CHARACTERISTIC CYCLE AND WILD LEFSCHETZ THEOREMS

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ABSTRACT. By relying on a new approach to Lefschetz type questions based on Beilinson's singular support and Saito's characteristic cycle, we prove an instance of the wild Lefschetz theorem envisioned by Deligne. Our main tool are new finiteness results for the characteristic cycles of perverse sheaves.

CONTENTS

1. Introduction

This paper is a contribution to Lefschetz type theorems in positive characteristic by means of Beilinson's singular support and Saito's characteristic cycle.

Let $X \subset \mathbb{P}_{\mathbb{C}}$ be a smooth complex connected projective variety of dimension ≥ 2. By a fundamental result of Lefschetz, for every sufficiently generic hyperplane H, the induced map at the level of the fundamental groups

$$\pi_1^{\mathrm{top}}(X \cap H) \to \pi_1^{\mathrm{top}}(X)$$

is surjective. When X is quasi-projective, Lefschetz's theorem still holds by works of Goresky-MacPherson [GM88] and Hamm-Lê [HL85]. A natural question is to know if a similar statement holds for the étale fundamental group over some algebraically closed field k of characteristic $p > 0$. See [E17] for a survey of this question. In the projective case, a positive answer was given by Grothendieck [Gro68]. In the quasi-projective case however, the answer is already negative for $U = \mathbb{A}^2_k$. When U is the complement of a strict normal crossing divisor D in a smooth projective variety $X \subset \mathbb{P}_k$, the Lefschetz theorem can nonetheless be saved if instead of considering the full étale fundamental group $\pi_1^{\text{ét}}(U)$ one considers its quotient $\pi_1^{t}(U)$ classifying étale covers of U with *tame* ramification along D [D12, EK16]. Beyond the tame case, wild ramification needs to be bounded. A natural way to do this is by means of effective

Cartier divisors supported on D. If R is such divisor, we say that $\mathcal{L} \in \mathrm{Loc}(U, \overline{\mathbb{Q}}_\ell)$ has *log conductors bounded by* R if for every morphism $f : C \to X$ over k where C is a smooth curve over k not mapped inside D and for every $x \in f^{-1}(D)$, the higher logarithmic ramification slope of $\mathcal{L}|_{C\times_X U}$ at x in the sense of [Ser68] is smaller than the multiplicity of f^*R at x. Deligne asked in [Del16] as a replacement for surjectivity the following

Question 1.1. *Given $r \geq 0$ and an effective Cartier divisor R supported on D, can one find a hypersurface $H \subset \mathbb{P}_k$ such that for every $\mathcal{L} \in \mathrm{Loc}(U, \overline{\mathbb{Q}}_\ell)$ with rank $\leq r$ and log conductors bounded by R, the sheaves $\mathcal{L}$ and $\mathcal{L}|_{U\cap H}$ have the same monodromy groups?*

Question 1.1 was answered positively in the rank 1 case in [KS14] when the compactification X is smooth and D is a strict normal crossing divisor and in [ES21] when X is normal and D is an effective Cartier divisor. In this paper, we show the following

Theorem 1.2 (Theorem 6.14)**.** *Let X be a projective scheme of pure dimension $n \geq 2$ over k algebraically closed. Let $j : U \hookrightarrow X$ be an open immersion with U smooth connected over k such that $D := X - U$ is an effective Cartier divisor. Then, there is a closed immersion $i : X \hookrightarrow \mathbb{P}_k$ and a function $N : \mathbb{N}^2 \to \mathbb{N}$ such that for every $r \geq 0$, every effective Cartier divisor R supported on D and every finite extension $L/\mathbb{Q}_\ell$, there is a dense open subset $V \subset (\mathbb{P}_k^\vee)^{N(\deg R, r)}$ such that for every $(H_1, \dots, H_{N(\deg R,r)}) \in V$ and every $\mathcal{L} \in \mathrm{Loc}(U, L)$ of rank $\leq r$ and log conductors bounded by R, there is $1 \leq i \leq N(\deg R, r)$ such that $\mathcal{L}$ and $\mathcal{L}|_{U\cap H_i}$ have the same monodromy groups.*

Let us explain the differences with Deligne's original question. Firstly, Theorem 1.2 applies to finite coefficients over $\mathbb{Q}_\ell$ while Deligne asks for $\overline{\mathbb{Q}}_\ell$-coefficients. Observe that the objects of $\mathrm{Loc}(U, L)$ with log conductors bounded by R are the representations of a quotient $\pi_1(U, R)$ of $\pi_1(U)$ classifying finite étale covers with ramification bounded by R (see [Hir17]). In general, Artin-Schreier's theory implies that the pro-p abelian quotient of $\pi_1(U, R)$ may not topologically finitely generated. Thus, $\pi_1(U, R)$ is not topologically finitely generated in general. Hence, defining a representation of $\pi_1(U, R)$ amounts to specify an infinite number of data even for finite coefficients. What Theorem 1.2 says is that these data are captured by curves. Secondly, Deligne's question is about the existence of a single hypersurface of possibly high degree depending on the ramification, whereas Theorem 1.2 sticks to hyperplanes and allows some flexibility in their choice. The price to pay for using only hyperplanes is that several of them are needed.

Remark 1.3. If one is interested by cutting down the dimension further with higher codimension projective subspaces, one can construct a moduli of multi-flags (see Appendix) containing a dense open subset of points realizing the Lefschetz theorem. This moduli provides some flexibility useful in practice. Using it, one can indeed show that the projective subspaces realizing the Lefschetz theorem can be taken in some prescribed dense open subset of the Grassmannian. See Theorem 6.14 for the full statement.

The proofs of Theorem 1.2 rely on a new approach to questions of Lefschetz type based on Beilinson's singular support and Saito's characteristic cycle. Namely given a quasi-projective variety $U \subset \mathbb{P}_k$, a hyperplane H and $\mathcal{L} \in \mathrm{Loc}(U, \overline{\mathbb{Q}}_\ell)$, we give a criterion for $\mathcal{L}$ and $\mathcal{L}|_{U\cap H}$ to have the same monodromy group in terms of the transversality of $H \hookrightarrow \mathbb{P}_k$ with respect to the singular support of some auxiliary sheaf constructed from $\mathcal{L}$ (see Lemma 6.12 and Proposition 5.1). The whole point of Theorem 1.2 is then to produce enough hyperplanes satisfying this transversality criterion, or equivalently to show that the singular supports under scrutiny can be controlled in some sense.

To explain how to do this, let us recall that for a smooth scheme of finite type X over a perfect field k of characteristic $p > 0$ and for a finite field Λ of characteristic $\ell \neq p$, [Bei16] associates to every object $\mathcal{K} \in D^b_c(X, \Lambda)$ a closed conical subset $SS(\mathcal{K})$ of the cotangent bundle $\mathbb{T}^*X$ of X, called the *singular support of* $\mathcal{K}$ and recording the hyperplanes through which the sections of $\mathcal{K}$ don't propagate. One can thus view $SS(\mathcal{K})$ as a measure of how far $\mathcal{K}$ is from being locally constant. The propagation defect of sections through a hyperplane H is reflected by the non triviality of some monodromy action on the complex of vanishing cycles of $\mathcal{K}$. The *characteristic cycle* $CC(\mathcal{K})$ *of* $\mathcal{K}$ is a cycle supported on $SS(\mathcal{K})$ measuring the wild ramification of this monodromy action through the so-called Milnor formula [Sai17b, Theorem 5.9].

Thus, for every $\mathcal{K}$ as above, basic numerical data can be attached : the number of irreducible components of $SS(\mathcal{K})$ and the multiplicities of $CC(\mathcal{K})$. To make sense of a third basic set of data playing a crucial role in this paper, let us recall following [Bei16] that for a sufficiently nice closed immersion $i : X \hookrightarrow \mathbb{P}_k$ in some projective space (see Assumption 4.4) and for every $\mathcal{K} \in D^b_c(\mathbb{P}_k, \Lambda)$ supported on X with

$$CC(\mathcal{K}) = \sum_a m_a \cdot [C_a]$$

where the $C_a \subset \mathbb{T}^*\mathbb{P}_k$ are distinct closed irreducible conical subsets, the images D_a of the projectivisation $\mathbb{P}(C_a) \subset \mathbb{T}^*\mathbb{P}_k$ by the map $p^\vee : \mathbb{T}^*\mathbb{P}_k \to \mathbb{P}_k^\vee$ sending (x, H) to H is a divisor of $\mathbb{P}_k^\vee$ and the induced map $\mathbb{P}(C_a) \to D_a$ is generically radicial. In a nutshell, this says that sufficiently nice embeddings have enough hyperplanes to distinguish the components of CC for sheaves supported on X. Hence, one may further consider the degrees d_a of the maps $\mathbb{P}(C_a) \to D_a$ and the degrees of the divisors D_a. Let us package these numerical data into the *total degree of* $CC(\mathcal{K})$ *with respect to* $i : X \hookrightarrow \mathbb{P}_k$ defined by

$$\operatorname{totdeg}_i(CC(\mathcal{K})) := (-1)^{n-m} \sum_a d_a \cdot m_a \cdot \deg D_a$$

where $n = \dim X$ and $m = \dim \mathbb{P}_k$. From the perspective of Theorem 1.2, the degrees $\deg D_a$ are the numerical data we need to bound. As explained above, proving Theorem 1.2 indeed amounts to construct enough hyperplanes H transversal to some suitable singular supports $SS(\mathcal{K})$. On the other hand transversality of H with respect to $SS(\mathcal{K})$ is equivalent to ask for H to avoid all the D_a. By some basic algebraic geometry lemma (see Lemma 6.17), this can be achieved if we take enough hyperplanes at the cost of bounding the $\deg D_a$. Since for a perverse sheaf $\mathcal{K}$, the multiplicities of $CC(\mathcal{K})$ are positive integers [Sai17b, Lemma 5.14], it is enough to bound the total degree $\operatorname{totdeg}_i(CC(\mathcal{K}))$. To carry out this program, we provide the following cohomological interpretation of the total degree :

Theorem 1.4 (Theorem 4.11)**.** *Let X be a smooth projective scheme of pure dimension over an algebraically closed field k. Let $i : X \hookrightarrow \mathbb{P}_k$ be a closed immersion satisfying Assumption 4.4. Let Λ be a finite field of characteristic $\ell \neq p$. Then, for every $\mathcal{K} \in D^b_c(X, \Lambda)$, for every sufficiently generic hyperplane $H \subset \mathbb{P}_k$ and every sufficiently generic pencil $L \subset \mathbb{P}^\vee_k$, we have*

$$\mathrm{totdeg}_i(CC(i_*\mathcal{K})) = \chi(X_L, \mathcal{K}|_{X_L}) - 2 \cdot \chi(X \cap H, \mathcal{K}|_{X\cap H}) .$$

where X_L is the total space of the pencil L.

Hence, Theorem 1.4 *translates the problem of bounding the total degree into the problem of bounding some Euler-Poincaré characteristics*. That this latter boundedness holds under rank and ramification boundedness conditions follows from the main result of [HT25b]. To take a full advantage of it, one needs to bound the wild ramification not only for the extension by 0 of locally constant constructible sheaves but for arbitrary constructible complexes. To this end, effective Cartier divisor are insufficient as some wild ramification may hide in codimension > 1. To solve this problem, we introduced in [HT25a] some ramification boundedness using coherent sheaves instead of effective Cartier divisors. If $\mathbb{Q}[\mathrm{Coh}(X)]$ is the free $\mathbb{Q}$-vector space on the set of isomorphism classes of coherent sheaves on a scheme of finite type X over a field k, we set the following

Definition 1.5. For $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$, we say that $\mathcal{K} \in D^b_c(X, \Lambda)$ has *log conductors bounded by* $\mathcal{E}$ if for every $i \in \mathbb{Z}$, every morphism $f : C \to X$ where C is a smooth curve over k and every $x \in C$, the logarithmic conductor of $\mathcal{H}^i\mathcal{K}|_C$ at x is smaller than the length of the torsion part of $(f^*\mathcal{E})_x$ viewed as a module over $\mathcal{O}_{C,x}$.

