STANDARD t-STRUCTURES

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ABSTRACT. We provide a general construction of induced t-structures, that generalizes standard t-structures for ∞-categories of sheaves. More precisely, given a presentable ∞-category $\mathcal{X}$ and a presentable stable ∞-category $\mathcal{E}$ equipped with an accessible t-structure $\tau = (\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$, we show that $\mathcal{X} \otimes \mathcal{E}$ is equipped with a canonical t-structure whose coconnective part is given in $\mathcal{X} \otimes \mathcal{E}_{\leqslant 0}$. When $\mathcal{X}$ is an ∞-topos, we give a more explicit description of the connective part as well.

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1. INTRODUCTION

Let X be a topological space and let A be a connective $\mathbb{E}_1$-ring spectrum. Then the ∞-categories $\mathrm{Sh}(X; \mathrm{Mod}_A)$ and $\mathrm{Sh}^{\mathrm{hyp}}(X; \mathrm{Mod}_A)$ of sheaves and hypersheaves of A-modules on X inherit an induced t-structure, called the *standard t-structure*. See [Lur18, §1.3.2]. In this short note, using properties of the tensor product of presentable ∞-categories, we revisit this construction in greater generality. Given a presentable stable ∞-category $\mathcal{E}$ equipped with a t-structure $(\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$, we show that under very mild conditions, the ∞-categories $\mathrm{Sh}(X; \mathcal{E})$ and $\mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E})$ of $\mathcal{E}$-valued sheaves and hypersheaves inherit an induced t-structure, that generalizes the case $\mathcal{E} = \mathrm{Mod}_A$. Along the way, we establish a certain exactness property of this operation that seems interesting on its own right, and that we briefly describe now.

Given a presentable stable ∞-category $\mathcal{E}$ equipped with an accessible t-structure $(\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$, both $\mathcal{E}_{\geqslant 0}$ and $\mathcal{E}_{\leqslant -1}$ are presentable, and the square

$$\begin{array}{ccc} \mathcal{E}_{\geqslant 0} & \xrightarrow{i_{\geqslant n}} & \mathcal{E} \\ \big\downarrow & & \big\downarrow{\scriptstyle \tau_{\leqslant -1}} \\ 0 & \longrightarrow & \mathcal{E}_{\leqslant n-1} \end{array}$$

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is both a pullback and a pushout square in $\mathbf{Pr}^{\mathrm{L}}$ (see Recollection 2.4). Tensoring with a second presentable ∞-category $\mathcal{X}$, we obtain a pushout square

$$\begin{array}{ccc} \mathcal{X} \otimes \mathcal{E}_{\geqslant 0} & \xrightarrow{\mathcal{X}\otimes i_{\geqslant 0}} & \mathcal{X} \otimes \mathcal{E} \\ \downarrow & & \downarrow{\scriptstyle \mathcal{X}\otimes\tau_{\leqslant -1}} \\ 0 & \longrightarrow & \mathcal{X} \otimes \mathcal{E}_{\leqslant -1} \end{array} \tag{1.1}$$

However, in general this square is not a pullback. Nevertheless, $\mathcal{X} \otimes \tau_{\leqslant -1}$ is a localization functor (see Lemma 2.1) and we can therefore identify $\mathcal{X} \otimes \mathcal{E}_{\leqslant -1}$ with a full subcategory of $\mathcal{X} \otimes \mathcal{E}$. We have:

Theorem 1.2 (Propositions 2.8 and 3.8)**.** *Let $\mathcal{X}$ be a presentable ∞-category and let $\mathcal{E}$ be a presentable stable ∞-category equipped with an accessible t-structure $(\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$.*

(1) The presentable stable ∞-category $\mathcal{X} \otimes \mathcal{E}$ admits a t-structure of the form

$$\big((\mathcal{X} \otimes \mathcal{E})_{\geqslant 0}, \mathcal{X} \otimes \mathcal{E}_{\leqslant 0}\big) ,$$

where $(\mathcal{X} \otimes \mathcal{E})_{\geqslant 0}$ is the full subcategory of $\mathcal{X} \otimes \mathcal{E}$ generated under colimits and extensions by the essential image of the functor $\mathcal{X} \otimes i_{\geqslant 0} \colon \mathcal{X} \otimes \mathcal{E}_{\geqslant 0} \to \mathcal{X} \otimes \mathcal{E}$.

(2) If $\mathcal{X}$ is an ∞-topos, (1.1) *is a pullback square. In particular $\mathcal{X} \otimes \mathcal{E}_{\geqslant 0}$ becomes a full subcategory of $\mathcal{X} \otimes \mathcal{E}$, and the t-structure of the previous point simply becomes*

$$\big(\mathcal{X} \otimes \mathcal{E}_{\geqslant 0}, \mathcal{X} \otimes \mathcal{E}_{\leqslant 0}\big) .$$

We conclude the introduction by describing a heuristic speculation. When $\mathcal{X} \simeq \mathrm{PSh}(\mathcal{C})$ is a presheaf category, one has

$$\mathrm{PSh}(\mathcal{C}) \otimes \mathcal{Y} \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{Y}) ,$$

and in particular it follows that $\mathrm{PSh}(\mathcal{C}) \otimes (-)$ commutes with both limits and colimits in $\mathbf{Pr}^{\mathrm{L}}$. On the other hand, any presentable ∞-category $\mathcal{X}$ can be written as a localization of a presheaf category $\mathrm{PSh}(\mathcal{C}_0)$. If Vopěnka's principle holds, the left orthogonal complement of $\mathcal{X}$ inside $\mathrm{PSh}(\mathcal{C})$ is also be presentable, and it is therefore be possible to write it as the localization of a second presheaf category $\mathrm{PSh}(\mathcal{C}_1)$. Iterating this process, we construct a "resolution" of $\mathcal{X}$ by presheaf ∞-categories, which have an exact behavior with respect to the tensor product in $\mathbf{Pr}^{\mathrm{L}}$. We would then be able to define Tor-categories, and Theorem 1.2 would be stating that when $\mathcal{X}$ is an ∞-topos,

$$\mathsf{Tor}_1(\mathcal{X}, \mathcal{E}_{\leqslant 0}) = 0 .$$

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2. Standard t-structures for presentable ∞-categories

2.1. Reminders on the tensor product in $\mathbf{Pr}^{\mathrm{L}}$.

Before we begin, we recall some standard properties of the tensor product in $\mathbf{Pr}^{\mathrm{L}}$.

