this paragraph, we formalize the structure of the estimates appearing in the statement of Theorem 7.15.

Definition 7.8. Let X be a scheme of finite type over k. We say that a $\mathbb{Q}$-linear map $\mu : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q}$ is *admissible* if the following conditions are satisfied :

(1) For every $\mathcal{E} \in \mathrm{Coh}(X)$, we have $\mu(\mathcal{E}) \in \mathbb{N}$.

(2) For every $\mathcal{E}_1, \mathcal{E}_2 \in \mathrm{Coh}(X)$, we have $\mu(\mathcal{E}_1 \bigoplus \mathcal{E}_2) \leq \mu(\mathcal{E}_1) + \mu(\mathcal{E}_2)$.

Definition 7.9. Following [SP23, 0391], let us recall that a morphism $f : X \to S$ between schemes of finite type over k is *normal* if it is flat with geometrically normal fibres.

Example 7.10. Let $f : X \to S$ be a normal morphism between schemes of finite type over k. For $\mathcal{E} \in \mathrm{Coh}(X)$, we define a function $\chi_{\mathcal{E}} : S \to \mathbb{N}$ by

$$\chi_{\mathcal{E}} : S \to \mathbb{N}, \quad s \mapsto m(T(\mathcal{E}_{\bar{s}})),$$

where $\bar{s} \to S$ is an algebraic geometric point above $s \in S$ and where $m(T(\mathcal{E}_{\bar{s}}))$ is the maximal multiplicity of $T(\mathcal{E}_{\bar{s}})$. By [HT25b, Corollary 4.8], the quantity

$$\mu_f(\mathcal{E}) := \sup \chi_{\mathcal{E}}(S)$$

is finite. Then, the induced $\mathbb{Q}$-linear map

$$\mu_f : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q}$$

is admissible.

Lemma 7.11. *Let $f : X \to Y$ be a morphism of schemes of finite type over k and let $\mu : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q}$ be an admissible function. Then, $\mu \circ f^* : \mathbb{Q}[\mathrm{Coh}(Y)] \to \mathbb{Q}$ is admissible.*

Definition 7.12. Let $f : X \to S$ be a proper morphism between schemes of finite type over k with fibres of dimension $\leq n$. Let Λ be a finite field or a finite extension of $\mathbb{Q}_\ell$ with $\ell \neq p$. Let $\bar{s} \to S$ be an algebraic geometric point and let $\mathcal{C} \subset D^b_c(X_{\bar{s}}, \Lambda)$ be a full subcategory. Let $\mu : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q}$ be an admissible function and let $P \in \mathbb{N}[x]^{n+1}$. We say that (μ, P) *is a Betti bound for $\mathcal{C}$* if for every $j = 0, \ldots, 2n$, every $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$ and every $\mathcal{K} \in \mathcal{C}$ with log conductors bounded by $\mathcal{E}_{\bar{s}}$, we have

$$h^j(X_{\bar{s}}, \mathcal{K}) \leq P_{\min(j,2n-j)}(\mu(\mathcal{E})) \cdot \mathrm{Rk}_\Lambda \mathcal{K}\,. \tag{7.12.1}$$

Remark 7.13. If (7.12.1) is satisfied for some algebraic geometric point $\bar{s} \to S$ localized at $s \in S$, then it is satisfied for every algebraic geometric point $\bar{t} \to S$ localized at s.

Definition 7.14. Let P be a property of morphisms of schemes over k and let $n \geq 0$. A P-relative stratified scheme $(X/S, \Sigma)$ of relative dimension $\leq n$ will refer to a morphism $X \to S$ between schemes of finite type over k satisfying P such that the fibres of $X \to S$ have dimensions $\leq n$ and where X is endowed with a finite stratification Σ.

Theorem 7.15. *Let $(X/S, \Sigma)$ be a proper relative stratified scheme of relative dimension $\leq n$. Then, there is an admissible function $\mu : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q}$ and $P \in \mathbb{N}[x]^{n+1}$ with P_i of degree i for every $i = 0, \ldots, n$ and such that for every algebraic geometric point $\bar{s} \to S$, every finite field Λ of characteristic $\ell \neq p$, the couple (μ, P) is a Betti bound for $\mathrm{Cons}_{\Sigma_{\bar{s}}}(X_{\bar{s}}, \Lambda)$.*

Definition 7.16. A Betti bound (μ, P) as in Theorem 7.15 will be referred as a *Betti bound for* $(X/S, \Sigma)$.

For the sake of the proof, we need to formulate a priori weaker

Theorem 7.17. *Let $(X/S, \Sigma)$ be a proper relative stratified scheme of relative dimension $\leq n$. Then, there is a dominant quasi-finite map $T \to S$ between schemes of finite type over k such that the pullback $(X_T/T, \Sigma_T)$ admits a Betti bound.*

7.18. Dévissage. The goal of what follows is to show that Theorem 7.15 and Theorem 7.17 are equivalent and to reduce the proof of Theorem 7.17 to the case where $X \to S$ is a normal morphism.

Lemma 7.19. *Let $(X/S, \Sigma)$ be a proper relative stratified scheme of relative dimension $\leq n$. Assume the existence of a finite family A of jointly surjective quasi-finite morphisms $T_\alpha \to S$ over k such that for every $\alpha \in A$, the pullback $(X_\alpha/S_\alpha, \Sigma_\alpha)$ admits a Betti bound. Then, $(X/S, \Sigma)$ admits a Betti bound.*

Proof. For $\alpha \in A$, let $h_\alpha : X_\alpha \to X$ be the induced morphism and let (μ_α, P_α) be a Betti bound for $(X_\alpha/S_\alpha, \Sigma_\alpha)$. By Lemma 7.11, the function

$$\mu := \sum_{\alpha \in A} \mu_\alpha \circ h_\alpha^* : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q}$$

is admissible. If we put $P := \sum_{\alpha \in A} P_\alpha$, one easily checks that (μ, P) is a Betti bound for $(X/S, \Sigma)$. ☐

Definition 7.20. For $d \geq 0$, we say that Theorem 7.15 (resp. Theorem 7.17) holds in absolute dimension $\leq d$ if for every $n \geq 0$, Theorem 7.15 (resp. Theorem 7.17) holds for every proper relative stratified scheme $(X/S, \Sigma)$ of relative dimension $\leq n$ with $\dim X \leq d$.

Lemma 7.21. *Let $d \geq 0$. Then Theorem 7.15 holds in absolute dimension $\leq d$ if and only if Theorem 7.17 holds in absolute dimension $\leq d$.*

Proof. Immediate by Lemma 7.19 and the fact that the base scheme S is noetherian. ☐

Lemma 7.22. *Let $d \geq 1$ and let $(X/S, \Sigma)$ be a proper relative stratified scheme of relative dimension $\leq n$ with $\dim X \leq d$. Assume that*

(1) *Theorem 7.15 holds in absolute dimension $\leq d - 1$.*

(2) *There is a closed subscheme $i : Z \hookrightarrow X$ of dimension $\leq d - 1$ with complement $j : U \hookrightarrow X$, an admissible function $\mu' : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q}$ and $P' \in \mathbb{N}[x]^{n+1}$ with P'_i of degree i for $i = 0, \ldots, n$ such that for every algebraic geometric point $\bar{s} \to S$, every finite field Λ of characteristic $\ell \neq p$, every $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$ and every $\mathcal{F} \in \mathrm{Cons}(X_{\bar{s}}, \mathcal{E}_{\bar{s}}, \Lambda)$ extension by 0 of an object of $\mathrm{Cons}_{(j^*\Sigma)_{\bar{s}}}(U_{\bar{s}}, \Lambda)$, we have*

$$h^j(X_{\bar{s}}, \mathcal{F}) \leq P'_{\min(j, 2n-j)}(\mu'(\mathcal{E})) \cdot \mathrm{Rk}_\Lambda \mathcal{F}$$

for every $j = 0, \ldots, 2n$.

Then, $(X/S, \Sigma)$ admits a Betti bound.

Proof. Note that the fibres of $Z \to S$ have dimension $\leq n$. Let (μ_Z, P_Z) be a Betti bound for $(Z/S, i^*\Sigma)$. Put

$$\mu = \mu' + \mu_Z \circ i^* : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q} \text{ and } P = P' + P_Z \in \mathbb{N}^{n+1}[x]$$

and let us show that (μ, P) is a Betti bound for $(X/S, \Sigma)$. Let $\bar{s} \to S$ be an algebraic geometric point and let Λ be a finite field of characteristic $\ell \neq p$. Let $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$ and let $\mathcal{F} \in \mathrm{Cons}_{\Sigma_{\bar{s}}}(X_{\bar{s}}, \mathcal{E}_{\bar{s}}, \Lambda)$. We want to show that for every $j = 0, \ldots, 2n$, we have

$$h^j(X_{\bar{s}}, \mathcal{F}) \leq P_{\min(j,2n-j)}(\mu(\mathcal{E})) \cdot \mathrm{Rk}_\Lambda \mathcal{F} \,. \tag{7.22.1}$$

Consider the localization exact sequence

$$0 \to j_{\bar{s}!}\mathcal{F}|_{U_{\bar{s}}} \to \mathcal{F} \to i_{\bar{s}*}\mathcal{F}|_{Z_{\bar{s}}} \to 0 \,.$$