By combining [HT25b, Corollary 7.30] with Theorem 1.4, we get the following

Theorem 1.6 (Theorem 4.16)**.** *In the setting of Theorem 1.4, let Σ be a stratification of X and let $a \leq b$ be integers. Then, there is a function $\mu : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q}$ and $P \in \mathbb{N}[x]$ of degree $\dim X$ such that for every finite field Λ of characteristic $\ell \neq p$, every $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$ and every $\mathcal{K} \in D^{[a,b]}_\Sigma(X, \mathcal{E}, \Lambda)$, we have*

$$|\,\mathrm{totdeg}(CC(i_*\mathcal{K}))| \leq P(\mu(\mathcal{E})) \cdot \mathrm{Rk}_\Lambda \mathcal{K} .$$

In particular, if $\mathcal{K}$ is perverse, the number of irreducible components of $SS(\mathcal{K})$, the multiplicities of $CC(\mathcal{K})$ and the degrees of the D_a are smaller than $P(\mu(\mathcal{E})) \cdot \mathrm{Rk}_\Lambda \mathcal{K}$.

Hence, if we fix a stratification and bound the rank and the wild ramification, there is a universal bound for the degrees of the divisors $p^\vee(SS(i_*\mathcal{K})) \subset \mathbb{P}^\vee_k$, which we saw is what is needed to prove Theorem 1.2.

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Linear Overview. Section 2 provides an account of Abbes and Saito theory for the logarithmic conductor. In section 3, we introduce Beilinson's singular support and Saito's characteristic cycle for étale sheaves and recall some terminology from [HT25a] to state the cohomological boundedness result from [HT25b] we need. In section 4, we give the cohomological interpretation of the total degree for the characteristic cycle (Theorem 1.4) and deduce Theorem 1.6. The consequence is the existence of some dense open subset of some moduli of multi-flags whose points achieve simultaneous transversality in some sense (Definition 4.26) for all perverse sheaves with bounded rank and log conductors (Theorem 4.29). In section 5, we give a criterion for the restriction to a hyperplane section to distinguish two sheaves in terms of the singular support of their $\mathcal{H}$om-sheaf (Proposition 5.1). We derived from it the existence of some dense open subset of a moduli of multi-flags whose points achieve simultaneous recognition for all pair of perverse sheaves with bounded rank and log conductors (Theorem 5.7). The wild Lefschetz Theorem 1.2 is then deduced in section 6. The Appendix provides the basic language used to formulate generic statements instead of merely existence statements, thus giving some flexibility for future applications.

Notation 1.7. We introduce the following running notations.

- k denotes a perfect field of characteristic $p > 0$.
- The letter Λ will refer to a finite local ring of residue characteristic $\ell \neq p$.
- For a scheme X of finite type over k, we denote by $D^b_{ctf}(X, \Lambda)$ the derived category of complexes of Λ-sheaves of finite tor-dimension with bounded and constructible cohomology sheaves.
- $\mathrm{Loc}_{tf}(X, \Lambda)$ will denote the category of locally constant constructible sheaves of Λ-modules of finite tor-dimension over X. By [Wei94, Lemma 4.4.14], the germs of any $\mathcal{L} \in \mathrm{Loc}_{tf}(X, \Lambda)$ are automatically free Λ-modules of finite rank.
- $\mathrm{Perv}_{tf}(X, \Lambda)$ will denote the category of perverse sheaves of Λ-modules of finite tor-dimension over X for the middle perversity function.
- Let X be a scheme of finite type over k and let Λ be a field of characteristic $\neq p$. For $\mathcal{K} \in D^b_{ctf}(X, \Lambda)$, we put

$$\mathrm{Rk}_\Lambda \mathcal{K} := \max\{\mathrm{rk}_\Lambda \mathcal{H}^i \mathcal{K}_{\bar{x}}, \text{ where } i \in \mathbb{Z} \text{ and } \bar{x} \to X \text{ is algebraic geometric}\}\,.$$

- For $r \geq 0$, we let $D^{\leq r}_{ctf}(X, \Lambda) \subset D^b_{ctf}(X, \Lambda)$ be the full subcategory spanned by objects $\mathcal{K}$ such that $\mathrm{Rk}_\Lambda \mathcal{K} \leq r$, and similarly with perverse complexes.
- For a finite stratification Σ of X, we let $D^b_{\Sigma,tf}(X, \Lambda) \subset D^b_{tf}(X, \Lambda)$ be the full subcategory spanned by Σ-constructible complexes, and similarly with perverse complexes.

2. Conductors of étale sheaves

2.1. Ramification filtrations. Let K be a henselian discrete valuation field over k. Let $\mathcal{O}_K$ be the ring of integer of K, let $\mathfrak{m}_K$ be the maximal ideal of $\mathcal{O}_K$ and F the residue field of $\mathcal{O}_K$. Fix $K \subset K^{sep}$ a separable closure of K and let G_K be the Galois group of K^{sep} over K. Let $I_K \subset G_K$ be the inertia subgroup and let $P_K \subset I_K$ be the wild ramification subgroup.

Recollection 2.2. In [AS02], Abbes and Saito defined two decreasing filtrations $\{G^r_K\}_{r\in\mathbb{Q}_{>0}}$ and $\{G^r_{K,\log}\}_{r\in\mathbb{Q}_{\geq 0}}$ on G_K by closed normal subgroups. They are called the *the ramification filtration* and *the logarithmic ramification filtration* respectively. For $r\in\mathbb{Q}_{\geq 0}$, put

$$G^{r+}_K=\overline{\bigcup_{s>r}G^s_K}\quad\text{and}\quad G^{r+}_{K,\log}=\overline{\bigcup_{s>r}G^s_{K,\log}}.$$

Proposition 2.3 ([AS02, AS03, Sai08, Sai17a])**.** *The following properties hold :*

(1) For any $0<r\leq 1$, we have

$$G^r_K=G^0_{K,\log}=I_K \textit{ and } G^{+1}_K=G^{0+}_{K,\log}=P_K.$$

(2) For any $r\in\mathbb{Q}_{\geq 0}$, we have

$$G^{r+1}_K\subseteq G^r_{K,\log}\subseteq G^r_K.$$

If F is perfect, then for any $r\in\mathbb{Q}_{\geq 0}$, we have

$$G^r_{K,\mathrm{cl}}=G^r_{K,\log}=G^{r+1}_K.$$

where $G^r_{K,\mathrm{cl}}$ is the classical wild ramification subgroup as defined in [Ser68].

(3) For any $r\in\mathbb{Q}_{>0}$, the graded piece $G^r_{K,\log}/G^{r+}_{K,\log}$ is abelian, p-torsion and contained in the center of $P_K/G^{r+}_{K,\log}$.

Let M be a finitely generated Λ-module with a continuous P_K-action. The module M has decompositions

$$M=\bigoplus_{r\geq 1}M^{(r)}\quad\text{and}\quad M=\bigoplus_{r\geq 0}M^{(r)}_{\log} \tag{2.3.1}$$

into P_K-stable Λ-submodules where $M^{(1)}=M^{(0)}_{\log}=M^{P_K}$, and such that for every $r\in\mathbb{Q}_{>0}$,

$$(M^{(r+1)})^{G^{r+1}_K}=0 \text{ and } (M^{(r+1)})^{G^{(r+1)+}_K}=M^{(r+1)};$$
$$(M^{(r)}_{\log})^{G^r_{K,\log}}=0 \text{ and } (M^{(r)}_{\log})^{G^{r+}_{K,\log}}=M^{(r)}_{\log}.$$

The decompositions (2.3.1) are respectively called the *slope decomposition* and the *logarithmic slope decomposition* of M. The values r for which $M^{(r)}\neq 0$ (resp. $M^{(r)}_{\log}\neq 0$) are the *slopes* (resp. the *logarithmic slopes*) of M. We denote by $c_K(M)$ the largest slope of M and refer to $c_K(M)$ as the *conductor of* M. Similarly, we denote by $lc_K(M)$ the largest logarithmic slope of M and refer to $lc_K(M)$ as the *logarithmic conductor of* M. We say that M is *isoclinic* (resp. *logarithmic isoclinic*) if M has only one slope (resp. only one logarithmic slope).

The following is an immediate consequence of Proposition 2.3-(2).

Lemma 2.4. *Let M be a finitely generated Λ-module with a continuous P_K-action. Then,*

$$lc_K(M)\leq c_K(M)\leq lc_K(M)+1.$$

If M is free as a Λ-module, then so are the $M^{(r)}_{\log}$ and the $M^{(r)}$ in virtue of [Ka88, Lemma 1.5]. In that case, the *total dimension* of M is defined by

$$\mathrm{dimtot}_K(M) := \sum_{r\geq 1} r \cdot \mathrm{rk}_\Lambda M^{(r)}$$

and the *Swan conductor* of M is defined by

$$\mathrm{sw}_K(M) := \sum_{r\geq 0} r \cdot \mathrm{rk}_\Lambda M^{(r)}_{\log} .$$

Lemma 2.5 ([AS02]). *In the setting of Recollection 2.2, we have*

$$\mathrm{sw}_K(M) \leq \mathrm{dimtot}_K(M) \leq \mathrm{sw}_K(M) + \mathrm{rk}_\Lambda M .$$

If the residue field F is perfect, we have

$$\mathrm{lc}_K(M) + 1 = c_K(M) .$$
$$\mathrm{sw}_K(M) + \mathrm{rk}_\Lambda M = \mathrm{dimtot}_K(M) .$$

2.6. Conductor divisors. Let X be a normal scheme of finite type over k. Let Z be an integral Weil divisor and let $\eta \in Z$ be its generic point. Let K be the fraction field of $\hat{\mathcal{O}}_{X,\eta}$ and fix a separable closure K^{sep} of K. For $\mathcal{F} \in \mathrm{Cons}(X, \Lambda)$, the pull-back $\mathcal{F}|_{\operatorname{Spec} K}$ is a Λ-module of finite type with continuous G_K-action. Using the notations from Section 2.1, we put

$$c_Z(\mathcal{F}) := c_K(\mathcal{F}|_{\operatorname{Spec} K}) \text{ and } \mathrm{lc}_Z(\mathcal{F}) := \mathrm{lc}_K(\mathcal{F}|_{\operatorname{Spec} K}) .$$

Definition 2.7. Let X be a normal scheme of finite type over k and let $\mathcal{F} \in \mathrm{Cons}_{tf}(X, \Lambda)$. We define the *conductor divisor of $\mathcal{F}$* as the Weil divisor with rational coefficients given by

$$C_X(\mathcal{F}) := \sum_Z c_Z(\mathcal{F}) \cdot Z$$

and the *logarithmic conductor divisor of $\mathcal{F}$* as the Weil divisor with rational coefficients given by

$$LC_X(\mathcal{F}) := \sum_Z \mathrm{lc}_Z(\mathcal{F}) \cdot Z$$

where the sums run over the set of integral Weil divisors of X.

Remark 2.8. The above divisors are $\mathbb{Q}$-Weil divisors of X. We will sometimes abuse the notations and write $C(\mathcal{F})$ instead of $C_X(\mathcal{F})$ and similarly in the logarithmic case.

Definition 2.9. In the setting of Section 2.6, we define the *generic conductor* and the *generic logarithmic conductor of $\mathcal{L}$ along* D respectively by

$$c_D(\mathcal{L}) := \max_Z c_Z(\mathcal{L}) \text{ and } \mathrm{lc}_D(\mathcal{L}) := \max_Z \mathrm{lc}_Z(\mathcal{L}) .$$

where Z runs over the set of irreducible components of D.

3. Singular support and characteristic cycle of étale sheaves

3.1. The singular support. Let X be a smooth scheme of finite type over k. We denote by $\mathbb{T}^*X$ the cotangent bundle of X. Let $C \subset$ be a closed conical subset. For a point $x \in X$, we put $\mathbb{T}^*_x X = \mathbb{T}^*X \times_X x$ and $C_x = C \times_X x$.