Lemma 2.1. *Let $L \colon \mathcal{Y} \to \mathcal{Y}'$ be a localization functor in $\mathbf{Pr}^{\mathrm{L}}$. For every $\mathcal{X} \in \mathbf{Pr}^{\mathrm{L}}$, the induced functor $\mathrm{id}_{\mathcal{X}} \otimes L \colon \mathcal{X} \otimes \mathcal{Y} \to \mathcal{X} \otimes \mathcal{Y}'$ is again a localization.*

Proof. We have to check that the right adjoint to $\mathrm{id}_{\mathcal{X}} \otimes L$ is fully faithful. Write $j \colon \mathcal{Y}' \to \mathcal{Y}$ for the fully faithful right adjoint to L. Using the identifications

$$\mathcal{X} \otimes \mathcal{Y} \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{X}^{\mathrm{op}}, \mathcal{Y}) \qquad \text{and} \qquad \mathcal{X} \otimes \mathcal{Y}' \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{X}^{\mathrm{op}}, \mathcal{Y}')$$

provided by [Lur17, Proposition 4.8.1.17], we see that the right adjoint to $\mathrm{id}_{\mathcal{X}} \otimes L$ is identified with the functor induced by composition with j. In particular, it follows from [GHN17, Lemma 5.2] that it is fully faithful. $\square$

Recollection 2.2. Let $\mathcal{X}$ be a presentable ∞-category and let $X \in \mathcal{X}$ be an object. It follows from [Lur09, Corollary 4.4.4.9] that the functor $\mathrm{Map}_{\mathcal{X}}(X, -) \colon \mathcal{X} \to \mathbf{Spc}$ admits a left adjoint, which we denote by

$$i_X \coloneqq (-) \otimes X \colon \mathbf{Spc} \to \mathcal{X} \ .$$

The functor i_X is the unique colimit-preserving functor $\mathbf{Spc} \to \mathcal{X}$ sending the terminal object to X. Now let $\mathcal{Y}$ be a second presentable ∞-category and consider the induced functor

$$i_X \otimes \mathrm{id}_{\mathcal{Y}} \colon \mathcal{Y} \to \mathcal{X} \otimes \mathcal{Y} \ .$$

This functor takes an object $Y \in \mathcal{Y}$ to the elementary tensor $X \otimes Y$. Under the identifications

$$\mathcal{X} \otimes \mathcal{Y} \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{Y}^{\mathrm{op}}, \mathcal{X}) \qquad \text{and} \qquad \mathcal{Y} \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{Y}^{\mathrm{op}}, \mathbf{Spc})$$

provided by [Lur17, Proposition 4.8.1.17], we see that the right adjoint to $i_X \otimes \mathrm{id}_{\mathcal{Y}}$ can be explicitly described as the functor given by postcomposition with $\mathrm{Map}_{\mathcal{X}}(X, -)$. Unraveling the definitions, we can alternatively describe this right adjoint as the evaluation functor

$$\begin{aligned} \mathrm{ev}_X \colon \mathcal{X} \otimes \mathcal{Y} \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{X}^{\mathrm{op}}, \mathcal{Y}) &\to \mathcal{Y} \\ F &\mapsto F(X) \ . \end{aligned}$$

In particular, we deduce that for every $F \in \mathcal{X} \otimes \mathcal{Y}$, one has a natural identification

$$\mathrm{Map}_{\mathcal{X} \otimes \mathcal{Y}}(X \otimes Y, F) \simeq \mathrm{Map}_{\mathcal{Y}}(Y, F(X)) \ , \tag{2.3}$$

where F is viewed as a limit-preserving functor $F \colon \mathcal{X}^{\mathrm{op}} \to \mathcal{Y}$.

2.2. Standard t-structures.

Recollection 2.4. Let $\mathcal{E}$ be a presentable stable ∞-category equipped with an accessible t-structure $\tau = (\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$. For every $n \in \mathbb{Z}$, the full subcategory $\mathcal{E}_{\leqslant n}$ is an accessible localization of $\mathcal{E}$, and therefore it is itself presentable. Notice that

$$\begin{array}{ccc} \mathcal{E}_{\geqslant n} & \xrightarrow{i_{\geqslant n}} & \mathcal{E} \\ \downarrow & & \downarrow {\scriptstyle \tau_{\leqslant n-1}} \\ 0 & \longrightarrow & \mathcal{E}_{\leqslant n-1} \end{array}$$

is a pullback square. In particular, $\mathcal{E}_{\geqslant n}$ is presentable as well. It automatically follows that the above square is also a pushout in $\mathbf{Pr}^{\mathrm{L}}$.

The following lemma follows immediately from the fact that in a stable ∞-category a null sequence is a fiber sequence if and only if it is a cofiber sequence:

Lemma 2.5. *Let $f \colon \mathcal{E} \to \mathcal{Y}$ be a functor between presentable ∞-categories. Assume that $\mathcal{E}$ is stable, that $\mathcal{Y}$ is pointed (with zero object $0_{\mathcal{Y}}$) and that f is either left exact or right exact. Then*

$$\ker(f) \coloneqq \{ E \in \mathcal{E} \mid f(E) \simeq 0_{\mathcal{Y}} \}$$

is closed under extensions. In particular, if $\tau = (\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$ is a t-structure on $\mathcal{E}$, then for every $n \in \mathbb{Z}$ the full subcategories $\mathcal{E}_{\leqslant n}$ and $\mathcal{E}_{\geqslant n}$ are closed under extensions.

Notation 2.6. Let $\mathcal{X}$ be a presentable ∞-category and let $\mathcal{E}$ be a presentable stable ∞-category equipped with an accessible t-structure $\tau = (\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$. For each $n \in \mathbb{Z}$, set

$$i^{\mathcal{X}}_{\geqslant n} \coloneqq \mathrm{id}_{\mathcal{X}} \otimes i_{\geqslant n} \colon \mathcal{X} \otimes \mathcal{E}_{\geqslant n} \to \mathcal{X} \otimes \mathcal{E} \qquad \text{and} \qquad \tau^{\mathcal{X}}_{\leqslant n} \coloneqq \mathrm{id}_{\mathcal{X}} \otimes \tau_{\leqslant n} \colon \mathcal{X} \otimes \mathcal{E} \to \mathcal{X} \otimes \mathcal{E}_{\leqslant n} \ .$$