By Lemma 7.5-(1), the sheaf $\mathcal{F}|_{Z_{\bar{s}}}$ is an object of $\mathrm{Cons}_{(i^*\Sigma)_{\bar{s}}}(Z_{\bar{s}}, \Lambda)$ with log conductors bounded by $(i^*\mathcal{E})_{\bar{s}}$. Hence, for every $j = 0, \ldots, 2n$, we have

$$h^j(Z_{\bar{s}}, \mathcal{F}|_{Z_{\bar{s}}}) \leq P_{Z,\min(j,2n-j)}(\mu_Z(i^*\mathcal{E})) \cdot \mathrm{Rk}_\Lambda \mathcal{F}|_{Z_{\bar{s}}} \leq P_{Z,\min(j,2n-j)}(\mu(\mathcal{E})) \cdot \mathrm{Rk}_\Lambda \mathcal{F} \,.$$

Note that $j_{\bar{s}!}\mathcal{F}|_{U_{\bar{s}}}$ has log conductors bounded by $\mathcal{E}_{\bar{s}}$ by Lemma 7.5-(2). By assumption, we deduce that

$$h^j(X_{\bar{s}}, j_{\bar{s}!}\mathcal{F}|_{U_{\bar{s}}}) \leq P'_{\min(j,2n-j)}(\mu'(\mathcal{E})) \cdot \mathrm{Rk}_\Lambda j_{\bar{s}!}\mathcal{F}|_{U_{\bar{s}}} \leq P'_{\min(j,2n-j)}(\mu(\mathcal{E})) \cdot \mathrm{Rk}_\Lambda \mathcal{F} \,.$$

The conclusion thus follows. □

Lemma 7.23. *Let* $d \geq 1$ *and let* $(X/S, \Sigma)$ *be a proper relative stratified scheme of relative dimension* $\leq n$ *with* $\dim X \leq d$. *Assume that*

1. *Theorem 7.17 holds in absolute dimension* $\leq d - 1$.
2. *There is a proper morphism* $h : Y \to X$ *of schemes of finite type over* S *such that the fibres of* $Y \to S$ *have dimension* $\leq n$ *and Theorem 7.17 holds for* $(Y/S, \Sigma')$ *where* Σ' *is any stratification on* Y.
3. h *induces an isomorphism above a dense open subset* $U \subset X$.

Then, Theorem 7.17 holds for $(X/S, \Sigma)$.

Proof. Let Σ' be a refinement of Σ such that U and $Z := X - U$ are unions of strata of Σ'. It is enough to show that Theorem 7.17 holds for $(X/S, \Sigma')$. Hence, we are left to prove Lemma 7.23 in the case where U and $Z := X - U$ are unions of strata of Σ.

Let $g : T \to S$ be a quasi-finite dominant map as given by Theorem 7.17 for $(Y/S, h^*\Sigma)$. At the cost of pulling everything back to T, we can suppose that $(Y/S, h^*\Sigma)$ admits a Betti bound. In that case, we are going to show that so does (X, Σ).

By (1) combined with Lemma 7.21, we know that Theorem 7.15 holds in absolute dimension $\leq d - 1$. Hence, we are left to show that Z satisfies the condition (2) of Lemma 7.22. Since U is dense open in X, we have $\dim Z < \dim X = d$. Let $\bar{s} \to S$ be an algebraic geometric point, let Λ be a finite field of characteristic $\ell \neq p$, let $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$ and let $\mathcal{F} \in \mathrm{Cons}(X_{\bar{s}}, \mathcal{E}_{\bar{s}}, \Lambda)$ extension by 0 of an object of $\mathrm{Cons}_{(j^*\Sigma)_{\bar{s}}}(U_{\bar{s}}, \Lambda)$. Since U and Z are unions of strata, $\mathcal{F}$ is an object of $\mathrm{Cons}_{\Sigma_{\bar{s}}}(X_{\bar{s}}, \mathcal{E}_{\bar{s}}, \Lambda)$. Hence, $h_{\bar{s}}^*\mathcal{F}$ is an object of $\mathrm{Cons}_{(h^*\Sigma)_{\bar{s}}}(Y_{\bar{s}}, (h^*\mathcal{E})_{\bar{s}}, \Lambda)$. Since $h_{\bar{s}} : Y_{\bar{s}} \to X_{\bar{s}}$ is an isomorphism above $U_{\bar{s}}$, the proper base change implies that the unit map

$$\mathcal{F} \to Rh_{\bar{s}*}h_{\bar{s}}^*\mathcal{F}$$

is an isomorphism. Thus,

$$h^j(X_{\bar{s}}, \mathcal{F}) = h^j(Y_{\bar{s}}, h_{\bar{s}}^*\mathcal{F}) \,.$$

Hence if (μ_Y, P_Y) is a Betti bound for $(Y/S, h^*\Sigma)$, we deduce

$$h^j(X_{\bar{s}}, \mathcal{F}) \leq P_{Y,\min(j,2n-j)}(\mu_Y(h^*\mathcal{E})) \cdot \mathrm{Rk}_\Lambda \mathcal{F} .$$

Hence, Lemma 7.22 is satisfied with $\mu' = \mu_Y \circ h^*$ and $P' = P_Y$. ☐

Lemma 7.24. *Let $f : X \to S$ be a proper morphism of schemes of finite type over k of relative dimension $\leq n$. Then there is a dense open subset $V \subset S$ and a commutative triangle*

(7.24.1)
$$\begin{tikzcd} Y \arrow[rr, "h"] \arrow[dr, "g"'] & & X_V \arrow[dl, "f_V"] \\ & V & \end{tikzcd}$$

where $h : Y \to X_V$ is surjective projective and induces an isomorphism over a dense open subset $U \subset X_V$ with $h^{-1}(U)$ dense in Y and $g : Y \to V$ projective of relative dimension $\leq n$.

Proof. At the cost of shrinking S, we can suppose that S is irreducible with generic point η. At the cost of shrinking S further, we can suppose that the generic points of X lie over η. By the Chow lemma [GD61a, Corollaire 5.6.2], there is a surjective projective morphism $h : Y \to X$ over S such that h is an isomorphism over a dense open subset $U \subset X$ with $h^{-1}(U)$ dense in Y and such that $Y \to S$ is projective. In particular, for a generic point $\xi \in Y$, we necessarily have $\xi \in h^{-1}(U)$. Hence, there is a generic point $\nu \in U$ specializing on $h(\xi)$. Thus, $h^{-1}(\nu)$ specializes on ξ and we get $h(\xi) = \nu$, so that ξ lies over η as well. Hence, $h^{-1}(U)_\eta$ is dense in Y_η. By [SP23, Tag 0573], there is a dense open subset $V \subset S$ such that for every $s \in V$, the open set $(h^{-1}(U))_s$ is dense in Y_s. Thus, for every $s \in V$, we have

$$\dim Y_s = \dim(h^{-1}(U))_s = \dim U_s \leq n .$$

The conclusion thus follows. ☐

Lemma 7.25. *Let $d \geq 1$ and assume that*

1. *Theorem 7.17 holds in absolute dimension $\leq d - 1$.*
2. *For every $n \geq 0$, Theorem 7.17 holds for every projective relative stratified scheme $(X/S, \Sigma)$ of relative dimension $\leq n$ with $\dim X \leq d$.*

Then, Theorem 7.17 holds in absolute dimension $\leq d$.

Proof. Let $n \geq 0$ and let $(X/S, \Sigma)$ be a proper relative stratified scheme of relative dimension $\leq n$ with $\dim X \leq d$. At the cost of replacing S by a dense open subset, we can assume by Lemma 7.24 the existence of a proper morphism $h : Y \to X$ inducing an isomorphism over a dense open subset $U \subset X$ with $h^{-1}(U)$ dense in Y and $g : Y \to S$ projective of relative dimension $\leq n$. In particular,

$$\dim Y = \dim h^{-1}(U) = \dim U = \dim X \leq d .$$

By (2), Theorem 7.17 holds for $(Y/S, \Sigma')$ where Σ' is any stratification on Y. Then Lemma 7.25 follows from Lemma 7.23. ☐

Lemma 7.26. *Let $d \geq 1$ and assume that*

1. *Theorem 7.17 holds in absolute dimension $\leq d - 1$.*

(2) *For every* $n \geq 0$, *Theorem* 7.17 *holds for every projective normal relative stratified scheme* $(X/S, \Sigma)$ *of relative dimension* $\leq n$ *with* $\dim X \leq d$ *and* S *integral affine over* k.

Then, Theorem 7.17 *holds in absolute dimension* $\leq d$.

Proof. By Lemma 7.25, it is enough to show that for every $n \geq 0$, Theorem 7.17 holds for every projective relative stratified scheme $(X/S, \Sigma)$ of relative dimension $\leq n$ with $\dim X \leq d$. At the cost of replacing S by a dense open subset, we can assume that S is irreducible. Let η be the generic point of S and let $\overline{\eta} \to S$ be an algebraic geometric point lying over η.