Recollection 3.2. Let $h : U \to X$ be a morphism of smooth schemes of finite type over k. For $u \in U$, we say that $h : U \to X$ is C-*transversal at* u if

$$\ker dh_u \bigcap C_{h(u)} \subseteq \{0\} \subseteq \mathbb{T}^*_{h(u)}X$$

where $dh_u : \mathbb{T}^*_{h(u)}X \to \mathbb{T}^*_u U$ is the cotangent map of h at u. We say that $h : U \to X$ is C-*transversal* if it is C-transversal at every point of U. For a C-transversal morphism $h : U \to X$, we let $h^\circ C$ be the scheme theoretic image of $C \times_X U$ in $\mathbb{T}^*U$ by $dh : \mathbb{T}^*X \times_X U \to \mathbb{T}^*U$.

Let $f : X \to Y$ be a morphism of smooth schemes of finite type over k. For $x \in X$, we say that $f : X \to Y$ is C-*transversal at* x if

$$df_x^{-1}(C_x) \subseteq \{0\} \subseteq \mathbb{T}^*_{f(x)}Y$$

We say that $f : X \to Y$ is C-*transversal* if it is C-transversal at every point of X.

Let $(h, f) : Y \leftarrow U \to X$ be a pair of morphisms of between smooth schemes of finite type over k. We say that (h, f) is C-*transversal* if $h : U \to X$ is C-transversal and if $f : U \to Y$ is $h^\circ C$-transversal.

Definition 3.3. In the setting of Section 3.1, we say that $\mathcal{K} \in D^b_c(X, \Lambda)$ is *micro-supported on* C if for every C-transversal pair $(h, f) : Y \leftarrow U \to X$, the map $f : U \to Y$ is universally locally acyclic with respect to $h^*\mathcal{K}$.

Theorem 3.4 ([Bei16, Theorem 1.3])**.** *For every $\mathcal{K} \in D^b_c(X, \Lambda)$, there is a smallest closed conical subset $SS(\mathcal{K}) \subset \mathbb{T}^*X$ on which $\mathcal{K}$ is micro-supported. Furthermore, if X has pure dimension n, then $SS(\mathcal{K})$ has pure dimension n.*

Definition 3.5. The closed conical subset $SS(\mathcal{K})$ is the *singular support of* $\mathcal{K}$.

3.6. Base change and transversality. Let $f : Y \to X$ be a separated morphism of smooth schemes of finite over k. Following [Sai17b, § 8.2], for every $\mathcal{K} \in D^b_c(X, \Lambda)$, there is a canonical morphism

$$h_{f,\mathcal{K}} : f^*\mathcal{K} \otimes^L_\Lambda Rf^!\Lambda \to Rf^!\mathcal{K} \tag{3.6.1}$$

obtained by adjunction from the composition

$$Rf_!(f^*\mathcal{K} \otimes^L_\Lambda Rf^!\Lambda) \xrightarrow{\sim} \mathcal{K} \otimes^L_\Lambda Rf_!Rf^!\Lambda \to \mathcal{K}$$

where the first arrow is the projection formula and where the second arrow is induced by the adjunction $Rf_!Rf^!\Lambda \to \Lambda$. Let us recall the following

Proposition 3.7 ([Sai17b, Proposition 8.13])**.** *If $\mathcal{K}$ is of finite tor-dimension and if $f : Y \to X$ is $SS(\mathcal{K})$-transversal, then the canonical morphism* (3.6.1) *is an isomorphism.*

Definition 3.8. For a morphism $f : Y \to X$ between pure dimensional smooth schemes of finite type over k, we put $c_f = \dim Y - \dim X$.

The following proposition generalizes [HT21, Corollary 2.13].

Proposition 3.9 ([Sai22, Proposition 1.1.8]). *Let*

$$\begin{tikzcd} V \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\square"] & U \arrow[d, "g"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

be a cartesian diagram of smooth schemes of finite type over k *of pure dimension. Let* $\mathcal{K} \in D^b_{ctf}(U, \Lambda)$ *and assume that the following hold :*

(1) The morphism $f : Y \to X$ *is separated.*

(2) We have $c_f = c_{f'}$.

(3) $f' : V \to U$ *is* $SS(\mathcal{K})$*-transversal.*

Then, $f : Y \to X$ *is* $SS(Rg_*\mathcal{K})$*-transversal if and only if the base change morphism*

$$f^* Rg_* \mathcal{K} \to Rg'_* f'^* \mathcal{K}$$

is an isomorphism.

Proof. Since all schemes are smooth over k, the two complexes $Rf^! \Lambda$ and $Rf'^! \Lambda$ have locally constant cohomologies by the Poincaré duality. Since $c_f = c_{f'}$ and $V = Y \times_X U$ is smooth, the base change morphism $g'^* f^! \Lambda \to f'^! \Lambda$ is an isomorphism ([Sai22, Lemma 1.1.4]). Then, Proposition 3.9 is a direct consequence of Proposition 3.7 and [Sai22, Proposition 1.1.8]. □

3.10. The characteristic cycle. Let $f : X \to S$ be a morphism between smooth schemes of finite type over k where S is a curve over k. Let $x \in X$ be a closed point and put $s = f(x)$. Note that any local trivialization of $\mathbb{T}^*S$ in a neighborhood of s gives rise to a local section of $\mathbb{T}^*X$ in a neighborhood of x by applying $df : \mathbb{T}^*S \times_S X \to \mathbb{T}^*X$. We abusively denote by df this section.

We say that x is an *at most* C*-isolated characteristic point* for $f : X \to S$ if $f : X \setminus \{x\} \to S$ is C-transversal. In that case, the intersection of a cycle A supported on C with $[df]$ is supported at most at a single point in $\mathbb{T}^*_x X$. Since C is conical, the intersection number $(A, [df])_{\mathbb{T}^*X, x}$ is independent of the chosen local trivialization for $\mathbb{T}^*S$ in a neighborhood of s.

Theorem 3.11 ([Sai17b, Theorem 5.9]). *Let* X *be a smooth scheme of finite type over* k. *For every* $\mathcal{K} \in D^b_{ctf}(X, \Lambda)$*, there is a unique cycle* $CC(\mathcal{K})$ *of* $\mathbb{T}^*X$ *supported on* $SS(\mathcal{K})$ *such that for every étale morphism* $h : U \to X$*, for every morphism* $f : U \to S$ *with* S *a smooth curve over* k*, for every at most* $h^\circ(SS(\mathcal{K}))$*-isolated characteristic point* $u \in U$ *for* $f : U \to S$*, we have the following Milnor type formula*

$$-\operatorname{dimtot}(R\Phi_{\overline{u}}(h^*\mathcal{K}, f)) = (h^* CC(\mathcal{K}), [df])_{T^*U, u}, \tag{3.11.1}$$

where $R\Phi_{\overline{u}}(h^*\mathcal{K}, f)$ *denotes the stalk of the vanishing cycle of* $h^*\mathcal{K}$ *with respect to* $f : U \to S$ *at a geometric point* $\overline{u} \to U$ *above* u.

Definition 3.12. The cycle $CC(\mathcal{K})$ from Theorem 3.11 is the *characteristic cycle of* $\mathcal{K}$.

Remark 3.13 ([Sai17b, Lemma 5.14])**.** When $\mathcal{K}$ is perverse, $SS(\mathcal{K})$ and $CC(\mathcal{K})$ have the same support and the multiplicities of $CC(\mathcal{K})$ are positive integers.

We store for future use the following

Lemma 3.14. *Let X be a scheme of finite type over a field k. Let $j : U \hookrightarrow X$ be an affine open immersion with U smooth over k. Let $i : X \hookrightarrow Y$ be a closed immersion in a smooth scheme of finite type over k. Then, for every $\mathcal{P} \in \mathrm{Perv}_{tf}(U, \Lambda)$, we have*

$$SS(i_* j_! \mathcal{P}) = SS(i_* Rj_* \mathcal{P}) .$$

Proof. Since $j : U \hookrightarrow X$ is an affine open immersion, the complexes $i_* j_! \mathcal{P}$ and $i_* Rj_* \mathcal{P}$ are perverse by [BBDG18, Corollaire 4.1.10]. Hence $SS(i_* j_! \mathcal{P}) = \mathrm{Supp}\, CC(i_* j_! \mathcal{P})$ and $SS(i_* Rj_* \mathcal{P}) = \mathrm{Supp}\, CC(i_* Rj_* \mathcal{P})$ by Remark 3.13. Since $CC(i_* j_! \mathcal{P}) = CC(i_* Rj_* \mathcal{P})$ by [Sai17b, Lemma 5.13-3], the conclusion follows. $\square$

The following index formula provides a positive characteristic analogue of Kashiwara-Dubson's formula for $\mathcal{D}$-modules.

Theorem 3.15 ([Sai17b, Theorem 7.13])**.** *Let X be a smooth projective variety over an algebraically closed field k. For every $\mathcal{K} \in D^b_{ctf}(X, \Lambda)$, we have*

$$\chi(X, \mathcal{K}) = (CC(\mathcal{K}), \mathbb{T}^*_X X)_{\mathbb{T}^* X} .$$

3.16. Bounding the ramification with coherent sheaves. Let X be a scheme of finite type over k. We denote by $\mathbb{Q}[\mathrm{Coh}(X)]$ the free $\mathbb{Q}$-vector space on the set of isomorphism classes of coherent sheaves on X. Observe that the pullback along every morphism $f : Y \to X$ of schemes of finite type over k induces a morphism of $\mathbb{Q}$-vector spaces

$$f^* : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q}[\mathrm{Coh}(Y)] .$$

Assume now that X is normal and let $\mathcal{E} \in \mathrm{Coh}(X)$. If $X^1 \subset X$ denotes the set of codimension 1 points of X, we define a Weil divisor on X by the formula

$$T(\mathcal{E}) := \sum_{\eta \in X^1} \mathrm{length}_{\mathcal{O}_{X,\eta}} (\mathcal{E}|^{\mathrm{tors}}_{X_\eta}) \cdot \overline{\{\eta\}}$$

where $X_\eta = \mathrm{Spec}\, \mathcal{O}_{X,\eta}$ and where $\mathcal{E}|^{\mathrm{tors}}_{X_\eta}$ is the torsion part of $\mathcal{E}|_{X_\eta}$.

Example 3.17. If R is an effective Cartier divisor of X with ideal sheaf $\mathcal{I}_R$ and if $\mathcal{E} = \mathcal{O}_X / \mathcal{I}_R$, then $T(\mathcal{E}) = R$.

If $\mathrm{Weil}(X)_{\mathbb{Q}}$ is the space of $\mathbb{Q}$-Weil divisors on X, the map $R : \mathrm{Coh}(X) \to \mathrm{Weil}(X)_{\mathbb{Q}}$ induces a map of $\mathbb{Q}$-vector spaces

$$R : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathrm{Weil}(X)_{\mathbb{Q}} .$$

Definition 3.18. Let X be a scheme of finite type over k. Let $\mathcal{K} \in D^b_c(X, \Lambda)$ and $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$. We say that $\mathcal{K}$ *has log conductors bounded by* $\mathcal{E}$ if for every morphism $f : C \to X$ over k where C is a smooth curve over k, we have

$$LC(\mathcal{H}^i \mathcal{K}|_C) \leq T(f^* \mathcal{E})$$

for every $i \in \mathbb{Z}$. We denote by $D^b_c(X, \mathcal{E}, \Lambda)$ the full subcategory of $D^b_c(X, \Lambda)$ spanned by objects having log conductors bounded by $\mathcal{E}$.

The following is our main example of sheaf with explicit bound on the log conductors.

Proposition 3.19 ([HT25a, Proposition 5.7]). *Let X be a normal scheme of finite type over k. Let D be an effective Cartier divisor of X and put $j : U := X - D \hookrightarrow X$. Let $\mathcal{L} \in \mathrm{Loc}_{\mathrm{tf}}(U, \Lambda)$ and $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$.*

(1) If $j_!\mathcal{L}$ has log conductors bounded by $\mathcal{E}$, then $\mathrm{LC}_X(j_!\mathcal{L}) \leq T(\mathcal{E})$.

(2) If X is smooth over k, then $j_!\mathcal{L}$ has log conductors bounded by $(\mathrm{lc}_D(\mathcal{L}) + 1) \cdot \mathcal{O}_D$.

Definition 3.20. Let X be a scheme of finite type over k. We say that a $\mathbb{Q}$-linear map $\mu : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q}$ is *admissible* if the following conditions are satisfied :

(1) For every $\mathcal{E} \in \mathrm{Coh}(X)$, we have $\mu(\mathcal{E}) \in \mathbb{N}$.