Then the square

$$\begin{array}{ccc} \mathcal{X} \otimes \mathcal{E}_{\geqslant n} & \xrightarrow{i^{\mathcal{X}}_{\geqslant n}} & \mathcal{X} \otimes \mathcal{E} \\ \downarrow & & \downarrow{\scriptstyle \tau^{\mathcal{X}}_{\leqslant n-1}} \\ 0 & \longrightarrow & \mathcal{X} \otimes \mathcal{E}_{\leqslant n-1} \end{array}$$

is a pushout in $\mathbf{Pr}^{\mathrm{L}}$; however, it typically is not a pullback. It follows from Lemma 2.1 that the right adjoint to $\tau^{\mathcal{X}}_{\leqslant n-1}$ is fully faithful. Under the identifications

$$\mathcal{X} \otimes \mathcal{E} \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{X}^{\mathrm{op}}, \mathcal{E}) \qquad \text{and} \qquad \mathcal{X} \otimes \mathcal{E}_{\leqslant n-1} \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{X}^{\mathrm{op}}, \mathcal{E}_{\leqslant n-1}) \ ,$$

we see that the right adjoint to $\tau^{\mathcal{X}}_{\leqslant n-1}$ is given by composition with $i_{\leqslant n-1}$. In particular, $\mathcal{X} \otimes \mathcal{E}_{\leqslant n-1}$ is naturally identified with the full subcategory of $\mathrm{Fun}^{\mathrm{R}}(\mathcal{X}^{\mathrm{op}}, \mathcal{E})$ spanned by those right adjoints $F \colon \mathcal{X}^{\mathrm{op}} \to \mathcal{E}$ that factor through $\mathcal{E}_{\leqslant n-1}$. Define

$$(\mathcal{X} \otimes \mathcal{E})_{\geqslant n} \coloneqq \ker\left(\tau^{\mathcal{X}}_{\leqslant n-1} \colon \mathcal{X} \otimes \mathcal{E} \to \mathcal{X} \otimes \mathcal{E}_{\leqslant n-1}\right) \ .$$

Lemma 2.7. *In the setting of Notation 2.6:*

(1) *For each $n \in \mathbb{Z}$, both $\mathcal{X} \otimes \mathcal{E}_{\leqslant n}$ and $(\mathcal{X} \otimes \mathcal{E})_{\geqslant n}$ are closed under extensions in $\mathcal{X} \otimes \mathcal{E}$.*

(2) *Let $\mathcal{X}_\bullet \colon I \to \mathbf{Pr}^{\mathrm{L}}$ be a diagram with limit $\mathcal{X}$. Assume that for every transition morphism $i \to j$, the induced functor $\mathcal{X}_i \to \mathcal{X}_j$ is both a left and a right adjoint. Then for each $n \in \mathbb{Z}$, the natural functors*

$$\mathcal{X} \otimes \mathcal{E}_{\leqslant n} \to \lim_{i \in I} \mathcal{X}_i \otimes \mathcal{E}_{\leqslant n}$$

and

$$\lim_{i \in I} (\mathcal{X}_i \otimes \mathcal{E})_{\geqslant 0} \to \Big(\big(\lim_{i \in I} \mathcal{X}_i \big) \otimes \mathcal{E} \Big)_{\geqslant 0}$$

are equivalences.

Proof. First we prove (1). The claim about $(\mathcal{X} \otimes \mathcal{E})_{\geqslant n}$ is a simple consequence of the definitions and Lemma 2.5. We now deal with $\mathcal{X} \otimes \mathcal{E}_{\leqslant n}$. Consider a fiber sequence

$$\begin{array}{ccc} F' & \longrightarrow & F \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & F'' \end{array}$$

in $\mathcal{X} \otimes \mathcal{E}$. Assume first that both F' and F'' belong to $\mathcal{X} \otimes \mathcal{E}_{\leqslant n}$. Under the identification $\mathcal{X} \otimes \mathcal{E} \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{X}^{\mathrm{op}}, \mathcal{E})$, observe that limits are computed objectwise. In other words, for every $X \in \mathcal{X}$, the induced square

$$\begin{array}{ccc} F'(X) & \longrightarrow & F(X) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & F''(X) \end{array}$$

is a pullback in $\mathcal{E}$. The assumption implies that both $F'(X)$ and $F''(X)$ belong to $\mathcal{E}_{\leqslant n}$, so the same is true of $F(X)$. In other words, $F \in \mathcal{X} \otimes \mathcal{E}_{\leqslant n}$.

We now prove (2). Since limits commute with limits, it is enough to prove the statement concerning $\mathcal{X} \otimes \mathcal{E}_{\leqslant n}$. However, the assumption on the diagram and [Lur09, Proposition 5.5.3.13

& Theorem 5.5.3.18] imply that the limit can equally be computed in $\mathbf{Pr}^{\mathrm{R}}$. The conclusion now follows from [Lur17, Remark 4.8.1.24] (see also [HPT24, Lemma 4.2.2]). □

The following is our main result about a t-structure on $\mathcal{X}\otimes\mathcal{E}$ when $\mathcal{X}$ is an arbitrary presentable ∞-category.

Proposition 2.8. *In the setting of Notation 2.6, there exists a unique t-structure*

$$\big((\mathcal{X}\otimes\mathcal{E})_{\geqslant 0}, (\mathcal{X}\otimes\mathcal{E})_{\leqslant 0}\big)$$

on $\mathcal{X}\otimes\mathcal{E}$ whose connective part coincides with the full subcategory $(\mathcal{X}\otimes\mathcal{E})_{\geqslant 0}$ of $\mathcal{X}\otimes\mathcal{E}$ introduced in Notation 2.6. In addition:

1. *We have $(\mathcal{X}\otimes\mathcal{E})_{\leqslant 0} = \mathcal{X}\otimes\mathcal{E}_{\leqslant 0}$ as full subcategories of $\mathcal{X}\otimes\mathcal{E}$.*
2. *The connective part $(\mathcal{X}\otimes\mathcal{E})_{\geqslant 0}$ is generated under colimits and extensions by objects of the form $X\otimes E$ for $X\in\mathcal{X}$ and $E\in\mathcal{E}_{\geqslant 0}$.*

Proof. It follows from Lemma 2.7-(1) that $(\mathcal{X}\otimes\mathcal{E})_{\geqslant 0}$ is a full subcategory of $\mathcal{X}\otimes\mathcal{E}$ closed under colimits and extensions. In particular, [Lur17, Proposition 1.4.4.11-(1)] applies, providing the existence (and uniqueness) of the required t-structure. The definition of $(\mathcal{X}\otimes\mathcal{E})_{\leqslant 0}$ shows that

$$\mathcal{X}\otimes\mathcal{E}_{\leqslant -1} \subseteq (\mathcal{X}\otimes\mathcal{E})_{\leqslant -1} \ .$$