Consider the normalization map $\alpha_{\overline{\eta}} : \widetilde{X}^{\mathrm{red}}_{\overline{\eta}} \to X^{\mathrm{red}}_{\overline{\eta}}$ and the reduction map $\beta_{\overline{\eta}} : X^{\mathrm{red}}_{\overline{\eta}} \to X_{\overline{\eta}}$. By spreading out $\alpha_{\overline{\eta}}$ and $\beta_{\overline{\eta}}$, there is a commutative diagram with cartesian squares

$$\begin{tikzcd}
\widetilde{X}^{\mathrm{red}}_{\overline{\eta}} \arrow[r] \arrow[d, "\alpha_{\overline{\eta}}"] & Y_T \arrow[d, "\alpha"] & \\
X^{\mathrm{red}}_{\overline{\eta}} \arrow[r] \arrow[d, "\beta_{\overline{\eta}}"] & Z_T \arrow[d, "\beta"] & \\
X_{\overline{\eta}} \arrow[r] \arrow[d, "f_{\overline{\eta}}"] & X_T \arrow[r] \arrow[d, "f_T"] & X \arrow[d, "f"] \\
\overline{\eta} \arrow[r] & T \arrow[r] & S
\end{tikzcd}$$

where $T \to S$ is quasi-finite flat with T integral affine over k. Since $\beta_{\overline{\eta}} : X^{\mathrm{red}}_{\overline{\eta}} \to X_{\overline{\eta}}$ is radicial, finite and surjective, [GD64, Théorème 8.10.5] implies that at the cost of shrinking T, we can suppose that β is radicial, finite and surjective as well. In particular, β is a universal homeomorphism by [GD64, Corollaire 18.12.11]. Since $\alpha_{\overline{\eta}} : \widetilde{X}^{\mathrm{red}}_{\overline{\eta}} \to X^{\mathrm{red}}_{\overline{\eta}}$ is finite and an isomorphism over a dense open subset of $U_{\overline{\eta}} \subset X^{\mathrm{red}}_{\overline{\eta}}$ with $\alpha^{-1}_{\overline{\eta}}(U_{\overline{\eta}})$ dense in $\widetilde{X}^{\mathrm{red}}_{\overline{\eta}}$, we can similarly at the cost of replacing T by a further quasi-finite flat scheme T' over T integral affine over k suppose that α is finite and an isomorphism over a dense open subset of $U \subset Z_T$ with $\alpha^{-1}(U)$ dense in Y_T.

To show that Theorem 7.17 holds for $(X/S, \Sigma)$, it is enough to show that it holds for $(X_T/T, \Sigma_T)$. By invariance of the étale topos under universal homeomorphism, it is enough to show that it holds for $(Z_T/T, \beta^*\Sigma_T)$. By Lemma 7.23, it is enough to show that it holds for $(Y_T/T, \Sigma')$ where Σ' is any stratification on Y_T. We have

$$\dim Y_T = \dim \alpha^{-1}(U) = \dim U = \dim Z_T = \dim X_T \leq \dim X \leq d\,.$$

Note that Y_T is projective over T of relative dimension $\leq n$. Hence, we are left to prove that for every $n \geq 0$, Theorem 7.17 holds for every projective relative stratified scheme $(X/S, \Sigma)$ of relative dimension $\leq n$ with $\dim X \leq d$, with S integral affine over k where $X_{\overline{\eta}}$ is normal. Since S is reduced, at the cost of shrinking S, we can suppose by generic flatness that X is flat over S. By [GD64, Théorème 12.2.4] and at the cost of shrinking S further, we can suppose that the morphism $X \to S$ is normal. The conclusion then follows by assumption (2). ☐

Lemma 7.27. *Let* $f : X \longrightarrow S$ *be a projective morphism of schemes of finite type over* k *with* S *irreducible and with geometrically reduced generic fibre of pure dimension* n. *Let* $D \subset X$ *be a divisor. Then, there is a dense open subset* $V \subset S$ *and a finite surjective morphism* $h : X_V \longrightarrow \mathbb{P}^n_V$ *sending* D_V *to the hyperplane* H_V *such that* $h : X_V \longrightarrow \mathbb{P}^n_V$ *is finite étale over* $\mathbb{A}^n_V := \mathbb{P}^n_V - H_V$.

Proof. At the cost of shrinking S, we can suppose that every generic point of D is mapped to the generic point η of S. By [Ke05, Theorem 1], there exists a finite surjective morphism $h_\eta : X_\eta \longrightarrow \mathbb{P}^n_\eta$ sending D_η in a hyperplane $H_\eta \subset \mathbb{P}_\eta$ and such that $h_\eta : X_\eta \longrightarrow \mathbb{P}_\eta$ is finite étale above $\mathbb{A}^n_\eta := \mathbb{P}^n_\eta - H_\eta$. At the cost of shrinking S, we can suppose that the morphism $h_\eta : X_\eta \longrightarrow \mathbb{P}^n_\eta$ spreads out as a finite surjective morphism $h : X \longrightarrow \mathbb{P}^n_S$ over S such that $h : X \longrightarrow \mathbb{P}_S$ is finite étale above $\mathbb{A}^n_S := \mathbb{P}^n_S - H_S$. Since every generic point of D is mapped to η and since h_η sends D_η in H_η, we have $D \subset h^{-1}(H_S)$. Lemma 7.27 is thus proved. □

Lemma 7.28. *Let* $n \geq 0$ *and let* $P \in \mathbb{N}[x]^{2n}$ *with* P_i *of degree smaller than* $\min(i, 2n-i)$ *for* $i = 0, \ldots, 2n$. *Then, there is* $P' \in \mathbb{N}[x]^n$ *with* P'_i *of degree* i *for* $i = 0, \ldots, n$ *such that for every* $i = 0, \ldots, 2n$ *we have* $P_i \leq P'_{\min(i,2n-i)}$.

Proof. For $0 \leq i \leq n$, put $P'_i = x^i + P_i + P_{2n-i}$. □

7.29. Proof of Theorem 7.15. We argue by recursion on d that Theorem 7.15 holds in absolute dimension $\leq d$. By Lemma 7.21, it is enough to show that Theorem 7.17 holds in absolute dimension $\leq d$. If $d = 0$, the claim is obvious. Assume that $d \geq 1$. By Lemma 7.26, it is enough to show that for every $n \geq 0$, Theorem 7.17 holds for every projective normal relative stratified scheme $(X/S, \Sigma)$ of relative dimension $\leq n$ with $\dim X \leq d$ and S integral affine over k .

In that case, X_η is normal. By [SP23, Tag 0357], X_η is thus a disjoint union of normal integral schemes $X_{1,\eta}, \ldots, X_{m,\eta}$. For $i = 1, \ldots, m$, let X_i be the closure of $X_{i,\eta}$ in X. At the cost of shrinking S, we can suppose by [SP23, Tag 054Y] that $X = \bigcup_{i=1}^m X_i$. Note that for every $i \neq j$, the intersection $X_i \cap X_j$ does not meet X_η. Hence, at the cost of shrinking S further, we have $X = \bigsqcup_{i=1}^m X_i$. Thus, at the cost of considering the $f|_{X_i} : X_i \longrightarrow S$, we can further suppose that the generic fibre of $f : X \longrightarrow S$ is integral. Let $0 \leq N \leq n$ be its dimension. At the cost of shrinking S, we can also suppose that X is integral.

Let $D \subset X$ be a divisor containing the strata of Σ of dimension $< \dim X$. By Lemma 7.27, at the cost of shrinking S, we can suppose the existence of a finite surjective morphism $h : X \longrightarrow \mathbb{P}^N_S$ sending D to the hyperplane H_S such that $h : X \longrightarrow \mathbb{P}^N_S$ is finite étale over $\mathbb{A}^N_S := \mathbb{P}^N_S - H_S$. Put $i : Z := h^{-1}(H_S) \hookrightarrow X$ and note that $\dim Z < \dim X \leq d$. To conclude, it is enough to check that Z satisfies the conditions of Lemma 7.22. Let $j : U := h^{-1}(\mathbb{A}^N_S) \hookrightarrow X$ be the inclusion. Let $\bar{s} \longrightarrow S$ be an algebraic geometric point, let Λ be a finite field of characteristic $\ell \neq p$, let $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$ and let $\mathcal{F} \in \mathrm{Cons}(X_{\bar{s}}, \mathcal{E}_{\bar{s}}, \Lambda)$ extension by 0 of an object of $\mathrm{Cons}_{(j^*\Sigma)_{\bar{s}}}(U_{\bar{s}}, \Lambda) = \mathrm{Loc}(U_{\bar{s}}, \Lambda)$, where the last equality follows from the fact that all the strata of $j^*\Sigma$ are open subsets of U. Since finite direct images are exact, we have

$$h^\bullet(X_{\bar{s}}, \mathcal{F}) = h^\bullet(\mathbb{P}^N_{\bar{s}}, h_{\bar{s}*}\mathcal{F})$$

where $h_{\bar{s}*}\mathcal{F}$ is the extension by 0 of a locally constant constructible sheaf on $\mathbb{A}^N_{\bar{s}}$. Hence,

$$h^j(X_{\bar{s}}, \mathcal{F}) = 0$$

for $0 \leq j < N$ and $j > 2N$ and if $\delta \geq 1$ is the generic degree of $h : X \to \mathbb{P}^N_S$, Corollary 6.4 implies that

$$h^j(X_{\bar{s}}, \mathcal{F}) \leq \mathfrak{b}_{2N-j}(\mathrm{lc}_{H_{\bar{s}}}(h_{\bar{s}*}\mathcal{F})) \cdot \delta \cdot \mathrm{Rk}_\Lambda \mathcal{F}$$

for $N \leq j \leq 2N$. By Theorem 2.15 applied to $h_{\bar{s}} : X_{\bar{s}} \to \mathbb{P}_{\bar{s}}$, we have

$$\mathrm{lc}_{H_{\bar{s}}}(h_{\bar{s}*}\mathcal{F}) \leq \mathrm{lc}_{H_{\bar{s}}}(h_{\bar{s}*}(j_{\bar{s}!}\Lambda)) + \delta \cdot \mathrm{lc}_{Z_{\bar{s}}}(\mathcal{F}) \,.$$