(2) For every $\mathcal{E}_1, \mathcal{E}_2 \in \mathrm{Coh}(X)$, we have $\mu(\mathcal{E}_1 \bigoplus \mathcal{E}_2) \leq \mu(\mathcal{E}_1) + \mu(\mathcal{E}_2)$.

Definition 3.21. Let P be a property of morphisms of schemes over k and let $n \geq 0$. A P-relative stratified scheme $(X/S, \Sigma)$ of relative dimension $\leq n$ will refer to a morphism $X \to S$ between schemes of finite type over k satisfying P such that the fibres of $X \to S$ have dimensions $\leq n$ and where X is endowed with a finite stratification Σ.

The following theorem is one of the main result of [HT25b] :

Theorem 3.22 ([HT25b, Corollary 7.30]). *Let $(X/S, \Sigma)$ be a proper relative stratified scheme of relative dimension $\leq n$ and let $a \leq b$ be integers. Then, there is an admissible function $\mu : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q}$ and $P \in \mathbb{N}[x]$ of degree n such that for every algebraic geometric point $\bar{s} \to S$, every finite field Λ of characteristic $\ell \neq p$, every $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$ and every $\mathcal{K} \in D^{[a,b]}_{\Sigma_{\bar{s}}}(X_{\bar{s}}, \mathcal{E}_{\bar{s}}, \Lambda)$, we have*

$$\sum_{j \in \mathbb{Z}} h^j(X_{\bar{s}}, \mathcal{K}) \leq P(\mu(\mathcal{E})) \cdot \mathrm{Rk}_\Lambda \mathcal{K}\,.$$

4. Characteristic cycle boundedness

4.1. Geometric situation. Let S be a scheme of finite type over k, let E be a locally free sheaf of $\mathcal{O}_S$-modules on S. We let Q_S be the relative universal hyperplane in $\mathbb{P}_S(E)$ and F_S the relative universal projective line in $\mathbb{P}_S(E^\vee)$. Let $f : X \to S$ be a projective morphism over k and let $i : X \hookrightarrow \mathbb{P}_S(E)$ be a closed immersion. Consider the commutative diagram with cartesian squares

$$\begin{tikzcd}
X_L \arrow[r] \arrow[d] & (X \times_{\mathbb{P}_S(E)} Q_S) \times_{\mathbb{P}_S(E^\vee)} F_S \arrow[r] \arrow[d] \arrow[ddd, bend right, "q_X^\vee"'] \arrow[rr, bend left, "q_X"] & X \times_{\mathbb{P}_S(E)} Q_S \arrow[r, "p_X"] \arrow[d] \arrow[dd, bend right, "p_X^\vee"'] & X \arrow[d, "i"] \\
\mathbb{P}(E)_L \arrow[r] \arrow[d, "p_L^\vee"] & F_S \times_{\mathbb{P}_S(E^\vee)} Q_S \arrow[r] \arrow[d] & Q_S \arrow[r, "p"] \arrow[d, "p^\vee"] & \mathbb{P}_S(E) \\
L \arrow[r] \arrow[d] & F_S \arrow[r] \arrow[d] & \mathbb{P}_S(E^\vee) & \\
\{L\} \arrow[r] & \mathbb{G}_S(E^\vee, 2) & &
\end{tikzcd}$$

where $L \in \mathbb{G}_S(E^\vee, 2)$ is a pencil lying above an algebraic geometric point $\bar{s} \to S$ lying over $s \in S$. Let $\bar{\eta}_s \to \mathbb{P}_S(E^\vee)$ be an algebraic geometric point over the generic point of $\mathbb{P}_{\bar{s}}(E^\vee_{\bar{s}})$. Let $\bar{\xi}_s \to \mathbb{G}_S(E^\vee, 2)$ be an algebraic geometric point over the generic point of $\mathbb{G}_{\bar{s}}(E^\vee_{\bar{s}}, 2)$. We consider the cartesian squares

$$\begin{tikzcd}
X_{\bar{\eta}_s} \arrow[r] \arrow[d] & X \times_{\mathbb{P}_S(E)} Q_S \arrow[d, "p_X^\vee"] \\
\bar{\eta}_s \arrow[r] & \mathbb{P}_S(E^\vee)
\end{tikzcd}
\qquad
\begin{tikzcd}
X_{\bar{\xi}_s} \arrow[r] \arrow[d] & (X \times_{\mathbb{P}_S(E)} Q_S) \times_{\mathbb{P}_S(E^\vee)} F_S \arrow[d, "q_X^\vee"] \\
\bar{\xi}_s \arrow[r] & \mathbb{G}_S(E^\vee, 2).
\end{tikzcd}$$

Remark 4.2. We think about $X_{\bar{\eta}_s}$ as the generic hyperplane section of $X_{\bar{s}}$ and about $X_{\bar{\xi}_s}$ as the generic pencil of $X_{\bar{s}}$.

Remark 4.3. Assume that $S = \operatorname{Spec} k$, in which case we drop the subscript S, let Λ be a finite field of characteristic $\ell \neq p$ and let $\mathcal{K} \in D^b_c(X, \Lambda)$. Since $\chi(X_{\bar{\eta}}, \mathcal{K}|_{X_{\bar{\eta}}})$ is the generic rank of $Rp^\vee_{X*}(\mathcal{K}|_{X \times_{\mathbb{P}(E)} Q})$, we have

$$\chi(X_{\bar{\eta}}, \mathcal{K}|_{X_{\bar{\eta}}}) = \chi(X \cap H, \mathcal{K}|_{X \cap H})$$

for every sufficiently generic hyperplan $H \subset \mathbb{P}(E)$. Similarly, since $\chi(X_{\bar{\xi}}, \mathcal{K}|_{X_{\bar{\xi}}})$ is the generic rank of $Rq^\vee_{X*}(\mathcal{K}|_{(X \times_{\mathbb{P}(E)} Q) \times_{\mathbb{P}(E^\vee)} F})$, we have

$$\chi(X_{\bar{\xi}}, \mathcal{K}|_{X_{\bar{\xi}}}) = \chi(X_L, \mathcal{K}|_{X_L})$$

for every sufficiently generic pencil $L \subset \mathbb{P}(E^\vee)$.

Assumption 4.4 ([Bei16, 4.3])**.** Let S be a scheme of finite type over k. Let X be a smooth projective scheme over S and let $\mathcal{L}$ be a very ample line bundle on X relative to S with corresponding closed immersion $i : X \hookrightarrow \mathbb{P}_S(E)$, where E is a locally free sheaf of $\mathcal{O}_S$-modules on S. For every algebraic geometric point $\bar{s} \to S$ and every closed points $u, v \in X_{\bar{s}}$, the composition

$$\Gamma(\mathbb{P}_{\bar{s}}(E_{\bar{s}}), \mathcal{O}_{\mathbb{P}_{\bar{s}}(E_{\bar{s}})}(1)) \to \Gamma(X_{\bar{s}}, \mathcal{L}|_{X_{\bar{s}}}) \to \mathcal{L}_u / \mathfrak{m}_u^2 \mathcal{L}_u \oplus \mathcal{L}_v / \mathfrak{m}_v^2 \mathcal{L}_v$$

is surjective.

4.5. Recollection. In the setting of Section 4.1, assume that X is a projective space $\mathbb{P}$ over k and let $L \subset \mathbb{P}^\vee$ be a pencil with axis $A \subset \mathbb{P}$. In that case, the diagram from Section 4.1 simplifies into the following cartesian diagram

$$\begin{tikzcd} \mathbb{P}_L \arrow[r] \arrow[d, "p_L^\vee"] \arrow[rr, bend left, "\pi"] & Q \arrow[r, "p"] \arrow[d, "p^\vee"] & \mathbb{P} \\ L \arrow[r] & \mathbb{P}^\vee & \end{tikzcd}$$

where $\pi : \mathbb{P}_L \to \mathbb{P}$ is the blow-up of $\mathbb{P}$ along A. We have

$$\mathbb{P}_L^\circ := \mathbb{P}_L - \pi^{-1}(A) = \mathbb{P} - A \subset Q - \mathbb{P}(T_A^*\mathbb{P})$$

with $\pi^{-1}(A)$ identifying to $\mathbb{P}(T_A^*\mathbb{P})$ via the closed immersion $\mathbb{P}_L \to Q$. For a closed conical subset $C \subset T^*\mathbb{P}$, we put

$$Z_L(C) := \mathbb{P}_L \cap \mathbb{P}(C) \subset \mathbb{P}_L .$$

Lemma 4.6 ([Sai17b, Lemma 3.10]). *Let $C \subset \mathbb{P}$ be a closed conical subset of pure dimension $\dim \mathbb{P}$. For every $H \in \mathbb{P}^\vee$, the following are equivalent :*

(1) *the inclusion $H \hookrightarrow \mathbb{P}$ is C-transversal.*

(2) *the map $p^\vee : Q \to \mathbb{P}^\vee$ is $p^\circ(C)$-transversal at every point $(x, H), x \in H$.*

(3) *H lies in $\mathbb{P}^\vee - p^\vee(\mathbb{P}(C))$.*

Proposition 4.7 ([Bei16]). *Let X be a smooth scheme of finite type of pure dimension n over k. Let $i : X \hookrightarrow \mathbb{P}$ be an immersion satisfying Assumption 4.4. Let $C \subset T^*X$ be a closed conical subset of pure dimension n. Assume that every irreducible component $C_a \subset C$ satisfies $i_\circ C_a \not\subseteq T_{\mathbb{P}}^*\mathbb{P}$. Then*

(1) *If we put*

$$D_a := \overline{p^\vee(\mathbb{P}(i_\circ C_a))} ,$$

then D_a is a divisor of $\mathbb{P}^\vee$ and the induced map $\mathbb{P}(i_\circ C_a) \to D_a$ is generically radicial.

(2) *If $C_a \neq C_b$, then $D_a \neq D_b$.*

Remark 4.8. As shown in [Bei16], the generic degree of $\mathbb{P}(i_\circ C_a) \to D_a$ is 1 when $p \neq 2$ and 1 or 2 if $p = 2$.

The following lemma is a rephrasing of [SY17, Lemma 2.3]. See also [UYZ20, Lemma 4.9] where the properly transveral condition is taken care of.

Lemma 4.9. *Let X be a smooth projective scheme of pure dimension n over k infinite. Let $i : X \hookrightarrow \mathbb{P}$ be a closed immersion satisfying Assumption 4.4. Let $C \subset T^*X$ be a closed conical subset of pure dimension n such that every irreducible component C_a satisfies $i_\circ C_a \not\subseteq T_{\mathbb{P}}^*\mathbb{P}$ and put $D = p^\vee(\mathbb{P}(i_\circ C))$. Then, for any sufficiently generic pencil L, we have :*

(1) *The map $\pi : \mathbb{P}_L \to \mathbb{P}$ is properly $i_\circ C$-transversal.*

(2) $Z_L(i_\circ C)$ *is a subset of* $\deg D$ *points of* $\mathbb{P}_L^\circ$ *mapping bijectively to* $D \cap L$.

(3) the map $p_L^\vee : \mathbb{P}_L \to L$ *is* $\pi^\circ i_\circ C$*-transversal away from* $Z_L(i_\circ C)$.

(4) the points of $Z_L(i_\circ C)$ *are isolated* $\pi^\circ i_\circ C$*-characteristic points for* $p_L^\vee : \mathbb{P}_L \to L$.

Corollary 4.10. *Let* X *be a smooth projective scheme of pure dimension over* k *infinite. Let* $i : X \hookrightarrow \mathbb{P}$ *be a closed immersion satisfying Assumption 4.4. Let* $\mathcal{K} \in D^b_{ctf}(X, \Lambda)$ *and put* $D = p^\vee(\mathbb{P}(SS(i_*\mathcal{K})))$*. Then, for any sufficiently generic pencil* L*, we have :*

(1) The map $\pi : \mathbb{P}_L \to \mathbb{P}$ *is properly* $SS(i_*\mathcal{F})$*-transversal.*

(2) $Z_L(SS(i_*\mathcal{K}))$ *is a subset of* $\deg D$ *points of* $\mathbb{P}_L^\circ$ *mapping bijectively to* $D \cap L$.