To prove that equality holds, let $F\in(\mathcal{X}\otimes\mathcal{E})_{\leqslant -1}$, and view F as a right adjoint functor $\mathcal{X}^{\mathrm{op}}\to\mathcal{E}$. We have to prove that F factors through $\mathcal{E}_{\leqslant -1}$. Observe that the functoriality of the tensor product of ∞-categories implies that the composite

$$\mathcal{X}\otimes\mathcal{E}_{\geqslant 0} \xrightarrow{\mathrm{id}_{\mathcal{X}}\otimes i_{\geqslant 0}} \mathcal{X}\otimes\mathcal{E} \xrightarrow{\mathrm{id}_{\mathcal{X}}\otimes\tau_{\leqslant -1}} \mathcal{X}\otimes\mathcal{E}_{\leqslant -1}$$

is zero. In other words, the functor $\mathrm{id}_{\mathcal{X}}\otimes i_{\geqslant 0}\colon \mathcal{X}\otimes\mathcal{E}_{\geqslant 0}\to\mathcal{X}\otimes\mathcal{E}$ factors through $(\mathcal{X}\otimes\mathcal{E})_{\geqslant 0}$. It follows that every object of the form $X\otimes E$, for $X\in\mathcal{X}$ and $E\in\mathcal{E}_{\geqslant 0}$, belongs to $(\mathcal{X}\otimes\mathcal{E})_{\geqslant 0}$. In particular, the assumption $F\in(\mathcal{X}\otimes\mathcal{E})_{\leqslant -1}$ guarantees that

$$\mathrm{Map}_{\mathcal{E}}(E, F(X)) \simeq \mathrm{Map}_{\mathcal{X}\otimes\mathcal{E}}(X\otimes E, F) \simeq 0 \ ,$$

where the first equivalence follows from Recollection 2.2, specifically (2.3). Since this holds for every $X\in\mathcal{X}$ and every $E\in\mathcal{E}_{\geqslant 0}$, we deduce that F factors through $\mathcal{E}_{\leqslant -1}$. Thus, $\mathcal{X}\otimes\mathcal{E}_{\leqslant -1} = (\mathcal{X}\otimes\mathcal{E})_{\leqslant -1}$.

We now prove item (2). Let $\mathcal{C}\subset\mathcal{X}\otimes\mathcal{E}$ be the smallest full subcategory closed under colimits and extensions and containing objects of the form $X\otimes E$ for $X\in\mathcal{X}$ and $E\in\mathcal{E}_{\geqslant 0}$. Recall from [Lur17, Proposition 1.4.4.11-(2)] that $\mathcal{C}$ is automatically presentable, and that therefore it gives rise to a t-structure $\tau' = (\mathcal{C},\mathcal{D})$ on $\mathcal{X}\otimes\mathcal{E}$. The same argument given above immediately implies that $\mathcal{D}\subseteq\mathcal{X}\otimes\mathcal{E}_{\leqslant 0}$. Conversely, let $F\in\mathcal{X}\otimes\mathcal{E}_{\leqslant 0}$ and let $\mathcal{C}_F$ be the full subcategory of $\mathcal{X}\otimes\mathcal{E}$ spanned by the objects G such that $\mathrm{Map}_{\mathcal{X}\otimes\mathcal{E}}(G,F)\simeq 0$. By definition, $\mathcal{C}_F$ is closed under colimits, and Lemma 2.5 implies that $\mathcal{C}_F$ is closed under extensions as well. Moreover, $\mathcal{C}_F$ contains every object of the form $X\otimes E$ for $X\in\mathcal{X}$ and $E\in\mathcal{E}_{\geqslant 1}$. Thus, $\mathcal{C}_F$ contains $\mathcal{C}[-1]$. It follows that $F\in\mathcal{D}$, and hence that $\mathcal{D}=\mathcal{X}\otimes\mathcal{E}_{\leqslant 0}$. The uniqueness of the t-structure implies then that $\mathcal{C} = (\mathcal{X}\otimes\mathcal{E})_{\geqslant 0}$, whence the conclusion. □

Definition 2.9. In the setting of Notation 2.6, we refer to

$$\tau^{\mathcal{X}} := \big((\mathcal{X}\otimes\mathcal{E})_{\geqslant 0}, (\mathcal{X}\otimes\mathcal{E})_{\leqslant 0}\big)$$

as the *standard t-structure* on $\mathcal{X}\otimes\mathcal{E}$ induced by the t-structure $\tau = (\mathcal{E}_{\geqslant 0},\mathcal{E}_{\leqslant 0})$ on $\mathcal{E}$.

Example 2.10. Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{E}=\mathbf{Sp}$ be the ∞-category of spectra, equipped with its standard t-structure. In [Lur18, Proposition 1.3.2.7] it is shown that $\mathcal{X}\otimes\mathbf{Sp}\simeq\mathrm{Sh}(\mathcal{X};\mathbf{Sp})$

is equipped with a t-structure $(\mathrm{Sh}(\mathcal{X};\mathbf{Sp})_{\geqslant 0}, \mathrm{Sh}(\mathcal{X};\mathbf{Sp})_{\leqslant 0})$. In addition, [Lur18, Remark 1.3.2.6] provides a natural identification

$$\mathrm{Sh}(\mathcal{X};\mathbf{Sp})_{\leqslant 0} \simeq \mathrm{Sh}(\mathcal{X};\mathbf{Sp}_{\leqslant 0}) \simeq \mathcal{X} \otimes \mathbf{Sp}_{\leqslant 0} \ .$$

It follows that the t-structure on $\mathcal{X} \otimes \mathbf{Sp}$ coincides with the one provided by Proposition 2.8. In particular, it follows from [Lur18, Proposition 1.3.2.7] that this t-structure is compatible with filtered colimits and right complete.

Corollary 2.11. *Let $f^* \colon \mathcal{X} \to \mathcal{Y}$ be a functor in $\mathbf{Pr}^{\mathrm{L}}$. Then the induced functor*

$$f^*_{\mathcal{E}} \coloneqq f^* \otimes \mathrm{id}_{\mathcal{E}} \colon \mathcal{X} \otimes \mathcal{E} \to \mathcal{Y} \otimes \mathcal{E}$$

is right t-exact. If in addition f^* is a left exact left adjoint between ∞-topoi and $\mathcal{E} = \mathbf{Sp}$, then $f^*_{\mathcal{E}}$ is also left t-exact.