By [HT25a, Corollary 5.8] applied to $\mathbb{P}^N_S \to S$ and $H_S \subset \mathbb{P}^N_S$ and to the sheaf $h_* j_! \Lambda$, we have furthermore

$$\mathrm{lc}_{H_{\bar{s}}}(h_{\bar{s}*}j_{\bar{s}!}\Lambda) \leq \mathrm{lc}_{H_{\bar{\eta}}}(h_{\bar{\eta}*}j_{\bar{\eta}!}\Lambda) \,.$$

Put $\alpha := \lceil \mathrm{lc}_{H_{\bar{\eta}}}(h_{\bar{\eta}*}j_{\bar{\eta}!}\Lambda) \rceil \in \mathbb{N}$. On the other hands, Proposition 7.4 applied to the normal scheme $X_{\bar{s}}$ and to the effective Cartier divisor $Z_{\bar{s}}$ gives

$$\mathrm{lc}_{Z_{\bar{s}}}(\mathcal{F}) \leq c(T(\mathcal{E}_{\bar{s}})) \leq \mu_f(\mathcal{E})$$

where $c(T(\mathcal{E}_{\bar{s}}))$ is the maximal multiplicity of $T(\mathcal{E}_{\bar{s}})$ and where $\mu_f : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q}$ is defined in Example 7.10. Hence, for $N \leq j \leq 2N$ we have

$$h^j(X_{\bar{s}}, \mathcal{F}) \leq \mathfrak{b}_{2N-j}(\alpha + \delta \cdot \mu_f(\mathcal{E})) \cdot \delta \cdot \mathrm{Rk}_\Lambda \mathcal{F} \,.$$

If we put $P_j = 0$ for $j \in [0, 2n] \setminus [N, 2N]$ and $P_j = \delta \cdot \mathfrak{b}_{2N-j}(\alpha + \delta \cdot (-)) \in \mathbb{N}[x]$ for $N \leq j \leq 2N$, we have

$$h^j(X_{\bar{s}}, \mathcal{F}) \leq P_j(\mu_f(\mathcal{E})) \cdot \mathrm{Rk}_\Lambda \mathcal{F}$$

for $0 \leq j \leq 2n$. Lemma 7.28 gives the existence of $P' \in \mathbb{N}[x]^n$ with P'_i of degree i for $i = 0, \ldots, n$ such that for every $i = 0, \ldots, 2n$ we have $P_i \leq P'_{\min(i, 2n-i)}$. Hence, for $j = 0, \ldots, 2n$, we have

$$h^j(X_{\bar{s}}, \mathcal{F}) \leq P'_{\min(j, 2n-j)}(\mu_f(\mathcal{E})) \cdot \mathrm{Rk}_\Lambda \mathcal{F} \,.$$

The proof of Theorem 7.15 is thus complete.

Corollary 7.30. *Let $(X/S, \Sigma)$ be a proper relative stratified scheme of relative dimension $\leq n$ and let $a \leq b$ be integers. Then, there is an admissible function $\mu : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q}$ and $P \in \mathbb{N}[x]$ of degree n such that for every algebraic geometric point $\bar{s} \to S$, every finite field Λ of characteristic $\ell \neq p$, every $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$ and every $\mathcal{K} \in D^{[a,b]}_{\Sigma_{\bar{s}}}(X_{\bar{s}}, \mathcal{E}_{\bar{s}}, \Lambda)$, we have*

$$\sum_{j \in \mathbb{Z}} h^j(X_{\bar{s}}, \mathcal{K}) \leq P(\mu(\mathcal{E})) \cdot \mathrm{Rk}_\Lambda \mathcal{K} \,.$$

Proof. Immediate from Theorem 7.15 using the hypercohomology spectral sequence. □

Corollary 7.31. *Let $(X/S, \Sigma)$ be a proper relative stratified scheme of relative dimension $\leq n$ and let $a \leq b$ be integers. Then, there is an admissible function $\mu : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q}$ and $P \in \mathbb{N}[x]$ of degree n such that for every $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$, every finite field Λ of characteristic $\ell \neq p$ and every $\mathcal{K} \in D^{[a,b]}_{\Sigma}(X, \mathcal{E}, \Lambda)$, we have*

$$\mathrm{Rk}_\Lambda Rf_* \mathcal{K} \leq P(\mu(\mathcal{E})) \cdot \mathrm{Rk}_\Lambda \mathcal{K} \,.$$

Proof. Immediate by proper base change theorem combined with Lemma 7.6 and Corollary 7.30. ☐

Theorem 7.32. *Let* (X, Σ) *be a proper stratified scheme of dimension* n *over an algebraically closed field* k*. Then, there is an admissible function* $\mu : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q}$ *and* $P \in \mathbb{N}[x]^{n+1}$ *with* P_i *of degree* i *for* $i = 0, \dots, n$ *such that for every finite field* Λ *of characteristic* $\ell \neq p$*, every* $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$*, every* $j = -n, \dots, n$ *and every* $\mathcal{P} \in \mathrm{Perv}_\Sigma(X, \mathcal{E}, \Lambda)$*, we have*

$$h^j(X, \mathcal{P}) \leq P_{\min(n+j,n-j)}(\mu(\mathcal{E})) \cdot \mathrm{Rk}_\Lambda \mathcal{P} .$$

Proof. For $0 \leq i \leq n$, let $\iota_i : X_i \hookrightarrow X$ be the closure of the union of the strata of Σ of dimension at most i. Then, $\mathcal{H}^{-i}\mathcal{P}$ is supported on X_i and its restriction to X_i is an object of $\mathrm{Cons}_{\iota_i^*\Sigma}(X_i, \iota_i^*\mathcal{E}, \Lambda)$. If (μ^i, P^i) is a Betti bound for $(X_i, \iota_i^*\Sigma)$ as in Theorem 7.15, we thus have

$$H^j(X, \mathcal{H}^{-i}\mathcal{P}) \leq P^i_{\min(j,2i-j)}(\mu^i(\iota_i^*\mathcal{E})) \cdot \mathrm{Rk}_\Lambda \mathcal{H}^{-i}\mathcal{P}$$

for every $0 \leq j \leq 2i$. Consider the admissible function

$$\mu := \sum_{i=1}^{n} \mu^i \circ \iota_i^* : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q} .$$

The hypercohomology spectral sequence reads

$$H^p(X, \mathcal{H}^q\mathcal{P}) \Rightarrow H^{p+q}(X, \mathcal{P}) .$$

Assume that $0 \leq i \leq n$. Then, the contributions to $H^{-i}(X, \mathcal{P})$ come from $H^m(X, \mathcal{H}^{-i-m}\mathcal{P})$ for $0 \leq m \leq n-i$. Thus,

$$\begin{aligned} h^{-i}(X, \mathcal{P}) &\leq \sum_{m=0}^{n-i} h^m(X, \mathcal{H}^{-i-m}\mathcal{P}) \\ &\leq \sum_{m=0}^{n-i} P^{i+m}_{\min(m,2i+m)}(\mu^{i+m}(\iota_{i+m}^*\mathcal{E})) \cdot \mathrm{Rk}_\Lambda \mathcal{H}^{-i-m}\mathcal{P} \\ &\leq \sum_{m=0}^{n-i} P^{i+m}_m(\mu(\mathcal{E})) \cdot \mathrm{Rk}_\Lambda \mathcal{P} . \end{aligned}$$

Observe that $\mathfrak{e}_i := \sum_{m=0}^{n-i} P^{i+m}_m$ has degree $n - i = \min(n-i, n+i)$. On the other hand, the contributions to $H^i(X, \mathcal{P})$ come from $H^m(X, \mathcal{H}^{i-m}\mathcal{P})$ for $i \leq m \leq n+i$. Thus,

$$\begin{aligned} h^{i}(X, \mathcal{P}) &\leq \sum_{m=i}^{n+i} h^m(X, \mathcal{H}^{i-m}\mathcal{P}) \\ &\leq \sum_{m=i}^{n+i} P^{m-i}_{\min(m,m-2i)}(\mu^{m-i}(\iota_{m-i}^*\mathcal{E})) \cdot \mathrm{Rk}_\Lambda \mathcal{H}^{i-m}\mathcal{P} \\ &\leq \sum_{m=2i}^{n+i} P^{i+m}_{m-2i}(\mu(\mathcal{E})) \cdot \mathrm{Rk}_\Lambda \mathcal{P} . \end{aligned}$$

Observe that $\mathfrak{f}_i := \sum_{m=2i}^{n+i} P_{m-2i}^{i+m}$ has degree $n-i = \min(n+i, n-i)$. Thus, the admissible function $\mu : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q}$ and the sequence

$$P := (\mathfrak{e}_n + \mathfrak{f}_n, \dots, \mathfrak{e}_0 + \mathfrak{f}_0) \in \mathbb{N}[x]^{n+1}$$

do the job. ☐

Remark 7.33. Theorem 7.32 holds more generally for complexes satisfying the support condition.