(3) the map $p_L^\vee : \mathbb{P}_L \to L$ *is* $SS(\pi^* i_*\mathcal{K})$*-transversal away from* $Z_L(SS(i_*\mathcal{K}))$.

(4) the points of $Z_L(SS(i_*\mathcal{K}))$ *are isolated* $SS(\pi^* i_*\mathcal{K})$*-characteristic points for* $p_L^\vee : \mathbb{P}_L \to L$.

Theorem 4.11. *Let* X *be a smooth projective scheme of pure dimension* n *over* k *infinite. Let* $i : X \hookrightarrow \mathbb{P}$ *be a closed immersion in some projective space over* k *of dimension* m *satisfying Assumption 4.4. Let* Λ *be a finite field of characteristic* $\ell \neq p$*, let* $\mathcal{K} \in D^b_c(X, \Lambda)$ *and write*

$$SS(\mathcal{K}) = \bigcup_a C_a \text{ and } CC(\mathcal{K}) = \sum_a m_a \cdot [C_a]$$

where the $C_a \subset \mathbb{T}^*X$ *are distinct closed irreducible conical subsets. Put* $D_a = p^\vee(\mathbb{P}(i_\circ C_a))$ *and let* d_a *be the generic inseparable degree of* $\mathbb{P}(i_\circ C_a) \to D_a$*. Then,*

$$\chi(X_{\overline{\xi}}, \mathcal{K}|_{X_{\overline{\xi}}}) = 2 \cdot \chi(X_{\overline{\eta}}, \mathcal{K}|_{X_{\overline{\eta}}}) + (-1)^{n-m} \sum_a d_a \cdot m_a \cdot \deg D_a .$$

Proof. By [Sai17b, Lemma 5.13], we have

$$SS(i_*\mathcal{K}) = \bigcup_a i_\circ C_a \text{ and } CC(i_*\mathcal{K}) = (-1)^{n-m} \cdot \sum_a m_a \cdot [i_\circ C_a]$$

Let L be a sufficiently generic pencil. By Proposition 4.7-(2), the $D_a \cap L$ are two by two disjoint. Hence, so are the $Z_L(i_\circ(C_a))$. In particular,

$$Z_L(SS(i_*\mathcal{F})) = \bigsqcup_a Z_L(i_\circ C_a) \subset \mathbb{P}_L^\circ$$

where $Z_L(i_\circ C_a)$ consists in exactly $\deg D_a$ points mapping bijectively to $D_a \cap L$ in virtue of Lemma 4.9-(2). Furthermore, Corollary 4.10-(3) implies that $p_L^\vee : \mathbb{P}_L \to L$ is $SS(\pi^* i_*\mathcal{K})$-transversal away from $Z_L(SS(i_*\mathcal{K}))$. Since $\pi^* i_*\mathcal{F}$ is micro-supported on $SS(\pi^* i_*\mathcal{K})$ and since $p_L^\vee : \mathbb{P}_L \to L$ is proper, [Sai17b, Lemma 4.3] implies that $Rp^\vee_{L*}\pi^* i_*\mathcal{K}$ has locally constant cohomology sheaves on $L - \bigcup_a (D_a \bigcap L)$. Thus,

$$CC(Rp^\vee_{L*}\pi^* i_*\mathcal{K}) = -\operatorname{rk}_\Lambda(Rp^\vee_{L*}\pi^* i_*\mathcal{K}) \cdot [\mathbb{T}^*_L L] + \sum_a \sum_{H \in D_a \bigcap L} n_H \cdot [\mathbb{T}^*_H L]$$

where the n_H are integers. By proper base change, we have

$$Rp^\vee_{L*}\pi^* i_*\mathcal{K} \simeq (Rp^\vee_{X*} p_X^*(\mathcal{K}))|_L .$$

For a sufficiently generic pencil L, we thus have

$$\mathrm{rk}_\Lambda(Rp^\vee_{L*}\pi^* i_*\mathcal{K}) = \mathrm{rk}_\Lambda(Rp^\vee_{X*}p^*_X(\mathcal{K})) = \chi(X_{\overline{\eta}}, \mathcal{K}|_{X_{\overline{\eta}}}) .$$

By Theorem 3.15, we deduce that for a sufficiently generic pencil L, we have

$$\chi(X_{\overline{\xi}}, \mathcal{K}|_{X_{\overline{\xi}}}) = 2\cdot\chi(X_{\overline{\eta}}, \mathcal{K}|_{X_{\overline{\eta}}}) + \sum_a \sum_{H\in D_a\bigcap L} n_H .$$

To conclude the proof of Theorem 4.11, we are thus left to show that for a sufficiently generic pencil L, we have $n_H = d_a \cdot m_a$ for every $H \in D_a \bigcap L$. Theorem 3.11 applied to $\mathrm{id}_L : L \to L$ at $H \in D_a \bigcap L$ yields

$$n_H = -\operatorname{dimtot} R\Phi_H(Rp^\vee_{L*}\pi^* i_*\mathcal{K}, \mathrm{id}_L)$$

On the other hand, the compatibility of the vanishing cycles with proper push-forward yields a canonical equivalence

$$R\Phi_H(Rp^\vee_{L*}\pi^* i_*\mathcal{K}, \mathrm{id}_L) \simeq R\Gamma(H\times\{H\}, R\Phi(\pi^* i_*\mathcal{K}, p^\vee_L)) .$$

Let $(x, H) \in Z_L(SS(i_\circ C_a))$ be the unique point of $Z_L(SS(i_\circ C_a))$ lying over H. Recall that no other point from $Z_L(SS(i_*\mathcal{K}))$ lies over H. Hence, $p^\vee_L : \mathbb{P}_L \to L$ is $SS(\pi^* i_*\mathcal{F})$-transversal at every point of $H\times\{H\}$ distinct from (x, H). Thus $p^\vee_L : \mathbb{P}_L \to L$ is locally acyclic with respect to $\pi^* i_*\mathcal{K}$ at every point of $H \times \{H\}$ distinct from (x, H). Hence, the restriction of $R\Phi(\pi^* i_*\mathcal{K}, p^\vee_L)$ to $H \times \{H\}$ is supported on (x, H). Thus,

$$n_H = -\operatorname{dimtot} R\Phi_{(x,H)}(\pi^* i_*\mathcal{K}, p^\vee_L) .$$

Since (x, H) is an isolated $SS(\pi^* i_*\mathcal{K})$-characteristic point for $p^\vee_L : \mathbb{P}_L \to L$ in virtue of Corollary 4.10-(4), Theorem 3.11 yields

$$\begin{aligned} n_H &= (CC(\pi^* i_*\mathcal{F}), dp^\vee_L)_{\mathbb{T}^*\mathbb{P}_L,(x,H)} \\ &= (CC(i_*\mathcal{F}), dp^{\vee\circ}_L)_{\mathbb{T}^*\mathbb{P},x} \\ &= (-1)^{n-m}\cdot m_a \cdot (i_\circ C_a, dp^{\vee\circ}_L)_{\mathbb{T}^*\mathbb{P},x} \\ &= (-1)^{n-m}\cdot d_a \cdot m_a \end{aligned}$$

where the second equality follows from the fact that (x, H) lies in $\mathbb{P}^\circ_L$, where the third equality follows from the fact that $p^\vee_L : \mathbb{P}_L \to L$ is $i_\circ C_b$-transversal at (x, H) for every $b \neq a$, and where the last equality follows from [Sai17b, Lemma 5.4]. □

Definition 4.12. Let X be a smooth projective scheme of pure dimension n over k and let $i : X \hookrightarrow \mathbb{P}^m_k$ be a closed immersion over k satisfying Assumption 4.4. Let CC be a n-cycle of $\mathbb{T}^*X$ and write

$$CC = \sum_a m_a \cdot [C_a]$$

where the $C_a \subset \mathbb{T}^*X$ are distinct closed irreducible conical subsets. Define $D_a = p^\vee(\mathbb{P}(i_\circ C_a))$ and let d_a be the generic degree of $\mathbb{P}(i_\circ C_a) \to D_a$. We define the *total degree of CC* by

$$\operatorname{totdeg}_i(CC) := (-1)^{n-m}\sum_a d_a \cdot m_a \cdot \deg D_a .$$

Remark 4.13. In the setting of Definition 4.12, if $CC = CC(\mathcal{P})$ where $\mathcal{P} \in \mathrm{Perv}_{tf}(X, \Lambda)$, then the m_a are positive and $SS(\mathcal{P})$ and $CC(\mathcal{P})$ have the same support by Remark 3.13. Thus, we have

$$\deg p^{\vee}(\mathbb{P}(SS(i_*\mathcal{P})) \leq \mathrm{totdeg}_i(CC(\mathcal{P}))\ .$$

Before drawing the consequences of the Betti number estimates for the characteristic cycle, let us recall the following

Lemma 4.14 ([UYZ20, Lemma 5.6]). *Let X be a smooth scheme of finite type over k. Let Λ be a finite local ring of residue characteristic $\ell \neq p$ with residue field Λ_0. For every $\mathcal{K} \in D^b_{ctf}(X, \Lambda)$, we have*

$$SS(\mathcal{K}) = SS(\mathcal{K} \otimes^L_{\Lambda} \Lambda_0) \quad \textit{and} \quad CC(\mathcal{K}) = CC(\mathcal{K} \otimes^L_{\Lambda} \Lambda_0)\ .$$

The above lemma suggests to introduce the following

Definition 4.15. Let X be a scheme of finite type over k and let Λ be a finite local ring of residue characteristic $\ell \neq p$ with residue field Λ_0. For $\mathcal{K} \in D^b_{ctf}(X, \Lambda)$, we put

$$\mathrm{Rk}_{\Lambda}\, \mathcal{K} := \mathrm{Rk}_{\Lambda_0}\, \mathcal{K} \otimes^L_{\Lambda} \Lambda_0\ .$$

Theorem 4.16. *Let $(X/S, \Sigma)$ be a relative smooth projective stratified scheme of relative dimension $\leq n$. Let $i : X \hookrightarrow \mathbb{P}_S(E)$ be a closed immersion over S satisfying Assumption 4.4. Let $a \leq b$ be integers. Then, there is an admissible function $\mu : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q}$ and $P \in \mathbb{N}[x]$ of degree n such that for every finite local ring Λ of residue characteristic $\ell \neq p$, every algebraic geometric point $\bar{s} \to S$, every $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$ and every $\mathcal{K} \in D^{[a,b]}_{\Sigma_{\bar{s}},tf}(X_{\bar{s}}, \mathcal{E}_{\bar{s}}, \Lambda)$, we have*

$$|\,\mathrm{totdeg}_{i_{\bar{s}}}(CC(\mathcal{K}))| \leq P(\mu(\mathcal{E})) \cdot \mathrm{Rk}_{\Lambda}\, \mathcal{K}\ . \tag{4.16.1}$$

In particular, if $\mathcal{K}$ is perverse, the number of irreducible components of $SS(\mathcal{K})$ and the multiplicities of $CC(\mathcal{K})$ are smaller than $P(\mu(\mathcal{E})) \cdot \mathrm{Rk}_{\Lambda}\, \mathcal{K}$.