Proof. Write $f_* \colon \mathcal{Y} \to \mathcal{X}$ for the right adjoint to f^*. Under the identifications

$$\mathcal{X} \otimes \mathcal{E} \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{X}^{\mathrm{op}}, \mathcal{E}) \qquad \text{and} \qquad \mathcal{Y} \otimes \mathcal{E} \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{Y}^{\mathrm{op}}, \mathcal{E}) \ ,$$

we see that the right adjoint $f_*^{\mathcal{E}}$ to $f^*_{\mathcal{E}}$ is given by composition with f_*. It immediately follows that $f_*^{\mathcal{E}}$ takes $\mathcal{Y} \otimes \mathcal{E}_{\leqslant 0}$ to $\mathcal{X} \otimes \mathcal{E}_{\leqslant 0}$, and therefore that $f^*_{\mathcal{E}}$ is right t-exact. The second half of the statement follows combining Example 2.10 with [Lur18, Remark 1.3.2.8]. ☐

3. Standard t-structures for ∞-topoi

Let $\mathcal{E}$ be a presentable stable ∞-category equipped with an accessible t-structure $\tau = (\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$. We now investigate the natural comparison functor

$$i^{\mathcal{X}}_{\geqslant 0} \colon \mathcal{X} \otimes \mathcal{E}_{\geqslant 0} \to (\mathcal{X} \otimes \mathcal{E})_{\geqslant 0} \ .$$

The main result of this section is that when $\mathcal{X}$ is an ∞-topos and the t-structure τ is right complete, this functor is an equivalence (Proposition 3.8).

We begin with the following general criterion, that holds without extra assumptions:

Lemma 3.1. *Let $\mathcal{E}$ be a presentable stable ∞-category equipped with an accessible t-structure $\tau = (\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$ and let $\mathcal{X}$ be a presentable ∞-category. Then the following conditions are equivalent:*

1. *The functor $i^{\mathcal{X}}_{\geqslant 0} \colon \mathcal{X} \otimes \mathcal{E}_{\geqslant 0} \to \mathcal{X} \otimes \mathcal{E}$ is fully faithful and the essential image of $i^{\mathcal{X}}_{\geqslant 0}$ is closed under extensions.*
2. *The functor $i^{\mathcal{X}}_{\geqslant 0} \colon \mathcal{X} \otimes \mathcal{E}_{\geqslant 0} \to (\mathcal{X} \otimes \mathcal{E})_{\geqslant 0}$ is an equivalence.*
3. *There exists an integer $n \in \mathbb{Z}$ such that the ∞-category $\mathcal{X} \otimes \mathcal{E}_{\geqslant n}$ is prestable and that*

$$i^{\mathcal{X}}_{\geqslant n} \colon \mathcal{X} \otimes \mathcal{E}_{\geqslant n} \to \mathcal{X} \otimes \mathcal{E}$$

is fully faithful.

Proof. The equivalence (1)⇔(2) is immediate from the definition of the connective part $(\mathcal{X} \otimes \mathcal{E})_{\geqslant 0}$ of the standard t-structure. The implication (2)⇒(3) is clear.

To see that (3)⇒(2), without loss of generality, we can suppose $n = 0$. In virtue of Proposition 2.8-(2), we see that $(\mathcal{X} \otimes \mathcal{E})_{\geqslant 0}$ is generated under colimits and extensions by the essential image of $i^{\mathcal{X}}_{\geqslant 0}$. Thus, to prove that the inclusion $(\mathcal{X} \otimes \mathcal{E})_{\geqslant 0} \subseteq \mathcal{X} \otimes \mathcal{E}_{\geqslant 0}$ holds, it suffices to prove that the essential image of $i^{\mathcal{X}}_{\geqslant 0}$ is closed under extensions. To see this, let $F' \to F \to F''$ be a fiber sequence in $\mathcal{X} \otimes \mathcal{E}$ and assume that F' and F'' belong to the essential image of $i^{\mathcal{X}}_{\geqslant 0}$ (and hence of $i^{\mathcal{X}}_{\geqslant -1}$). Let $\alpha \colon F' \to F''[1]$ be the map classifying the given extension. Since $\mathcal{X} \otimes \mathcal{E}$ is stable, we can write

$$F \simeq \mathrm{fib}\big(F'' \xrightarrow{\alpha} F'[1]\big) \ ,$$

where α is a morphism in $\mathcal{X} \otimes \mathcal{E}$. By assumption, we can write

$$F' \simeq i^{\mathcal{X}}_{\geqslant 0}(U') \qquad \text{and} \qquad F'' \simeq i^{\mathcal{X}}_{\geqslant 0}(U'') \ .$$

Since $i^{\mathcal{X}}_{\geqslant 0}$ commutes with colimits, it commutes in particular with suspensions, so that

$$F'[1] \simeq i^{\mathcal{X}}_{\geqslant 0}(U'[1]) \ ,$$

where the suspension $U'[1]$ is computed in $\mathcal{X} \otimes \mathcal{E}_{\geqslant 0}$. The full faithfulness of $i^{\mathcal{X}}_{\geqslant 0}$ guarantees that we can write $\alpha \simeq i^{\mathcal{X}}_{\geqslant 0}(\beta)$, where $\beta \colon U'' \to U'[1]$ is a morphism in $\mathcal{X} \otimes \mathcal{E}_{\geqslant 0}$. Set

$$U \coloneqq \mathrm{fib}(\beta) \in \mathcal{X} \otimes \mathcal{E}_{\geqslant 0} \ .$$

Since this ∞-category is prestable by assumption, we deduce that the pullback diagram

$$\begin{array}{ccc} U & \longrightarrow & U'' \\ \big\downarrow & & \big\downarrow{\scriptstyle \beta} \\ 0 & \longrightarrow & U'[1] \end{array}$$

is also a pushout. In particular, it is taken to a pushout by $i^{\mathcal{X}}_{\geqslant 0}$ and, since $\mathcal{X} \otimes \mathcal{E}$ is stable, we deduce that in fact

$$i^{\mathcal{X}}_{\geqslant 0}(U) \simeq \mathrm{fib}(i^{\mathcal{X}}_{\geqslant 0}(\beta)) \simeq \mathrm{fib}(\alpha) \simeq F \ .$$

Thus, F belongs as well to the essential image of $i^{\mathcal{X}}_{\geqslant 0}$, whence the conclusion. □

We now record an easy application of Lemma 3.1. For this, the reader may wish to review the definition of a *projectively generated* presentable ∞-category in [Lur09, Definition 5.5.8.23] or [Hai22, Recollection 2.4].