When X is smooth and $\mathcal{K}$ is the extension by 0 of a locally constant constructible sheaf on the complement of an effective Cartier divisor, Theorem 7.32 implies the following

Theorem 7.34. *Let X be a proper smooth scheme of finite type of dimension n over an algebraically closed field k. Let D be a reduced effective Cartier divisor of X and put $j : U := X - D \hookrightarrow X$. Then, there exists $P \in \mathbb{N}[x]^{n+1}$ with P_i of degree i for $i = 0, \dots, n$ such that for every finite field Λ of characteristic $\ell \neq p$, every $j = 0, \dots, 2n$ and every $\mathcal{L} \in \mathrm{Loc}(U, \Lambda)$ we have*

$$h^j(U, \mathcal{L}) \leq P_{\min(j, 2n-j)}(\mathrm{lc}_D(\mathcal{L})) \cdot \mathrm{rk}_\Lambda \mathcal{L} \, .$$

Proof. By Poincaré duality, it is enough to consider the analogous statement for h^j_c. Put $\Sigma := \{U, D\}$ and let $j : U \hookrightarrow X$ be the inclusion. Let (μ, P) be a Betti bound for (X, Σ) whose existence follows from Theorem 7.15. Since X is smooth over k, Proposition 7.4-2 implies that $j_!\mathcal{L}$ has log conductors bounded by $(\mathrm{lc}_D(\mathcal{L}) + 1) \cdot \mathcal{O}_D$. On the other hand,

$$\mu((\mathrm{lc}_D(\mathcal{L}) + 1) \cdot \mathcal{O}_D) = \mu(\mathcal{O}_D) \cdot (\mathrm{lc}_D(\mathcal{L}) + 1) \, .$$

We conclude by using the sequence $P(\mu(\mathcal{O}_D) \cdot ((-) + 1))$ instead of P. ☐

Recollection 7.35. Let X be a scheme of finite type over a field k of characteristic $p > 0$ separably closed or finite and let $\ell \neq p$. Then, for every $\mathcal{K} \in D^b_c(X, \overline{\mathbb{Q}}_\ell)$, there is a finite extension $L/\mathbb{Q}_\ell$ and an integral representative $\mathcal{K}_\bullet = (\mathcal{K}_m)_{m \geq 0}$ for $\mathcal{K}$. If we put $\Lambda_m := \mathcal{O}_L/\mathfrak{m}_L^m$, the sheaf $\mathcal{K}_m$ is an object of $D^b_{cft}(X, \Lambda_m)$ such that $\Lambda_m \otimes^L_{\Lambda_{m+1}} \mathcal{K}_{m+1} \simeq \mathcal{K}_m$.

The following definition upgrades Definition 7.3 to $\overline{\mathbb{Q}}_\ell$-coefficients.

Definition 7.36. Let X be a scheme of finite type over a field k of characteristic $p > 0$ separably closed or finite and let $\ell \neq p$. Let $\mathcal{K} \in D^b_c(X, \overline{\mathbb{Q}}_\ell)$ and $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$. We say that $\mathcal{K}$ *has log conductors bounded by* $\mathcal{E}$ if there is a finite extension $L/\mathbb{Q}_\ell$ and an integral representative $\mathcal{K}_\bullet = (\mathcal{K}_m)_{m \geq 0}$ for $\mathcal{K}$ such that for every $m \geq 0$, the complex $\mathcal{K}_m \in D^b_{cft}(X, \Lambda_m)$ has log conductors bounded by $\mathcal{E}$ in the sense of Definition 7.3. We denote by $D^b_c(X, \mathcal{E}, \overline{\mathbb{Q}}_\ell)$ the full subcategory of $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ spanned by objects having log conductors bounded by $\mathcal{E}$.

Remark 7.37. Theorem 7.15, Corollary 7.30, Theorem 7.32 and Theorem 7.34 admit immediate variants for $\overline{\mathbb{Q}}_\ell$ coefficients with the help of the next lemma.

Lemma 7.38 ([FW04, Lemma 1.8]). *Let X be a separated scheme of finite type over an algebraically closed field k. For every $\mathcal{K} \in D^b_c(X, \overline{\mathbb{Q}}_\ell)$ represented by $\mathcal{K}_\bullet$ in the sense of Recollection 7.35, we have*

$$\dim_{\Lambda_0} H^n(X, \mathcal{K}_0) \geq \dim_{\overline{\mathbb{Q}}_\ell} H^n(X, \mathcal{K}) .$$

for any $n \in \mathbb{Z}$.

Proof. By [Fu11, 10.1.5, 10.1.17], there is a bounded complex M of free $\mathcal{O}_L$-modules of finite ranks such that, for any $i \geq 0$, the complex $M_i = M \otimes_{\mathcal{O}_L} \Lambda_i$ represents $R\Gamma(X, \mathcal{K}_i)$. Let π_L be a uniformizer of $\mathcal{O}_L$. We take an exact sequence of bounded complexes of $\mathcal{O}_L$-modules

$$0 \to M \xrightarrow{\cdot \pi_L} M \to M_0 \to 0 .$$

Then, for each integer $n \geq 0$, we have a long exact sequence

$$H^{n-1}(X, \mathcal{K}_0) \to H^n(M) \xrightarrow{\cdot \pi_L} H^n(M) \to H^n(X, \mathcal{K}_0) ,$$

which induces an injection $H^n(M) \otimes_{\mathcal{O}_L} \Lambda_0 \to H^n(X, \mathcal{K}_0)$. Hence, for any $n \in \mathbb{Z}$, we have

$$\begin{aligned} \dim_{\Lambda_0} H^n(X, \mathcal{K}_0) &\geq \dim_{\Lambda}(H^n(M) \otimes_{\mathcal{O}_L} \Lambda_0) \\ &\geq \dim_{\overline{\mathbb{Q}}_\ell}(H^n(M) \otimes_{\mathcal{O}_L} \overline{\mathbb{Q}}_\ell) \\ &= \dim_{\overline{\mathbb{Q}}_\ell}(H^n(X, \mathcal{K})) . \end{aligned}$$

□

8. Betti numbers of inverse and higher direct images

Recollection 8.1. Let X be a projective scheme of finite type over k and let $i : X \hookrightarrow \mathbb{P}_k(E)$ be a closed immersion. For $1 \leq a \leq \operatorname{rk} E$, we let $\overline{\eta}$ be a geometric generic point of $\mathbb{G}_k(E, a)$ and let $F \subset \mathbb{P}_{\overline{\eta}}(E_{\overline{\eta}})$ be the corresponding projective subspace of dimension $a-1$ over $\overline{\eta}$. Following [SFFK21, Definition 6.3], we define the *complexity to* i *of* $\mathcal{K} \in D^b_c(X, \overline{\mathbb{Q}}_\ell)$ by

$$c_i(\mathcal{K}) := \max_{1 \leq a \leq n} \sum_{j \in \mathbb{Z}} h^j(X_{\overline{\eta}} \cap F, \mathcal{K}|_{X_{\overline{\eta}} \cap F}) .$$

In particular, we have

$$\sum_{j \in \mathbb{Z}} h^j(X, \mathcal{K}) \leq c_i(\mathcal{K}) .$$

Lemma 8.2. *Let $f : X \to S$ be a projective morphism over k and let $i : X \hookrightarrow \mathbb{P}_k(E)$ be a closed immersion. For every $\mathcal{K} \in D^b_c(X, \overline{\mathbb{Q}}_\ell)$, there is $N(\mathcal{K}) \geq 0$ such that for every $s \in S$, we have $c_{i_s}(\mathcal{K}|_{X_s}) \leq N(\mathcal{K})$.*

Proof. This is a direct consequence of the proper base change theorem and the constructibility of the higher direct images applied to the families of sections by projective subspaces. □

Theorem 8.3. *Let* $(X/S, \Sigma)$ *be a projective relative stratified scheme of relative dimension* $\leq n$ *and let* $i : X \hookrightarrow \mathbb{P}_S(E)$ *be a closed immersion. Let* $a \leq b$ *be integers. Then, there is an admissible function* $\mu : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q}$ *and* $P \in \mathbb{N}[x]$ *of degree* n *such that for every algebraic geometric point* $\bar{s} \to S$*, every prime* $\ell \neq p$*, every* $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$ *and every* $\mathcal{K} \in D^{[a,b]}_{\Sigma_{\bar{s}}}(X_{\bar{s}}, \mathcal{E}_{\bar{s}}, \overline{\mathbb{Q}}_\ell)$*, we have*

$$c_{i_{\bar{s}}}(\mathcal{K}) \leq P(\mu(\mathcal{E})) \cdot \mathrm{Rk}_{\overline{\mathbb{Q}}_\ell} \mathcal{K} \, .$$

Proof. For $2 \leq r \leq \mathrm{rk}\, E$, consider the commutative diagram

$$\begin{tikzcd}
X(E,r) \arrow[r] \arrow[d] \arrow[dd, bend right, "p^\vee_{r,X}"'] & X \arrow[d, "i"] \\
\mathrm{Fl}_S(E,1,r) \arrow[r] \arrow[d] & \mathbb{P}_S(E) \arrow[d] \\
\mathbb{G}_S(E,r) \arrow[r] & S
\end{tikzcd}$$

with cartesian upper square where $\mathrm{Fl}_S(E,1,r)$ is the universal family of dimension r projective subspaces of $\mathbb{P}_S(E)$. Then, Theorem 8.3 follows from Corollary 7.30 applied to $p^\vee_{r,X} : X(E,r) \to \mathbb{G}_S(E,r)$. ☐

Observation 8.4. Every continuity from [SFFK21] translates via Theorem 8.3 into some estimate involving the rank and the wild ramification only.

We formulate below two instances of the above observation.