Proof. The final claim is a consequence of (4.16.1) and Remark 3.13. By Lemma 4.14 and by Definition 4.15, both sides of (4.16.1) depend on $\mathcal{K}$ only via its reduction to the residue field of Λ. Hence, we can suppose that Λ is a finite field of characteristic $\ell \neq p$. By Theorem 4.11, we have

$$|\,\mathrm{totdeg}_{i_{\bar{s}}}(CC(\mathcal{K}))| \leq |\chi(X_{\bar{\xi}_s}, \mathcal{K}|_{X_{\bar{\xi}_s}})| + 2 \cdot |\chi(X_{\bar{\eta}_s}, \mathcal{K}|_{X_{\bar{\eta}_s}})|$$

where $\bar{\eta}_s \to \mathbb{P}_{\bar{s}}(E^{\vee}_{\bar{s}})$ is an algebraic geometric point over the generic point of $\mathbb{P}_{\bar{s}}(E^{\vee}_{\bar{s}})$ and where $\bar{\xi}_s \to \mathbb{G}_S(E^{\vee}, 2)$ is an algebraic geometric point over the generic point of $\mathbb{G}_{\bar{s}}(E^{\vee}_{\bar{s}}, 2)$. By Theorem 3.22 applied to the proper morphism

$$p^{\vee}_X : X \times_{\mathbb{P}_S(E)} Q_S \to \mathbb{P}_S(E^{\vee})$$

of relative dimension $\leq n-1$ endowed with $p^*_X\Sigma$, there is an admissible function $\mu_1 : \mathbb{Q}[\mathrm{Coh}(X \times_{\mathbb{P}_S(E)} Q_S)] \to \mathbb{Q}$ and $P_1 \in \mathbb{N}[x]$ of degree $n-1$ independent of $\bar{s}$ and $\mathcal{K}$ such that

$$|\chi(X_{\bar{\eta}_s}, \mathcal{K}|_{X_{\bar{\eta}_s}})| \leq P_1(\mu_1(p^*_X\mathcal{E})) \cdot \mathrm{Rk}_{\Lambda}\, \mathcal{K}\ .$$

By Theorem 3.22 applied to the proper morphism

$$q_X^\vee : (X \times_{\mathbb{P}_S(E)} Q_S) \times_{\mathbb{P}_S(E^\vee)} F_S \to \mathbb{G}_S(E^\vee, 2)$$

of relative dimension $\leq n$ endowed with $q_X^*\Sigma$, there is an admissible function $\mu_2 : \mathbb{Q}[\mathrm{Coh}((X \times_{\mathbb{P}_S(E)} Q_S) \times_{\mathbb{P}_S(E^\vee)} F_S)] \to \mathbb{Q}$ and $P_2 \in \mathbb{N}[x]$ of degree n independent of $\bar{s}$ and $\mathcal{K}$ such that

$$|\chi(X_{\bar{\xi}_s}, \mathcal{K}|_{X_{\bar{\xi}_s}})| \leq P_2(\mu_2(q_X^*(\mathcal{E}))) \cdot \mathrm{Rk}_\Lambda \mathcal{K}\,.$$

The conclusion thus follows by putting $\mu := \mu_1 \circ p_X^* + \mu_2 \circ q_X^*$ and $P := P_1 + P_2$. $\square$

When X is smooth and $\mathcal{K}$ is the extension by 0 of a locally constant constructible sheaf, Theorem 4.16 admits the following slightly simpler formulation :

Theorem 4.17. *Let X/k be a smooth projective scheme of pure dimension n over k. Let $D \subset X$ be an effective Cartier divisor and put $j : U := X - D \hookrightarrow X$. Let $i : X \hookrightarrow \mathbb{P}^m_k$ be a closed immersion over k satisfying Assumption 4.4. Then, there is an additive function $\mu : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q}$ and $P \in \mathbb{N}[x]$ of degree n such that for every finite local ring Λ of residue characteristic $\ell \neq p$ and every $\mathcal{L} \in \mathrm{Loc}(U, \Lambda)$, we have*

$$|\,\mathrm{totdeg}_i(CC(j_!\mathcal{L}))| \leq P(\mathrm{lc}_D(\mathcal{L})) \cdot \mathrm{rk}_\Lambda \mathcal{L}\,.$$

In particular, the number of irreducible components of $SS(j_!\mathcal{L})$ and the multiplicities of $CC(j_!\mathcal{L})$ are smaller than $P(\mathrm{lc}_D(\mathcal{L})) \cdot \mathrm{rk}_\Lambda \mathcal{L}$.

Proof. Combine Theorem 4.16 with Proposition 3.19. $\square$

Construction 4.18. The families of interest to which we are going to apply Theorem 4.16 are the families of sections by projective subspaces. Let $f : X \to S$ be a smooth projective morphism with fibers of pure dimension $n \geq 0$ between schemes of finite type over k. Let $i : X \hookrightarrow \mathbb{P}_S(E)$ be a closed immersion over S satisfying Assumption 4.4. For $2 \leq r \leq \mathrm{rk}\, E$, consider the commutative diagram

$$\begin{tikzcd}
X(E,r) \arrow[r, "p_{r,X}"] \arrow[d, "i_r"] \arrow[dd, bend right=60, "p^\vee_{r,X}"'] & X \arrow[d, "i"] \\
\mathrm{Fl}_S(E,1,r) \arrow[r] \arrow[d] & \mathbb{P}_S(E) \arrow[d] \\
\mathbb{G}_S(E,r) \arrow[r] & S
\end{tikzcd}$$

with cartesian upper square.

Lemma 4.19. *In the setting of Construction 4.18, there is a dense open subset $U_r \subset \mathbb{G}_S(E,r)$ above which $p^\vee_{r,X} : X(E,r) \to \mathbb{G}_S(E,r)$ is smooth of pure relative dimension.*

Proof. Since $\mathbb{G}_S(E,r) \to S$ is flat, the going-down property implies that the inverse image of a dense open subset of S is dense in $\mathbb{G}_S(E,r)$. Thus, we can always replace S by a dense open subset. Hence, we can suppose that S is irreducible. By going-down again, $\mathbb{G}_S(E,r)$ is irreducible as well. Thus, we are left to find a non empty open subset $U_r \subset \mathbb{G}_S(E,r)$ above which $p^\vee_{r,X}$ is smooth of pure relative dimension. By generic flatness, there is a dense open subset $V_r \subset \mathbb{G}_S(E,r)$ above which $p^\vee_{r,X}$ is flat. Choose a closed point $s \in S$ in the image of V_r. In particular, $V_r \cap \mathbb{G}_s(E_s,r)$ is a dense open subset of $\mathbb{G}_s(E_s,r)$. Let $\bar{s} \to S$ be a separably closed point lying over s. By Bertini's theorem [Jou83, Corollaire 6.11.2] applied to $i_{\bar{s}} : X_{\bar{s}} \hookrightarrow \mathbb{P}_{\bar{s}}(E_{\bar{s}})$, there is a rational point $\overline{F} \in (V_r \cap \mathbb{G}_s(E_s,r))(\bar{s})$ such that the fiber of

$$p^\vee_{r,X_{\bar{s}}} : X_{\bar{s}}(E_{\bar{s}},r) \to \mathbb{G}_{\bar{s}}(E_{\bar{s}},r)$$

over $\overline{F}$ is smooth of pure dimension. Let $F \in V_r \cap \mathbb{G}_s(E_s,r)$ be the image of $\overline{F}$. Since smoothness is a local property for the étale topology, the fiber of

$$p^\vee_{r,X_s} : X_s(E_s,r) \to \mathbb{G}_s(E_s,r)$$

over F is smooth of pure dimension. By openness of the smooth locus [GD64, Théorème 12.2.4] for proper flat morphisms of finite presentation, we deduce the existence of a non empty open subset $U_r \subset V_r$ above which $p^\vee_{r,X}$ is smooth of pure relative dimension. $\square$

Lemma 4.20. *Let X be a smooth projective variety over k and let $i : X \hookrightarrow \mathbb{P}_k(E)$ be a closed immersion satisfying Assumption 4.4. Let $F \subset \mathbb{P}_k(E)$ be a projective subspace such that $X \cap F$ is smooth. Then, $X \cap F \hookrightarrow F$ satisfies Assumption 4.4 again.*

Proof. Let $\mathcal{L}$ be the very ample line bundle on X corresponding to $i : X \hookrightarrow \mathbb{P}_k(E)$. Let $u, v \in X \cap F$ be closed points. By assumption, the top horizontal arrow of the following commutative diagram

$$\begin{array}{ccccc}
\Gamma(\mathbb{P}_k(E), \mathcal{O}_{\mathbb{P}_k(E)}(1)) & \longrightarrow & \Gamma(X,\mathcal{L}) & \longrightarrow & \mathcal{L}_u/\mathfrak{m}_u^2\mathcal{L}_u \oplus \mathcal{L}_v/\mathfrak{m}_v^2\mathcal{L}_v \\
\downarrow & & \downarrow & & \downarrow \\
\Gamma(F, \mathcal{O}_F(1)) & \longrightarrow & \Gamma(X\cap F, \mathcal{L}|_F) & \longrightarrow & (\mathcal{L}|_F)_u/\mathfrak{m}_u^2(\mathcal{L}|_F)_u \oplus (\mathcal{L}|_F)_v/\mathfrak{m}_v^2(\mathcal{L}|_F)_v
\end{array}$$

is surjective. Since the right vertical arrow is surjective as well, so is the bottom horizontal arrow. $\square$

Lemma 4.21. *In the setting of Construction 4.18, let $U_r \subset \mathbb{G}_S(E,r)$ be an open subset above which $p^\vee_{r,X} : X(E,r) \to \mathbb{G}_S(E,r)$ is smooth. Let $E' \in \mathrm{Coh}(\mathbb{G}_S(E,r))$ be the universal locally free quotient of rank r of E. Then, via the canonical identification*

$$\mathrm{Fl}_S(E,1,r) \simeq \mathbb{P}_{\mathbb{G}_S(E,r)}(E')$$

supplied by Lemma 6.26, the closed immersion

$$i_a : X(E,r) \hookrightarrow \mathrm{Fl}_S(E,1,r)$$

over $\mathbb{G}_S(E,r)$ satisfies Assumption 4.4 above U_r.

Proof. Let $\bar{x} \to U_r$ be an algebraic geometric point lying above $\bar{s} \to S$ and let $F \subset \mathbb{P}_{\bar{s}}(E_{\bar{s}})$ be the corresponding projective subspace of dimension $r-1$. Then, the pullback of i_r over $\bar{x}$ reads as $X_{\bar{s}} \cap F \hookrightarrow F$. Thus, Lemma 4.21 follows from Lemma 4.20. $\square$

Lemma 4.22. *Let $(X/S, \Sigma)$ be a relative smooth projective stratified scheme of relative dimension $\leq n$. Let $i : X \hookrightarrow \mathbb{P}_S(E)$ be a closed immersion over S satisfying Assumption 4.4. Let $\operatorname{rk} E - n \leq r \leq \operatorname{rk} E$ and $a \leq b$ be integers. Then, there is an admissible function $\mu_r : \mathbb{Q}[\operatorname{Coh}(X)] \to \mathbb{Q}$ and $P_r \in \mathbb{N}[x]$ of degree $n + r - \operatorname{rk} E$, there is a dense open subset $V_r \subset \mathbb{G}_S(E, r)$ such that for every finite local ring Λ of residue characteristic $\ell \neq p$, every algebraic geometric point $\bar{s} \to S$, every $F \in V_r(\bar{s})$, every $\mathcal{E} \in \mathbb{Q}[\operatorname{Coh}(X)]$ and every $\mathcal{K} \in D^{[a,b]}_{\Sigma|_{X_{\bar{s}} \cap F}, tf}(X_{\bar{s}} \cap F, \mathcal{E}|_{X_{\bar{s}} \cap F}, \Lambda)$, we have :*

(1) *The scheme $X_{\bar{s}} \cap F$ is smooth over $\bar{s}$.*

(2) $|\operatorname{totdeg}_{i_{\bar{s}}}(CC(\mathcal{K}))| \leq P_r(\mu_r(\mathcal{E})) \cdot \operatorname{Rk}_\Lambda \mathcal{K}$.