Corollary 3.2. *Let $\mathcal{E}$ be a presentable stable ∞-category equipped with an accessible t-structure $(\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$, and let $\mathcal{X}$ be a projectively generated presentable ∞-category. Then the functor*

$$i^{\mathcal{X}}_{\geqslant 0} \colon \mathcal{X} \otimes \mathcal{E}_{\geqslant 0} \to (\mathcal{X} \otimes \mathcal{E})_{\geqslant 0}$$

is an equivalence.

Proof. Write $\mathcal{X}_0 \subset \mathcal{X}$ for the full subcategory spanned by the compact projective objects. Then we have identifications

$$\mathcal{X} \otimes \mathcal{E}_{\geqslant 0} \simeq \mathrm{Fun}^{\times}(\mathcal{X}_0^{\mathrm{op}}, \mathcal{E}_{\geqslant 0}) \qquad \text{and} \qquad \mathcal{X} \otimes \mathcal{E} \simeq \mathrm{Fun}^{\times}(\mathcal{X}_0^{\mathrm{op}}, \mathcal{E}) \ ,$$

where $\mathrm{Fun}^{\times}(\mathcal{X}_0^{\mathrm{op}}, \mathcal{D})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{X}_0^{\mathrm{op}}, \mathcal{D})$ spanned by the functors that preserve finite products. Moreover, since $i_{\geqslant 0} \colon \mathcal{E}_{\geqslant 0} \hookrightarrow \mathcal{E}$ preserves finite products, under these identifications, the functor $i^{\mathcal{X}}_{\geqslant 0} = \mathrm{id}_{\mathcal{X}} \otimes i_{\geqslant 0}$ is given by postcomposition with $i_{\geqslant 0}$. See [Hai22, Variant 2.10].

In particular, since $i_{\geqslant 0}$ is fully faithful with essential image closed under extensions, we deduce that $i^{\mathcal{X}}_{\geqslant 0} \colon \mathcal{X} \otimes \mathcal{E}_{\geqslant 0} \to \mathcal{X} \otimes \mathcal{E}$ is fully faithful and the essential image closed under extensions. Lemma 3.1 completes the proof. □

3.1. **An unstable statement.** Let $\mathcal{X}$ be an ∞-topos. We write $\mathbf{1}_{\mathcal{X}}$ for the final object of $\mathcal{X}$ and we write

$$\mathcal{X}_* \coloneqq \mathcal{X}_{\mathbf{1}_{\mathcal{X}}/}$$

for the ∞-category of pointed objects of $\mathcal{X}$. For an integer $n \geqslant -2$, we consider the ∞-category

$$\mathcal{X}_*^{\leqslant n} \coloneqq (\mathcal{X}_*)_{\leqslant n}$$

of n-truncated objects in $\mathcal{X}_*$. Unraveling the definitions, we see that the inclusion $\mathcal{X}_*^{\leqslant n} \subseteq \mathcal{X}_*$ has a left adjoint that sends a pointed object (X, x) to the pointed object $(\tau_{\leqslant n}(X), x')$, where x' is the composite

$$\mathbf{1}_{\mathcal{X}} \xrightarrow{x} X \longrightarrow \tau_{\leqslant n} X \ .$$

We still denote this left adjoint by

$$\tau_{\leqslant n} \colon \mathcal{X}_* \to \mathcal{X}_*^{\leqslant n} \ .$$

For $k \geqslant -1$, we define $\mathcal{X}_*^{\geqslant k}$ as the fiber product

$$\begin{array}{ccc} \mathcal{X}_*^{\geqslant k} & \longrightarrow & \mathcal{X}_* \\ \downarrow & & \downarrow{\scriptstyle \tau_{\leqslant k-1}} \\ * & \xrightarrow[\mathbf{1}_{\mathcal{X}}]{} & \mathcal{X}_*^{\leqslant k-1} \ . \end{array}$$

The functoriality of the tensor product of ∞-categories immediately yields the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{X} \otimes \mathbf{Spc}_*^{\geqslant k} & \longrightarrow & \mathcal{X} \otimes \mathbf{Spc}_* & \xrightarrow{\mathrm{id}_{\mathcal{X}} \otimes \tau_{\leqslant k-1}} & \mathcal{X} \otimes \mathbf{Spc}_*^{\leqslant k-1} \\ {\scriptstyle \alpha_k}\downarrow & & \downarrow & & \downarrow \\ \mathcal{X}_*^{\geqslant k} & \longrightarrow & \mathcal{X}_* & \xrightarrow[\tau_{\leqslant k-1}^{\mathcal{X}}]{} & \mathcal{X}_*^{\leqslant k} \ . \end{array}$$

The central and the right vertical functors are equivalences (see [Lur17, Examples 4.8.1.21 & 4.8.1.22]), and they would be even if $\mathcal{X}$ were simply a presentable ∞-category. Since $\mathcal{X}$ is an ∞-topos, we furthermore see:

Proposition 3.3. *Let $\mathcal{X}$ be an ∞-topos. Then for each integer $k > 0$, the comparison functor*

$$\alpha_k \colon \mathcal{X} \otimes \mathbf{Spc}_*^{\geqslant k} \to \mathcal{X}_*^{\geqslant k}$$

is an equivalence.

Proof. Recall from [Lur17, Notation 5.2.6.11] the iterated bar-cobar adjunction

$$\mathrm{Bar}_{\mathcal{X}}^{(k)} \colon \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{X}) \leftrightarrows \mathcal{X}_* \colon \mathrm{CoBar}_{\mathcal{X}}^{(k)} \ .$$

(For $X \in \mathcal{X}_*$, the underlying object of $\mathrm{CoBar}_{\mathcal{X}}^{(k)}(X)$ is just the k-fold based loop object $\Omega^k X$.) By [Lur17, Theorem 5.2.6.15], this adjunction restricts to an equivalence

$$\mathrm{Bar}_{\mathcal{X}}^{(k)} \colon \mathrm{Mon}_{\mathbb{E}_k}^{\mathrm{gp}}(\mathcal{X}) \leftrightarrows \mathcal{X}_*^{\geqslant k} \colon \mathrm{CoBar}_{\mathcal{X}}^{(k)} \ .$$

Notice that

$$\begin{aligned} \mathcal{X} \otimes \mathrm{Mon}_{\mathbb{E}_k}^{\mathrm{gp}}(\mathbf{Spc}) &\simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{X}^{\mathrm{op}}, \mathrm{Mon}_{\mathbb{E}_k}^{\mathrm{gp}}(\mathbf{Spc})) \\ &\simeq \mathrm{Mon}_{\mathbb{E}_k}^{\mathrm{gp}}(\mathrm{Fun}^{\mathrm{R}}(\mathcal{X}^{\mathrm{op}}, \mathbf{Spc})) \\ &\simeq \mathrm{Mon}_{\mathbb{E}_k}^{\mathrm{gp}}(\mathcal{X}) \ . \end{aligned}$$