Theorem 8.5. *Let* $f : Y \to X$ *be a morphism between projective schemes over* k *algebraically closed. Let* Σ *be a stratification on* X *and let* $a \leq b$ *be integers. Then, there is an admissible function* $\mu : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q}$ *and* $P \in \mathbb{N}[x]$ *of degree* $\dim X$ *such that for every prime* $\ell \neq p$*, every* $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$ *and every* $\mathcal{K} \in D^{[a,b]}_{\Sigma}(X, \mathcal{E}, \overline{\mathbb{Q}}_\ell)$*, we have*

$$\sum_{j \in \mathbb{Z}} h^j(Y, f^*\mathcal{K}) \leq P(\mu(\mathcal{E})) \cdot \mathrm{Rk}_{\overline{\mathbb{Q}}_\ell} \mathcal{K} \, .$$

Proof. Choose closed immersion $i : X \hookrightarrow \mathbb{P}_k(E)$ and $i' : Y \hookrightarrow \mathbb{P}_k(E')$. By Theorem 8.3, there is an admissible function $\mu : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q}$ and $P \in \mathbb{N}[x]$ of degree $\dim X$ such that for every prime $\ell \neq p$, every $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$ and every $\mathcal{K} \in D^{[c,d]}_{\Sigma}(X, \mathcal{E}, \overline{\mathbb{Q}}_\ell)$, we have

$$c_i(\mathcal{K}) \leq P(\mu(\mathcal{E})) \cdot \mathrm{Rk}_{\overline{\mathbb{Q}}_\ell} \mathcal{K} \, .$$

By [SFFK21, Theorem 6.8], there is an integer C depending only on $\mathrm{rk}\, E$ and $\mathrm{rk}\, E'$ such that for every prime $\ell \neq p$ and every $\mathcal{K} \in D^b_c(X, \overline{\mathbb{Q}}_\ell)$, we have

$$\sum_{j \in \mathbb{Z}} h^j(Y, f^*\mathcal{K}) \leq c_{i'}(f^*\mathcal{K}) \leq C \cdot c_i(\mathcal{K}) \, .$$

The conclusion thus follows. ☐

Corollary 8.6. *Let* $f : Y \to X$ *be a morphism between projective schemes over* k *algebraically closed where* X *is smooth. Let* D *be an effective Cartier divisor of* X *and put* $U := X - D$ *and* $V := Y - f^{-1}(D)$*. Then there is* $P \in \mathbb{N}[x]$ *of degree* $\dim X$ *such that for every prime* $\ell \neq p$ *and every* $\mathcal{L} \in \mathrm{Loc}(U, \overline{\mathbb{Q}}_\ell)$*, we have*

$$\sum_{j\in\mathbb{Z}} h^j_c(V, \mathcal{L}|_V) \leq P(\mathrm{lc}_D(\mathcal{L})) \cdot \mathrm{rk}_{\overline{\mathbb{Q}}_\ell} \mathcal{L} \,.$$

Proof. Let $j : U \hookrightarrow X$ be the inclusion. Then, combine Proposition 7.4 and Theorem 8.5 applied to $j_!\mathcal{L}$. □

Theorem 8.7. *Let* $(X/S, \Sigma)$ *be a projective relative stratified scheme over* k *algebraically closed such that* S *is projective. Let* $a \leq b$ *be integers. Then, there is an admissible function* $\mu : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q}$ *and an increasing function* $N : \mathbb{Q}^+ \times \mathbb{Q}^+ \to \mathbb{N}^+$ *such that for every prime* $\ell \neq p$*, every* $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$ *and every* $\mathcal{K} \in D^{[a,b]}_\Sigma(X, \mathcal{E}, \overline{\mathbb{Q}}_\ell)$*, we have*

$$\sum_{i,j\in\mathbb{Z}} h^j(S, R^i f_* \mathcal{K}) \leq N(\mu(\mathcal{E}), \mathrm{Rk}_{\overline{\mathbb{Q}}_\ell} \mathcal{K}) \,.$$

Proof. Choose closed embeddings $i : X \hookrightarrow \mathbb{P}_k(E)$ and $i' : S \hookrightarrow \mathbb{P}_k(E')$. By Theorem 8.3, there is a sub-additive function $\mu : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q}$ and $P \in \mathbb{N}[x]$ of degree $\dim X$ such that for every prime $\ell \neq p$, every $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$ and every $\mathcal{K} \in D^{[a,b]}_\Sigma(X, \mathcal{E}, \overline{\mathbb{Q}}_\ell)$, we have

$$c_i(\mathcal{K}) \leq P(\mu(\mathcal{E})) \cdot \mathrm{Rk}_{\overline{\mathbb{Q}}_\ell} \mathcal{K} \,.$$

By [SFFK21, Proposition 6.24], there is an increasing function $M : \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{N}^+$ such that for every prime $\ell \neq p$ and every $\mathcal{K}' \in D^b_c(S, \overline{\mathbb{Q}}_\ell)$, we have

$$\sum_{j\in\mathbb{Z}} c_{i'}(\mathcal{H}^j \mathcal{K}') \leq M(\mathrm{rk}\, E', c_{i'}(\mathcal{K}')) \,.$$

By [SFFK21, Theorem 6.8], there is $C \geq 0$ depending only on (X, i),(S, i') and $X \to S$ such that for every $\mathcal{K} \in D^b_c(X, \overline{\mathbb{Q}}_\ell)$, we have

$$c_{i'}(Rf_* \mathcal{K}) \leq C \cdot c_i(\mathcal{K}) \,.$$

For every prime $\ell \neq p$, every $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$ and $\mathcal{K} \in D^{[a,b]}_\Sigma(X, \mathcal{E}, \overline{\mathbb{Q}}_\ell)$, we thus have

$$\begin{aligned}\sum_{j,j'\in\mathbb{Z}} h^j(S, R^{j'} f_* \mathcal{K}) &\leq \sum_{j\in\mathbb{Z}} c_{i'}(R^j f_* \mathcal{K}) \\ &\leq M(\mathrm{rk}\, E', c_{i'}(Rf_* \mathcal{K})) \\ &\leq M(\mathrm{rk}\, E', C \cdot c_i(\mathcal{K})) \\ &\leq M(\mathrm{rk}\, E', C \cdot \lceil P(\mu(\mathcal{E})) \cdot \mathrm{Rk}_{\overline{\mathbb{Q}}_\ell} \mathcal{K} \rceil) \,.\end{aligned}$$

The conclusion thus follows if we put $N(x, y) := M(\mathrm{rk}\, E', C \cdot \lceil P(x) \cdot y \rceil)$. □

Corollary 8.8. *Let* $f : X \to Y$ *be a projective morphism between projective schemes over* k *algebraically closed where* X *is smooth. Let* D *be an effective Cartier divisor of* X *and put*

$j : U := X - D \hookrightarrow X$. *Then, there is a function* $C : \mathbb{Q}^+ \times \mathbb{N}^+ \to \mathbb{N}^+$ *such that for every prime* $\ell \neq p$ *and every* $\mathcal{L} \in \mathrm{Loc}(U, \overline{\mathbb{Q}}_\ell)$*, we have*

$$\sum_{i,j \in \mathbb{Z}} h^j(Y, R^i f_* j_! \mathcal{L}) \leq C(\mathrm{lc}_D(\mathcal{L}), \mathrm{rk}_{\overline{\mathbb{Q}}_\ell} \mathcal{L}) .$$

Proof. Combine Proposition 7.4 and Theorem 8.7 applied to $j_! \mathcal{L}$. □

9. Relative Hermite-Minkowski for perverse sheaves

Notation 9.1. In this section, $\mathbb{F}$ denotes a finite field of characteristic $p > 0$ and $\overline{\mathbb{F}}$ an algebraic closure of $\mathbb{F}$. Let S be a scheme of finite type over $\mathbb{F}$. For a closed point $s \in S$, we denote by $\mathbb{F}(s)$ the residue field of s and put $\deg s := [\mathbb{F}(s), \mathbb{F}]$. For $n \geq 1$, there is up to isomorphism only one degree n extension of $\mathbb{F}(s)$ that we denote by $\mathbb{F}(s)_n$. We denote by S_n the pullback of S to $\mathbb{F}_n$.

Notation 9.2. We fix an isomorphism $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$. Purity will be understood with respect to this choice of isomorphism. Let X be a scheme of finite type over a finite field $\mathbb{F}$. For $\mathcal{P} \in \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$ and $n \geq 1$, we denote by

$$t_{\mathcal{P},n} : X(\mathbb{F}_n) \to \overline{\mathbb{Q}}_\ell$$

the trace function of $\mathcal{P}$. Reviewed as a complex-valued function via the above choice of isomorphism $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$, one can apply to it the usual hermitian product on $\mathbb{C}^{|X(\mathbb{F}_n)|}$.

Theorem 9.3 ([SFFK21, Theorem 7.13])**.** *Let* X *be a geometrically irreducible quasi-projective scheme of finite type over* $\mathbb{F}$ *and let* $i : X \hookrightarrow \mathbb{P}_{\mathbb{F}}(E)$ *be a closed immersion. Then, there is* $C_1, C_2 \geq 0$ *depending only on* $\mathrm{rk}\, E$ *such that for every prime* $\ell \neq p$*, the following hold :*

1. *For every* $\mathcal{P} \in \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$ *geometrically simple pure of weight 0, we have*

$$\left| \|t_{\mathcal{P},1}\|^2 - 1 \right| \leq C_1 \cdot c_i(\overline{\mathbb{Q}}_\ell) \cdot c_i(\mathcal{P})^2 \cdot |\mathbb{F}|^{-1/2} .$$

2. *For every* $\mathcal{P}, \mathcal{Q} \in \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$ *geometrically simple pure of weight 0 not geometrically isomorphic, we have*

$$\left| < t_{\mathcal{P},1}, t_{\mathcal{Q},1} > \right| \leq C_2 \cdot c_i(\overline{\mathbb{Q}}_\ell) \cdot c_i(\mathcal{P}) \cdot c_i(\mathcal{Q}) \cdot |\mathbb{F}|^{-1/2}$$

The following lemma could be proved by tracking the constants appearing in the Lang-Weil estimates [LW54]. We propose here a short proof based on [Del80].