Proof. We use the notations from Construction 4.18. By Lemma 4.19, there is a dense open subset $V_r \subset \mathbb{G}_S(E, r)$ above which $p^\vee_{r,X} : X(E, r) \to \mathbb{G}_S(E, r)$ is smooth projective of relative dimension $\leq n + r - \operatorname{rk} E$. By Lemma 4.21, the closed immersion

$$i_r : X(E, r) \hookrightarrow \operatorname{Fl}_S(E, 1, r)$$

satisfies Assumption 4.4 above V_r. By Theorem 4.16, applied to $p^\vee_{r,X} : X(E, r) \to \mathbb{G}_S(E, r)$ endowed with $p^*_{r,X}\Sigma$, there is an admissible function $\mu : \mathbb{Q}[\operatorname{Coh}(X(E, r))] \to \mathbb{Q}$ and $P \in \mathbb{N}[x]$ of degree $n + r - \operatorname{rk} E$ such that for every finite local ring Λ of residue characteristic $\ell \neq p$, every algebraic geometric point $\bar{s} \to S$, every $F \in V_r(\bar{s})$, every $\mathcal{E} \in \mathbb{Q}[\operatorname{Coh}(X)]$ and every $\mathcal{K} \in D^{[a,b]}_{\Sigma|_{X_{\bar{s}} \cap F}, tf}(X_{\bar{s}} \cap F, \mathcal{E}|_{X_{\bar{s}} \cap F}, \Lambda)$, we have

$$|\operatorname{totdeg}_{i_{\bar{s}}}(CC(\mathcal{K}))| \leq P(\mu(p^*_{r,X}\mathcal{E})) \cdot \operatorname{Rk}_\Lambda \mathcal{K} \,.$$

We then conclude by putting $\mu_r := \mu \circ p^*_{r,X}$ and $P_r = P$. □

From this point on, we will use the results from the Appendix. For the reader's convenience, we recall here the minimal terminology so that the Appendix can be used as a black box in the proofs below.

Definition 4.23. A *tree* is a finite poset Tr with an initial object 0 such that for every $v \in \operatorname{Tr}$, the subset $\operatorname{Tr}^{\leq v} := \{w \in \operatorname{Tr} \text{ with } w \leq v\}$ is totally ordered. For a vertex $v \in \operatorname{Tr}$, the natural number $d(v) := |\operatorname{Tr}^{\leq v}| - 1$ is the *depth* of v. The *depth* of Tr is the maximal depth of its vertices. A *branch* of Tr is a maximal totally ordered subset of Tr.

Construction 4.24. Let S be a scheme of finite type over k and let E be a non zero locally free sheaf of $\mathcal{O}_S$-modules on S. Let Tr be a tree. Recall from Construction 6.20 that the functor

$$\operatorname{Fl}_S(E, \operatorname{Tr}) : \operatorname{Sch}_S^{op} \to \operatorname{Set}$$

sending $f : T \to S$ to the set of isomorphism classes of diagrams

$$E_\bullet : \operatorname{Tr} \to \operatorname{Coh}(T)$$

where

(1) $E_0 = f^*E$ and for every $v \in \operatorname{Tr}$, the sheaf E_v is locally free of rank $\operatorname{rk} E - d(v)$

(2) The arrows of Tr are sent to epimorphisms of $\operatorname{Coh}(T)$

is a scheme of finite type over S.

Remark 4.25. The diagram $E_\bullet : \mathrm{Tr} \to \mathrm{Coh}(T)$ is equivalent to a collection of projective subspaces over T

$$F_v \subset \mathbb{P}_T(f^*E),\ v \in \mathrm{Tr}$$

where F_v has relative dimension $\operatorname{rk} E - d(v) - 1$ over T. We will tacitly use both descriptions.

Definition 4.26. Let X be a projective scheme over k. Let $i : X \hookrightarrow \mathbb{P}_k(E)$ be a closed immersion, let Tr be a tree and let $\mathcal{C} \subset D^b_{c,ft}(X, \Lambda)$ be a full subcategory. We say that $F_\bullet \in \mathrm{Fl}_k(E, \mathrm{Tr})(k)$ is *$\mathcal{C}$-transversal* if for every $\mathcal{K} \in \mathcal{C}$, the tree Tr admits a branch B such that for every consecutive vertices $v \leq w$ in B, the map $F_w \hookrightarrow F_v$ is $SS((i_*\mathcal{K})|_{F_v})$-transversal.

Remark 4.27. We will abuse the terminology and refer to a branch $B \subset \mathrm{Tr}$ as in Definition 4.26 as a *$\mathcal{K}$-transversal branch of* $F_\bullet$.

Remark 4.28. If $\mathcal{P} \in \mathrm{Perv}_{ft}(X, \Lambda)$ and if $F_\bullet \in \mathrm{Fl}_k(E, \mathrm{Tr})(k)$ admits a $\mathcal{P}$-transversal branch $B \subset \mathrm{Tr}$, then $\mathcal{P}|_{X\cap F_v}[-d(v)]$ is perverse for every $v \in B$ by [Sai17b, Lemma 8.6.5].

For the notion of ramified enough tree, let us refer to Definition 6.31.

Theorem 4.29. *Let $(X/S, \Sigma)$ be a relative smooth projective stratified scheme of relative dimension $\leq n$. Let $i : X \hookrightarrow \mathbb{P}_S(E)$ be a closed immersion over S satisfying Assumption 4.4. Then, there is an admissible function $\mu : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q}$ and $P \in \mathbb{N}[x]$ of degree n depending only on $(X/S, \Sigma, i)$ such that for every $1 \leq a \leq \operatorname{rk} E$, every $U_a \subset \mathbb{G}_S(E, a)$ dense open, every $r \geq 1$, every $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$ and every tree Tr ramified enough with respect to $(\operatorname{rk} E, P(\mu(\mathcal{E})) \cdot r)$, there is a dense open subset $U \subset \mathrm{Fl}_S(E, \mathrm{Tr})$ satisfying the following property : for every finite local ring Λ of residue characteristic $\ell \neq p$, every geometric algebraic point $\bar{s} \to S$ and every $F_\bullet \in U(\bar{s})$, we have :*

(1) $F_v \in U_{\operatorname{rk} E - d(v)}$ for every $v \in \mathrm{Tr}$.

(2) $F_\bullet$ is $\mathrm{Perv}^{\leq r}_{\Sigma_{\bar{s}}, ft}(X_{\bar{s}}, \mathcal{E}_{\bar{s}}, \Lambda)$-transversal.

Proof. By Lemma 4.22, there is an admissible function $\mu : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q}$, a polynomial $P \in \mathbb{N}[x]$ of degree n such that for every $a = 1, \dots, \operatorname{rk} E$, there is a dense open subset $V_a \subset \mathbb{G}_S(E, a)$ such that for every finite local ring Λ of residue characteristic $\ell \neq p$, every $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$, algebraic geometric point $\bar{s} \to S$, every $F \in V_a(\bar{s})$, the scheme $X_{\bar{s}} \cap F$ is smooth over $\bar{s}$ and for every $\mathcal{P} \in \mathrm{Perv}^{\leq r}_{\Sigma|_{X_{\bar{s}}\cap F}, ft}(X_{\bar{s}} \cap F, \mathcal{E}|_{X_{\bar{s}}\cap F}, \Lambda)$, we have

$$|\operatorname{totdeg}_{i_{\bar{s}}}(CC(\mathcal{P}))| \leq P(\mu(\mathcal{E})) \cdot r\,.$$

Let Tr be a tree ramifying enough with respect to $(\operatorname{rk} E, P(\mu(\mathcal{E})) \cdot r)$. By Lemma 6.35 applied to the $U_a \cap V_a$, there is a dense open subset $A \subset \mathrm{Fl}_S(E, \mathrm{Tr})$ such that for every algebraic geometric point $\bar{s} \to S$, every $F \in A(\bar{s})$ and every $v \in \mathrm{Tr}$, item (1) holds, the scheme $X_{\bar{s}} \cap F_v$ is smooth over $\bar{s}$ and for every $\mathcal{P} \in \mathrm{Perv}^{\leq r}_{\Sigma|_{X_{\bar{s}}\cap F_v}, ft}(X_{\bar{s}} \cap F_v, \mathcal{E}|_{X_{\bar{s}}\cap F_v}, \Lambda)$, we have

$$|\operatorname{totdeg}_{i_{\bar{s}}}(CC(\mathcal{P}))| \leq P(\mu(\mathcal{E})) \cdot r\,.$$

By Lemma 6.34, there is a dense open subset $B \subset \mathrm{Fl}_S(E, \mathrm{Tr})$ such that for every geometric algebraic point $\bar{s} \to S$, every $F_\bullet \in B(\bar{s})$ and every non maximal vertex $v \in \mathrm{Tr}$, no hypersurface of $F_v^\vee$ of degree at most $P(\mu(\mathcal{E})) \cdot r$ contains all the F_w, for w immediate successor of v. Put $U := A \cap B$ and let us show that U satisfies (2). Let Λ be a finite local ring of residue characteristic $\ell \neq p$, let $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$, let $\bar{s} \to S$ be a geometric algebraic point, let $F_\bullet \in U(\bar{s})$ and let $\mathcal{P} \in \mathrm{Perv}^{\leq r}_{\Sigma_{\bar{s}}, ft}(X_{\bar{s}}, \mathcal{E}_{\bar{s}}, \Lambda)$. We construct the branch $B \subset \mathrm{Tr}$ step by step by starting from the initial object 0. Suppose that we have found $v \in \mathrm{Tr}$ non maximal such that for every consecutive vertices $v' \leq w'$ smaller than v, the map $F_{w'} \hookrightarrow F_{v'}$ is $SS((i_{\bar{s}*}\mathcal{P})|_{F_{v'}})$-transversal. By Remark 4.28, the complex $(i_{\bar{s}*}\mathcal{P})|_{F_v}[-d(v)]$ is perverse. Hence, $\mathcal{P}|_{X_{\bar{s}} \cap F_v}[-d(v)]$ is an object of $\mathrm{Perv}^{\leq r}_{\Sigma|_{X_{\bar{s}} \cap F_v}, ft}(X_{\bar{s}} \cap F_v, \mathcal{E}|_{X_{\bar{s}} \cap F_v}, \Lambda)$. Since $F_\bullet$ lies in A, Remark 4.13 gives

$$\deg p^\vee(\mathbb{P}(SS((i_{\bar{s}*}\mathcal{P})|_{F_v}))) \leq |\mathrm{totdeg}(CC(\mathcal{P}|_{X_{\bar{s}} \cap F_v}[-d(v)]))| \leq P(\mu(\mathcal{E})) \cdot r \, .$$

Since $F_\bullet$ lies in B, the non maximal vertex v admits an immediate successor w such that

$$F_w \notin p^\vee(\mathbb{P}(SS((i_{\bar{s}*}\mathcal{P})|_{F_v}))) \subset F_v^\vee \, .$$

By Lemma 4.6, the inclusion $F_w \hookrightarrow F_v$ is thus $SS((i_{\bar{s}*}\mathcal{P})|_{F_v})$-transversal. □

Remark 4.30. The condition (1) in the statement of Theorem 4.29 provides some flexibility in the choice of the subspaces F_v.

The above remark suggests to introduce the following

Definition 4.31. Let U be a scheme of finite type over k of pure dimension $n \geq 2$ and let $i : U \hookrightarrow \mathbb{P}_k(E)$ be an immersion. Let Tr be a tree of depth $\leq n-1$. We say that $F_\bullet \in \mathrm{Fl}_k(E, \mathrm{Tr})(k)$ is in *good position with respect to* U if for every $v \in \mathrm{Tr}$, the space F_v is transverse to the smooth locus of U^{red} and the schemes U and $U \cap F_v$ have the same number of irreducible components.

Definition 4.32. Let (X, Σ) be a stratified scheme of finite type over k of pure dimension $n \geq 2$ and let $i : X \hookrightarrow \mathbb{P}_k(E)$ be an immersion. Let Tr be a tree of depth $\leq n-1$. We say that $F_\bullet \in \mathrm{Fl}_k(E, \mathrm{Tr})(k)$ is in *good position with respect to* (X, Σ) if it is in good position with respect to X and to every stratum of Σ.

Remark 4.33. Assume that k is algebraically closed. By Bertini's theorem [Jou83, Théorème 6.3] and Lemma 6.35, there is a dense open subset $V \subset \mathrm{Fl}_k(E, \mathrm{Tr})$ such that every $F_\bullet \in V(k)$ is in good position with respect to (X, Σ).

When X is not assume to be smooth any more, Theorem 4.29 admits the following immediate consequence enough for applications :

Theorem 4.34. *Let (X, Σ) be a projective stratified scheme over k algebraically closed. Then, there is a closed immersion $i : X \hookrightarrow \mathbb{P}_k(E)$, there is an admissible function $\mu : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q}$ and $P \in \mathbb{N}[x]$ depending only on (X, Σ, i) such that for every $r \geq 1$, every $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$ and every tree Tr ramified enough with respect to $(\mathrm{rk}\, E, P(\mu(\mathcal{E})) \cdot r)$, there is a dense open subset $U \subset \mathrm{Fl}_S(E, \mathrm{Tr})$ satisfying the following property : for every finite local ring Λ of residue characteristic $\ell \neq p$, every $F_\bullet \in U(k)$, we have*

(1) $F_\bullet$ is in good position with respect to (X, Σ).