Hence it suffices to argue that the diagram

$$\begin{array}{ccc} \mathcal{X} \otimes \mathrm{Mon}_{\mathbb{E}_k}^{\mathrm{gp}}(\mathbf{Spc}) & \xrightarrow{\sim} & \mathrm{Mon}_{\mathbb{E}_k}^{\mathrm{gp}}(\mathcal{X}) \\ {\scriptstyle \mathrm{id}_{\mathcal{X}} \otimes \mathrm{Bar}^{(k)}}\downarrow & & \downarrow{\scriptstyle \mathrm{Bar}_{\mathcal{X}}^{(k)}} \\ \mathcal{X} \otimes \mathbf{Spc}_*^{\geqslant k} & \xrightarrow[\alpha_k]{} & \mathcal{X}_*^{\geqslant k} \end{array}$$

commutes. (Here, $\mathrm{Bar}^{(k)}$ denotes the iterated bar construction for **Spc**.) Since all functors commute with colimits, it suffices to check that the diagram commutes after composition with the universal functor

$$\mathcal{X} \times \mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_k}(\mathbf{Spc}) \to \mathcal{X} \otimes \mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_k}(\mathbf{Spc})$$

that preserves colimits separately in each variable. For this, it is enough to observe that given $X \in \mathcal{X}$, the functor

$$X \otimes (-) \colon \mathbf{Spc} \to \mathcal{X}$$

commutes with colimits and therefore [Lur17, Example 5.2.3.11] supplies a canonical identification

$$X \otimes \mathrm{Bar}^{(k)}(-) \simeq \mathrm{Bar}^{(k)}_{\mathcal{X}}(X \otimes -) ,$$

which is functorial in X. The conclusion follows. □

Corollary 3.4. *Let $\mathcal{X}$ be an ∞-topos. Then the natural functor*

$$\mathcal{X} \otimes \mathbf{Sp}_{\geqslant 0} \to \mathcal{X} \otimes \mathbf{Sp}$$

is fully faithful.

Proof. Recall from [Lur17, Remark 5.2.6.26] that one has

$$\mathbf{Sp}_{\geqslant 0} \simeq \lim \big(\cdots \xrightarrow{\Omega} \mathbf{Spc}_*^{\geqslant n+1} \xrightarrow{\Omega} \mathbf{Spc}_*^{\geqslant n} \xrightarrow{\Omega} \cdots \xrightarrow{\Omega} \mathbf{Spc}_*^{\geqslant 1} \big) .$$

Similarly,

$$\mathbf{Sp} \simeq \lim \big(\cdots \xrightarrow{\Omega} \mathbf{Spc}_* \xrightarrow{\Omega} \mathbf{Spc}_* \xrightarrow{\Omega} \cdots \xrightarrow{\Omega} \mathbf{Spc}_* \big) .$$

Moreover, the inclusion $\mathbf{Sp}_{\geqslant 0} \hookrightarrow \mathbf{Sp}$ is induced by the fully faithful inclusions $\mathbf{Spc}_*^{\geqslant n} \hookrightarrow \mathbf{Spc}_*$, which assemble into a natural transformation of the above limit diagrams. Notice that both limits are taken in $\mathbf{Pr}^{\mathrm{R}}$ and therefore they are preserved by the functor $\mathcal{X} \otimes (-)$ (see [Lur17, Remark 4.8.1.24]). Thus, the claim follows at once from Proposition 3.3. □

Remark 3.5. Let $(\mathcal{C}, \tau)$ be an ∞-site and assume that $\mathcal{X} \simeq \mathrm{Sh}(\mathcal{C}, \tau)$. The functoriality of the tensor product in $\mathbf{Pr}^{\mathrm{L}}$ immediately implies that the diagram

$$\begin{array}{ccc} \mathrm{PSh}(\mathcal{C}) \otimes \mathbf{Sp}_{\geqslant 0} & \longrightarrow & \mathrm{PSh}(\mathcal{C}) \otimes \mathbf{Sp} \\ \downarrow & & \downarrow \\ \mathrm{Sh}(\mathcal{C}, \tau) \otimes \mathbf{Sp}_{\geqslant 0} & \longrightarrow & \mathrm{Sh}(\mathcal{C}, \tau) \otimes \mathbf{Sp} \end{array}$$

commutes, where the vertical arrows are the sheafification functors. Moreover, the formula for the sheafification provided in the proof of [Lur09, Proposition 6.2.2.7] (which holds with coefficients in any presentable ∞-category) shows that this square is horizontally right adjointable. In particular, one can deduce the full faithfulness provided by Corollary 3.4 for $\mathcal{X} = \mathrm{Sh}(\mathcal{C}, \tau)$ directly from the one for $\mathcal{X} = \mathrm{PSh}(\mathcal{C})$, which is straightforward since $\mathrm{PSh}(\mathcal{C}) \otimes (-) \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, -)$ preserves fully faithful left adjoints.

3.2. **Proof of the main theorem.**

Recollection 3.6. Let $\mathcal{Y}$ be a presentable ∞-category and let $\mathcal{D}$ be a presentable prestable ∞-category. Then combining [Lur18, Example C.1.5.6 and Theorem C.4.1.1] we deduce that

$$\begin{aligned} \mathcal{Y} \otimes \mathcal{D} &\simeq \mathcal{Y} \otimes (\mathbf{Sp}_{\geqslant 0} \otimes \mathcal{D}) \\ &\simeq (\mathcal{Y} \otimes \mathbf{Sp}_{\geqslant 0}) \otimes \mathcal{D} . \end{aligned}$$

Similarly, if $\mathcal{E}$ is a presentable stable ∞-category, [Lur17, Example 4.8.1.23] shows that

$$\begin{aligned} \mathcal{Y} \otimes \mathcal{E} &\simeq \mathcal{Y} \otimes (\mathbf{Sp} \otimes \mathcal{E}) \\ &\simeq (\mathcal{Y} \otimes \mathbf{Sp}) \otimes \mathcal{E} \\ &\simeq \mathrm{Sp}(\mathcal{Y}) \otimes \mathcal{E} \ . \end{aligned}$$

Corollary 3.7. *Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{D}$ be a Grothendieck prestable ∞-category. Then $\mathcal{X} \otimes \mathcal{D}$ is again a Grothendieck prestable ∞-category.*

Proof. Combining Recollection 3.6 and [Lur18, Theorem C.4.2.1], it is enough to deal with the case where $\mathcal{D} = \mathbf{Sp}_{\geqslant 0}$, and this case immediately follows from Corollary 3.4, Example 2.10, and [Lur18, Proposition C.1.2.9]. □

Proposition 3.8. *Let $\mathcal{X}$ be an ∞-topos. Let $\mathcal{E}$ be a presentable stable ∞-category equipped with an accessible t-structure $\tau = (\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$ which is compatible with filtered colimits and right complete. Then for every integer $n \in \mathbb{Z}$, the natural functor*

$$\mathcal{X} \otimes \mathcal{E}_{\geqslant n} \to (\mathcal{X} \otimes \mathcal{E})_{\geqslant n}$$

is an equivalence. In particular, the standard t-structure on $\mathcal{X} \otimes \mathcal{E}$ is compatible with filtered colimits and right complete.