Lemma 9.4. *Let* $f : X \to S$ *be a proper morphism of relative dimension* $\leq d$ *between schemes of finite type over* $\mathbb{F}$*. Then, there is* $C \in \mathbb{N}$ *such that for every closed point* $s \in S$*, we have*

$$|X_s(\mathbb{F}(s))_n| \leq C \cdot |\mathbb{F}(s)|^{nd} .$$

Proof. Put $C := \mathrm{Rk}\, Rf_* \overline{\mathbb{Q}}_\ell$. By proper base change, for every geometric point $\bar{s} \to S$ and every $0 \leq j \leq 2d$, we have

$$h^j(X_{\bar{s}}, \overline{\mathbb{Q}}_\ell) \leq C .$$

By [Del80], the complex $R^j f_* \overline{\mathbb{Q}}_\ell$ is mixed of weight $\leq j$. Hence for every $0 \leq j \leq 2d$ and for every closed point $s \in S$, the eigenvalues of the Frobenius Fr_s acting on $H^j(X_{\bar{s}}, \overline{\mathbb{Q}}_\ell)$

have weight $\leq j$. By the Lefschetz trace formula [Gro77, XII,XV],[Del77, Rapport], we deduce that for ever $n \geq 1$, we have

$$|X_s(\mathbb{F}(s)_n)| = \mathrm{Tr}(\mathrm{Fr}_s^n, H^\bullet(X_{\bar{s}}, \overline{\mathbb{Q}}_\ell)) \leq 2dC \cdot |\mathbb{F}(s)|^{nd} .$$

☐

Lemma 9.5. *Let* X *be a scheme of finite type over a field and let* $\mathcal{P} \in \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$*. If* $\mathcal{P}$ *is simple, then it is supported on some irreducible component of* X*.*

Proof. Immediate from [KW68, Corollary 5.5]. ☐

The following lemma is a relative variant of [SFFK21, Corollary 7.15]. An inspection of the proof of loc. cit. shows that it works almost verbatim in the relative setting as well. We detail why for the sake of completeness.

Lemma 9.6. *Let* $f : X \to S$ *be a projective morphism between schemes of finite type over* $\mathbb{F}$ *and let* $i : X \hookrightarrow \mathbb{P}_S(E)$ *be a closed immersion over* S*. Then, there is an increasing function* $N : \mathbb{R}^+ \times \mathbb{N}^+ \to \mathbb{N}^+$ *such that for every prime* $\ell \neq p$*, every* $c > 0$ *and every closed point* $s \in S$*, there is up to geometric isomorphism at most* $N(c, \deg s)$ *geometrically simple pure of weight 0 objects* $\mathcal{P} \in \mathrm{Perv}(X_s, \overline{\mathbb{Q}}_\ell)$ *with* $c_{i_s}(\mathcal{P}) \leq c$*.*

Proof. If $T \to S$ is a quasi-finite morphism where T is of finite type over $\mathbb{F}$, then [SP23, Tag 055B] implies the existence of $a \geq 1$ such that for every closed point $t \in T$ with $s = f(t) \in S$, we have $[\mathbb{F}(t) : \mathbb{F}(s)] \leq a$. In particular,

$$\deg t = \deg s \cdot [\mathbb{F}(t) : \mathbb{F}(s)] \leq a \cdot \deg s .$$

Hence if there is an increasing function $N : \mathbb{R}^+ \times \mathbb{N}^+ \to \mathbb{N}^+$ as in Lemma 9.6 for the pullback $f_T : X_T \to T$, for every prime $\ell \neq p$, every $c > 0$ and every closed point $t \in T$ with $s = f(t) \in S$, there is up to geometric isomorphism at most $N(c, a \cdot \deg s)$ geometrically simple pure of weight 0 objects $\mathcal{P} \in \mathrm{Perv}(X_s, \overline{\mathbb{Q}}_\ell)$ with $c_{i_s}(\mathcal{P}) \leq c$. Since S is noetherian, it is enough to show that Lemma 9.6 holds after pullback along a dominant quasi-finite morphism.

By [SP23, Tag 0551], we can thus assume that S is integral affine with generic point η and that the irreducible components $X_{\eta,1}, \dots, X_{\eta,m}$ of the generic fibre X_η are geometrically irreducible. For $1 \leq i \leq m$, let $X_i \subset X$ be the closure of $X_{\eta,i}$. At the cost of shrinking S, we can assume by [SP23, Tag 054Y] that $X = \bigcup_{i=1}^m X_i$. At the cost of shrinking S further, we can assume by [SP23, Tag 0559] that the $f|_{X_i} : X_i \to S$ have geometrically irreducible fibers. By Lemma 9.5, at the cost of considering the $f|_{X_i} : X_i \to S$ separately, we can assume that $f : X \to S$ has geometrically irreducible generic fibers. At the cost of performing a further quasi-finite pullback, we can assume that X_η admits a rational point. By spreading out, we can further assume that $f : X \to S$ admits a section. In particular for every $s \in S$, the set $X_s(\mathbb{F}(s))$ is not empty. By generic flatness, we can also assume that $f : X \to S$ is flat.

Let $\ell \neq p$ be a prime and let $s \in S$ be a closed point. By Theorem 9.3 applied to X_s, there is an integer C_1 depending only on $\mathrm{rk}\, E$ and not on s nor ℓ such that for every $\mathcal{P} \in \mathrm{Perv}(X_s, \overline{\mathbb{Q}}_\ell)$ with $c_{i_s}(\mathcal{P}) \leq c$ and every $n \geq 1$, we have

$$\left| ||t_{\mathcal{P},n}||^2 - 1 \right| \leq C_1 \cdot c_{i_s}(\overline{\mathbb{Q}}_\ell) \cdot c_{i_s}(\mathcal{P})^2 \cdot |\mathbb{F}(s)|^{-n/2} \leq C_2 \cdot c^2 \cdot 2^{-n/2}$$

where we used Lemma 8.2. Hence, there is $n_0(c) \geq 1$ increasing as a function of c such that for every $n \geq n_0(c)$, every prime $\ell \neq p$, every closed point $s \in S$ and every $\mathcal{P} \in \mathrm{Perv}(X_s, \overline{\mathbb{Q}}_\ell)$ with $c_{i_s}(\mathcal{P}) \leq c$, we have

$$3/4 < ||t_{\mathcal{P},n}||^2 < 5/4 \,.$$

By Theorem 9.3 applied to X_s, there is an integer C_3 depending only on rk E and not on s nor ℓ such that for every $\mathcal{P}, \mathcal{Q} \in \mathrm{Perv}(X_s, \overline{\mathbb{Q}}_\ell)$ geometrically simple pure of weight 0 and not geometrically isomorphic with $c_{i_s}(\mathcal{P}), c_{i_s}(\mathcal{Q}) \leq c$, every $n \geq 1$, we have

$$\big| < t_{\mathcal{P},n}, t_{\mathcal{Q},n} > \big| \leq C_3 \cdot c_{i_s}(\overline{\mathbb{Q}}_\ell) \cdot c_{i_s}(\mathcal{P}) \cdot c_{i_s}(\mathcal{Q}) \cdot |\mathbb{F}(s)|^{-n/2} \leq C_4 \cdot c^2 \cdot 2^{-n/2} \,.$$

Hence, there is $n_1(c) \geq 1$ increasing as a function of c such that for every $n \geq n_1(c)$, every prime $\ell \neq p$, every closed point $s \in S$ and every $\mathcal{P}, \mathcal{Q} \in \mathrm{Perv}(X_s, \overline{\mathbb{Q}}_\ell)$ geometrically simple pure of weight 0 and not geometrically isomorphic with $c_{i_s}(\mathcal{P}), c_{i_s}(\mathcal{Q}) \leq c$, we have

$$\big| < t_{\mathcal{P},n}, t_{\mathcal{Q},n} > \big| < 1/2 \,.$$

Put $n(c) := \max(n_0(c), n_1(c))$ and let $C, d \in \mathbb{N}$ as in Lemma 9.4 for $f : X \to S$. By assumption, for every closed point $s \in S$ and every $n \geq 1$, we have

$$1 \leq |X_s(\mathbb{F}(s)_n)| \leq C \cdot |\mathbb{F}(s)|^{nd} \,.$$

For $m \geq 1$, let $N(m) \in \mathbb{N}$ be the maximal number of unit vectors $x, y \in \mathbb{C}^m$ such that $| < x, y > | < 2/3$. By the above discussion, for every prime $\ell \neq p$ and every closed point $s \in S$, there are at most

$$N(|X_s(\mathbb{F}(s)_n)|) \leq N(C \cdot |\mathbb{F}(s)|^{n(c)d}) = N(C \cdot |\mathbb{F}|^{n(c)d \deg s})$$

geometrically simple pure of weight 0 objects $\mathcal{P} \in \mathrm{Perv}(X_s, \overline{\mathbb{Q}}_\ell)$ with $c_{i_s}(\mathcal{P}) \leq c$. ☐

Theorem 9.7. *Let* $(X/S, \Sigma)$ *be a relative projective stratified scheme over* $\mathbb{F}$. *Then, there is a function* $N : \mathbb{Q}[\mathrm{Coh}(X)] \times \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{N}^+$ *such that for every* $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$, *every* $r \geq 0$, *every prime* $\ell \neq p$ *and every closed point* $s \in S$, *there are up to geometric isomorphism at most* $N(\mathcal{E}, r, \deg s)$ *geometrically simple pure of weight 0 objects in* $\mathrm{Perv}^{\leq r}_{\Sigma_s}(X_s, \mathcal{E}_s, \overline{\mathbb{Q}}_\ell)$.

Proof. Follows immediately from Theorem 8.3 and Lemma 9.6. ☐

Remark 9.8. For $\mathcal{E} \in \mathrm{Coh}(X)$, the function $N : \mathbb{Q}[\mathrm{Coh}(X)] \times \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{N}^+$ constructed in the proof of Theorem 9.7 is so that the induced function $N_{\mathcal{E}} : \mathbb{Q}^+ \times \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{N}^+$ defined by

$$N_{\mathcal{E}}(c, r, d) = N(c \cdot \mathcal{E}, r, d)$$

is increasing.

The purity assumption in Theorem 9.7 can be removed at the cost of using the following consequence of the Langlands correspondence.

Theorem 9.9 ([Dri88, Dri89, Laf02]). *Let* U *be a connected normal scheme of finite type over* $\mathbb{F}_q$. *Let* $\mathcal{L} \in \mathrm{Loc}(U, \overline{\mathbb{Q}}_\ell)$ *and assume that* $\mathcal{L}$ *is simple with finite order determinant. Then* $\mathcal{L}$ *is pure of weight 0.*

Lemma 9.10. *Let* X *be a scheme of finite type over* $\mathbb{F}_q$. *Then, every simple perverse sheaf on* X *is geometrically isomorphic to a simple perverse sheaf which is pure of weight* 0.

Proof. Let $\mathcal{P} \in \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$ be a simple perverse sheaf. It is enough to show that there is a character $\chi : G_{\mathbb{F}_q} \to \overline{\mathbb{Q}}_\ell^\times$ such that the twist

$$\chi \cdot \mathcal{P} := p^*\chi \otimes_{\overline{\mathbb{Q}}_\ell} \mathcal{P}$$

is pure of weight 0, where $p : X \to \operatorname{Spec} \mathbb{F}_q$ is the structural morphism. We can suppose that X is reduced. By [KW68, Corollary 5.5], the complex $\mathcal{P}$ is of the form $i_* j_{!*} \mathcal{L}[d]$ where $i : Y \hookrightarrow X$ is an irreducible closed subset, where $j : U \hookrightarrow Y$ is a smooth dense open subset of dimension d and where $\mathcal{L} \in \mathrm{Loc}(U, \overline{\mathbb{Q}}_\ell)$ is simple. By [Fu11, Corollary 6.5.6], for any character χ coming from $\mathbb{F}_q$, we have

$$i_*(\chi \cdot j_{!*}\mathcal{L}[d]) \simeq \chi \cdot i_* j_{!*}\mathcal{L}[d] \simeq \chi \cdot \mathcal{P} \, .$$

Since i_* preserves pure complexes of weight 0 in virtue of [BBDG18, Stabilités 5.1.14], we can assume that $Y = X$. By [Del80, Proposition 1.3.4], the sheaf $\mathcal{L}$ admits a twist $\chi \cdot \mathcal{L}$ with finite order determinant. Then, Theorem 9.9 applied to $\chi \cdot \mathcal{L}$ on the smooth connected scheme U ensures that $\chi \cdot \mathcal{L}$ is pure of weight 0. At the cost of twisting $\chi \cdot \mathcal{L}$ further by a character of weight $-d$ coming from $\mathbb{F}_q$, there is a character χ coming from $\mathbb{F}_q$ such that $\chi \cdot \mathcal{L}$ is pure of weight $-d$. In particular $\chi \cdot \mathcal{L}[d]$ is pure of weight 0. On the other hand, we have

$$j_{!*}(\chi \cdot \mathcal{L}[d]) \simeq \chi \cdot j_{!*}\mathcal{L}[d] \simeq \chi \cdot \mathcal{P} \, .$$

Since $j_{!*}$ preserves pure perverse sheaves of a given weight by [BBDG18, Corollaire 5.4.3], the conclusion follows. ☐

Theorem 9.11. *Let* $(X/S, \Sigma)$ *be a relative projective stratified scheme over* $\mathbb{F}$. *Then, there is a function* $N : \mathbb{Q}[\mathrm{Coh}(X)] \times \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{N}^+$ *such that for every* $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$, *every* $r \geq 0$, *every prime* $\ell \neq p$ *and every closed point* $s \in S$, *there are up to geometric isomorphism at most* $N(\mathcal{E}, r, \deg s)$ *geometrically simple objects in* $\mathrm{Perv}_{\Sigma_s}^{\leq r}(X_s, \mathcal{E}_s, \overline{\mathbb{Q}}_\ell)$.

Proof. Combine Theorem 9.7 and Lemma 9.10. ☐

10. Appendix : concrete estimates for $\mathbb{A}_k^n$

Lemma 10.1. *In the setting of Definition 1.2, we have* $b_2(x) = x^2 + 7x + 9$ *and for every* $n \geq 3$, *we have the following inequality of functions on* $\mathbb{R}^+$ *:*

$$b_n(x) \leq (x + 3n - 3) \prod_{j=1}^{n-1} (x + 3j + 1) \, .$$

Proof. The equation $b_2(x) = x^2 + 7x + 9$ is from the definition. In the rest of the proof, we consider all polynomials with indeterminate x as functions defined in $\mathbb{R}_{\geq 0}$. For

$n \geq 3$, we have

$$\begin{aligned}
b_n(x) &= \sum_{\substack{i=0 \\ i \neq n \bmod 2}}^{n-1} (x+2) \cdot b_i(x+3) + \sum_{\substack{i=0 \\ i = n \bmod 2}}^{n-1} (x+3) \cdot b_i(x) + \sum_{\substack{i=0 \\ i \neq n \bmod 2}}^{n-1} b_i(x) \\
&\leq \sum_{\substack{i=0 \\ i \neq n \bmod 2}}^{n-1} (x+2) \cdot b_i(x+3) + \sum_{\substack{i=0 \\ i = n \bmod 2}}^{n-1} (x+3) \cdot b_i(x+3) + \sum_{\substack{i=0 \\ i \neq n \bmod 2}}^{n-1} b_i(x+3) \\
&\leq \sum_{\substack{i=0 \\ i \neq n \bmod 2}}^{n-1} (x+3) \cdot b_i(x+3) + \sum_{\substack{i=0 \\ i = n \bmod 2}}^{n-1} (x+3) \cdot b_i(x+3) \\
&= \left(\sum_{i=0}^{n-1} b_i(x+3) \right) \cdot (x+3).
\end{aligned}$$

Let $c_0(x) = 1, c_1(x) = x$ be polynomials in $\mathbb{Z}[x]$. For $n \geq 2$, we inductively define $c_n(x)$ by

$$c_n(x) = \left(\sum_{i=0}^{n-1} c_i(x+3) \right) \cdot (x+3).$$

For $n \geq 1$, we see that the coefficients of $c_n(x)$ are positive integers and $\deg(c_n(x)) = n$. In particular, we have $c_2(x) = x^2 + 7x + 12$. For $n \geq 3$, we have

$$\begin{aligned}
c_n(x) &= \left(\sum_{i=0}^{n-1} c_i(x+3) \right) \cdot (x+3) \\
&= \left(c_{n-1}(x+3) + \sum_{i=0}^{n-2} c_i(x+3) \right) \cdot (x+3) \\
&= \left(\left(\sum_{i=0}^{n-2} c_i(x+6) \right) \cdot (x+6) + \sum_{i=0}^{n-2} c_i(x+3) \right) \cdot (x+3) \\
&\leq \left(\sum_{i=0}^{n-2} c_i(x+6) \right) \cdot (x+7)(x+3) \\
&\leq \left(\sum_{i=0}^{n-2} c_i(x+6) \right) \cdot (x+6)(x+4) = c_{n-1}(x+3) \cdot (x+4).
\end{aligned}$$

Hence for $n \geq 3$, we have $c_n(x) \leq (x + 3n - 3) \prod_{i=1}^{n-1} (x + 3i + 1)$. Notice that $b_0(x) \leq c_0(x), b_1(x) \leq c_1(x), b_2(x) \leq c_2(x)$. Suppose that for every $0 \leq n \leq k$, we have

$b_k(x) \leq c_k(x)$. When $n = k+1$, we have

$$\begin{aligned} b_n(x) = b_{k+1}(x) &\leq \left(\sum_{i=0}^{k} b_i(x+3)\right) \cdot (x+3) \\ &\leq \left(\sum_{i=0}^{k} c_i(x+3)\right) \cdot (x+3) = c_{k+1}(x) = c_n(x). \end{aligned}$$

By induction, we have $b_n(x) \leq c_n(x)$ for every $n \geq 0$. Hence for every $n \geq 3$, we have

$$b_n(x) \leq c_n(x) \leq (x+3n-3)\prod_{i=1}^{n-1}(x+3i+1).$$

□

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