Proof. It is enough to treat the case $n = 0$. We know from Corollary 3.7 that $\mathcal{X} \otimes \mathcal{E}_{\geqslant 0}$ is a Grothendieck prestable ∞-category. In particular, [Lur18, Remark C.1.1.6 & Proposition C.1.2.9] imply that the natural functor

$$\mathcal{X} \otimes \mathcal{E}_{\geqslant 0} \to \mathrm{Sp}(\mathcal{X} \otimes \mathcal{E}_{\geqslant 0}) \simeq \mathcal{X} \otimes \mathcal{E}_{\geqslant 0} \otimes \mathbf{Sp} \simeq \mathcal{X} \otimes \mathrm{Sp}(\mathcal{E}_{\geqslant 0})$$

is fully faithful. On the other hand, since the t-structure τ is right complete, [Lur18, Remark C.3.1.5] provides a canonical equivalence $\mathrm{Sp}(\mathcal{E}_{\geqslant 0}) \simeq \mathcal{E}$. The first claim then follows from Lemma 3.1. Finally, [Lur18, Proposition C.1.4.1] guarantees that the unique t-structure on $\mathcal{X} \otimes \mathcal{E} \simeq \mathrm{Sp}(\mathcal{X} \otimes \mathcal{E}_{\geqslant 0})$ whose connective part is given by $\mathcal{X} \otimes \mathcal{E}_{\geqslant 0}$ is compatible with filtered colimits.

We are left to prove that the standard t-structure is right complete. For this, we have to check that the canonical functor

$$\operatorname{colim}\left(\cdots \to (\mathcal{X} \otimes \mathcal{E})_{\geqslant n} \to (\mathcal{X} \otimes \mathcal{E})_{\geqslant n-1} \to \cdots \right) \to \mathcal{X} \otimes \mathcal{E}$$

is an equivalence, where the colimit is computed in $\mathbf{Pr}^{\mathrm{L}}$. Using the equivalences

$$(\mathcal{X} \otimes \mathcal{E})_{\geqslant n} \simeq \mathcal{X} \otimes \mathcal{E}_{\geqslant n} \ ,$$

the conclusion follows immediately from the fact that the t-structure τ is right complete and the fact that $\mathcal{X} \otimes (-)$ commutes with colimits in $\mathbf{Pr}^{\mathrm{L}}$. □

Remark 3.9. In particular, Proposition 3.8 establishes the full faithfulness of the natural functor

$$\mathcal{X} \otimes \mathcal{E}_{\geqslant 0} \to \mathcal{X} \otimes \mathcal{E} \ .$$

Assume that $\mathcal{X} = \mathrm{Sh}(\mathcal{C}, \tau)$ is the ∞-topos of sheaves on some ∞-site $(\mathcal{C}, \tau)$. Then

$$\mathcal{X} \otimes \mathcal{E}_{\geqslant 0} \simeq \mathrm{Sh}(\mathcal{C}, \tau; \mathcal{E}_{\geqslant 0}) \qquad \text{and} \qquad \mathcal{X} \otimes \mathcal{E} \simeq \mathrm{Sh}(\mathcal{C}, \tau; \mathcal{E}) \ .$$

Notice that the natural functor

$$\mathrm{Sh}(\mathcal{C}, \tau; \mathcal{E}_{\geqslant 0}) \to \mathrm{Sh}(\mathcal{C}, \tau; \mathcal{E})$$

induced by the functoriality of the tensor product in $\mathbf{Pr}^{\mathrm{L}}$ implicitly involves sheafification. Indeed, if F is a sheaf with values in $\mathcal{E}_{\geqslant 0}$, we can view F as a presheaf with values in $\mathcal{E}$, but this presheaf is typically not a sheaf (as the constant sheaf on S^1 with coefficients in a commutative ring R

shows). Instead, the above comparison functor further sheafifies the resulting presheaf. As a result, even for sheaf ∞-topoi, it is not obvious that this functor is fully faithful.

Corollary 3.10. *Let $f^*\colon \mathcal{X} \to \mathcal{Y}$ be a left exact left adjoint between ∞-topoi. Let $\mathcal{E}$ be a presentable stable ∞-category equipped with an accessible t-structure $\tau = (\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$ which is compatible with filtered colimits and right complete. Then the induced functor*

$$f^* \otimes \mathrm{id}_{\mathcal{E}}\colon \mathcal{X} \otimes \mathcal{E} \to \mathcal{Y} \otimes \mathcal{E}$$

is t-exact.

Proof. We already know from Corollary 2.11 that $f^* \otimes \mathrm{id}_{\mathcal{E}}$ is right t-exact. To prove left t-exactness, we first recall that Proposition 3.8 shows that the t-structures on both $\mathcal{X} \otimes \mathcal{E}$ and $\mathcal{Y} \otimes \mathcal{E}$ are right complete. Therefore, $\mathcal{X} \otimes \mathcal{E} \simeq \mathrm{Sp}(\mathcal{X} \otimes \mathcal{E}_{\geqslant 0})$, and similarly for $\mathcal{Y} \otimes \mathcal{E}$. Invoking [Lur18, Proposition C.3.2.1], we see that $f^* \otimes \mathrm{id}_{\mathcal{E}}$ is left t-exact if and only if the induced functor

$$(f^* \otimes \mathrm{id}_{\mathcal{E}_{\geqslant 0}})\colon \mathcal{X} \otimes \mathcal{E}_{\geqslant 0} \to \mathcal{Y} \otimes \mathcal{E}_{\geqslant 0}$$

is left exact. Combining Recollection 3.6, Corollary 3.7 and [Lur18, Proposition C.4.4.1], we reduce ourselves to the case where $\mathcal{E} = \mathbf{Sp}$. In this case, the conclusion follows from the second half of Corollary 2.11. □

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