(2) $F_\bullet$ *is* $\mathrm{Perv}^{\leq r}_{\Sigma,\mathrm{ft}}(X,\mathcal{E},\Lambda)$*-transversal.*

5. Lefschetz recognition principle

Proposition 5.1. *Let* X *be a projective scheme of finite type of pure dimension* $n \geq 2$ *over* k *algebraically closed. Let* $j : U \hookrightarrow X$ *be an affine open immersion with* U *smooth over* k*. Let* $i : X \hookrightarrow \mathbb{P}_k$ *be a closed immersion. Let* $H \subset \mathbb{P}_k$ *be a hyperplane such that* $U \cap H$ *is smooth of pure dimension* $n-1$*. Let* $\mathcal{L}, \mathcal{M} \in \mathrm{Loc}_{\mathrm{tf}}(U,\Lambda)$ *such that* $H \hookrightarrow \mathbb{P}_k$ *is* $SS(i_* j_! \mathcal{H}om(\mathcal{L},\mathcal{M}))$*-transversal. Then, the canonical morphism*

$$\mathrm{Hom}(\mathcal{L},\mathcal{M}) \to \mathrm{Hom}(\mathcal{L}|_{U\cap H}, \mathcal{M}|_{U\cap H})$$

is an isomorphism.

Proof. Consider the diagram with cartesian squares

$$\begin{tikzcd} U\cap H \arrow[r] \arrow[d, "j'"] & U \arrow[d, "j"] \\ X\cap H \arrow[r] \arrow[d, "i'"] & X \arrow[d, "i"] \\ H \arrow[r] & \mathbb{P}_k . \end{tikzcd}$$

We have

$$\mathrm{Hom}(\mathcal{L},\mathcal{M}) = H^0(U, \mathcal{H}om(\mathcal{L},\mathcal{M})) = H^0(\mathbb{P}_k, i_* Rj_* \mathcal{H}om(\mathcal{L},\mathcal{M}))$$

and

$$\mathrm{Hom}(\mathcal{L}|_{U\cap H}, \mathcal{M}|_{U\cap H}) = H^0(H, i'_* Rj'_* \mathcal{H}om(\mathcal{L}|_{U\cap H}, \mathcal{M}|_{U\cap H}))$$

Note that $\mathcal{H}om(\mathcal{L},\mathcal{M})[n]$ is perverse of finite tor-dimension on U. Lemma 3.14 gives

$$SS(i_* Rj_* \mathcal{H}om(\mathcal{L},\mathcal{M})) = SS(i_* j_! \mathcal{H}om(\mathcal{L},\mathcal{M})) .$$

Hence, $H \hookrightarrow \mathbb{P}_k$ is $SS(i_* Rj_* \mathcal{H}om(\mathcal{L},\mathcal{M}))$-transversal. On the other hand, the map $U \cap H \hookrightarrow U$ is tautologically $SS(\mathcal{H}om(\mathcal{L},\mathcal{M}))$-transversal. By assumption, we have

$$\dim U \cap H - \dim U = -1 = \dim H - \dim \mathbb{P}_k$$

where all the above schemes are smooth of pure dimension. Hence, Proposition 3.9 implies that the base change

$$(i_* Rj_* \mathcal{H}om(\mathcal{L},\mathcal{M}))|_H \to i'_* Rj'_* \mathcal{H}om(\mathcal{L}|_{U\cap H}, \mathcal{M}|_{U\cap H})$$

is an isomorphism. Put $\mathbb{A}_k := \mathbb{P}_k - H$ and let $\jmath : \mathbb{A}_k \hookrightarrow \mathbb{P}_k$ is the inclusion. Then, there is a distinguished triangle

$$\jmath_! \jmath^* i_* Rj_* \mathcal{H}om(\mathcal{L},\mathcal{M}) \to i_* Rj_* \mathcal{H}om(\mathcal{L},\mathcal{M}) \to i'_* Rj'_* \mathcal{H}om(\mathcal{L}|_{U\cap H}, \mathcal{M}|_{U\cap H}) .$$

Hence, to conclude the proof of Proposition 5.1, we are left to show that

$$H^1_c(\mathbb{A}_k, \jmath^* i_* Rj_* \mathcal{H}om(\mathcal{L},\mathcal{M}))$$

vanishes. Since $\jmath^* i_* Rj_* \mathcal{H}om(\mathcal{L},\mathcal{M})[n]$ is perverse on $\mathbb{A}_k$ and since $\mathbb{A}_k$ has dimension at least 2, the sought-after vanishing thus follows from the weak Lefschetz theorem. □

Remark 5.2. In Proposition 5.1, observe that if H is in good position with respect to U in the sense of Definition 4.31, then $U \cap H$ is smooth of pure dimension $n-1$.

Lemma 5.3. *Let X be a projective scheme of finite type of pure dimension $n \geq 2$ over k algebraically closed. Let $j : U \hookrightarrow X$ be an affine open immersion with U smooth over k. Let $i : X \hookrightarrow \mathbb{P}_k(E)$ be a closed immersion and let $\mathcal{L}, \mathcal{M} \in \mathrm{Loc}_{\mathrm{tf}}(U, \Lambda)$. Let Tr be a tree of depth at most $n-1$ and let $F_\bullet \in \mathrm{Fl}_k(E, \mathrm{Tr})(k)$ in good position with respect to U and admitting a $j_! \mathcal{H}om(\mathcal{L}, \mathcal{M})$-transversal branch B. Then, the canonical morphism*

$$\mathrm{Hom}(\mathcal{L}, \mathcal{M}) \to \mathrm{Hom}(\mathcal{L}|_{U\cap F_v}, \mathcal{M}|_{U\cap F_v})$$

is an isomorphism for every $v \in B$.

Proof. Let $v, w \in B$ such that w is an immediate successor of v. It is enough to show that the canonical morphism

$$\mathrm{Hom}(\mathcal{L}|_{U\cap F_v}, \mathcal{M}|_{U\cap F_v}) \to \mathrm{Hom}(\mathcal{L}|_{U\cap F_w}, \mathcal{M}|_{U\cap F_w})$$

is an isomorphism. Consider the following commutative diagram with cartesian squares

$$\begin{tikzcd} U \cap F_v \arrow[r, "j_v"] \arrow[d] & X \cap F_v \arrow[r, "i_v"] \arrow[d] & F_v \arrow[d] \\ U \arrow[r, "j"] & X \arrow[r, "i"] & \mathbb{P}_k(E) . \end{tikzcd}$$

By assumption, the inclusion $F_w \hookrightarrow F_v$ is $SS((i_* j_!(\mathcal{H}om(\mathcal{L}, \mathcal{M})))|_{F_v})$-transversal. On the other hand, we have

$$(i_* j_!(\mathcal{H}om(\mathcal{L}, \mathcal{M})))|_{F_v} \simeq i_{v*} j_{v!}(\mathcal{H}om(\mathcal{L}|_{U\cap F_v}, \mathcal{M}|_{U\cap F_v})) .$$

The conclusion then follows from Proposition 5.1. □

Lemma 5.4. *Let X be a scheme of finite type over k and let $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$. Let $j : U \hookrightarrow X$ be an open immersion and let $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Loc}_{\mathrm{tf}}(U, \mathcal{E}, \Lambda)$. Then, $\mathcal{H}om(\mathcal{L}_1, \mathcal{L}_2) \in \mathrm{Loc}_{\mathrm{tf}}(U, \mathcal{E}, \Lambda)$.*

Proof. Since $\mathcal{L}_1, \mathcal{L}_2$ are locally constant constructible sheaves, the formation of the sheaf $j_! \mathcal{H}om(\mathcal{L}_1, \mathcal{L}_2)$ commutes with pullback. Hence, we reduce to an analogous statement where X is the spectrum of a strict henselian dvr over k, where the statement is obvious. □

Definition 5.5. Let X be a scheme of finite type over k. Let Λ be a finite extension of $\mathbb{Q}_\ell$ or a finite local ring of residue characteristic $\ell \neq p$. Let $i : X \hookrightarrow \mathbb{P}_k(E)$ be an immersion, let Tr be a tree and let $\mathcal{C} \subset D^b_{c,ft}(X, \Lambda)$ be a full subcategory. We say that $F_\bullet \in \mathrm{Fl}_k(E, \mathrm{Tr})(k)$ *realizes the Lefschetz recognition principle for* $\mathcal{C}$ if for every $\mathcal{K}_1, \mathcal{K}_2 \in \mathcal{C}$, there is a branch $B \subset \mathrm{Tr}$ such that for $a, b \in \{1, 2\}$, the canonical morphism

$$\mathrm{Hom}(\mathcal{K}_a, \mathcal{K}_b) \to \mathrm{Hom}(\mathcal{K}_a|_{U\cap F_v}, \mathcal{K}_b|_{U\cap F_v})$$

is an isomorphism for every $v \in B$.

Definition 5.6. A branch B as in Definition 5.5 will be said to *distinguish* $\mathcal{K}_1$ from $\mathcal{K}_2$. In this case for every $v \in B$, we have $\mathcal{K}_1 \simeq \mathcal{K}_2$ if and only if $\mathcal{K}_1|_{U\cap F_v} \simeq \mathcal{K}_2|_{U\cap F_v}$.

Theorem 5.7. *Let X be a projective scheme of pure dimension $n \geq 2$ over k algebraically closed. Let $j : U \hookrightarrow X$ be an affine open immersion with U smooth over k. Then, there is a closed immersion $i : X \hookrightarrow \mathbb{P}_k(E)$, there is an admissible function $\mu : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q}$ and $P \in \mathbb{N}[x]$ depending only on (X, U, i) such that for every $r \geq 1$, every $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$ and every tree Tr of depth $\leq n-1$ ramified enough with respect to $(\mathrm{rk}\, E, P(\mu(\mathcal{E})) \cdot 4r^2)$, there is a dense open subset $V \subset \mathrm{Fl}_S(E, \mathrm{Tr})$ satisfying the following property : for every finite local ring Λ of residue characteristic $\ell \neq p$, every $F_\bullet \in V(k)$ is in good position with respect to U and realizes the Lefschetz recognition principle for $\mathrm{Loc}_{\mathrm{ft}}^{\leq r}(U, \mathcal{E}, \Lambda)$.*

Proof. Put $\Sigma := \{U, D\}$, let $i : X \hookrightarrow \mathbb{P}_k(E)$ and $\mu : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q}$ and $P \in \mathbb{N}[x]$ as given by Theorem 4.34 applied to (X, Σ). Let $r \geq 1$ and $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$. Let Tr be a tree of depth smaller than $n-1$ ramifying enough with respect to $(\mathrm{rk}\, E, P(\mu(\mathcal{E})) \cdot 4r^2)$. Then, there is a dense open subset $V \subset \mathrm{Fl}_k(E, \mathrm{Tr})$ such that for every $F_\bullet \in V(k)$ and every finite local ring Λ of residue characteristic $\ell \neq p$, we have

(1) $F_\bullet$ is in good position with respect to U.

(2) $F_\bullet$ is $\mathrm{Perv}_{\Sigma,\mathrm{ft}}^{\leq 4r^2}(X, \mathcal{E}, \Lambda)$-transversal.

Let $F_\bullet \in V(k)$ and $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Loc}_{\mathrm{ft}}^{\leq r}(U, \mathcal{E}, \Lambda)$. By Lemma 5.3, it is enough to find a branch B which is $j_! \mathcal{H}om(\mathcal{L}_a, \mathcal{L}_b)$-transversal for every $a, b \in \{1, 2\}$. Since

$$\mathcal{P}(\mathcal{L}_1, \mathcal{L}_2) := \bigoplus_{a,b \in \{1,2\}} j_! \mathcal{H}om(\mathcal{L}_a, \mathcal{L}_b)[n]$$

is an object of $\mathrm{Perv}_{\Sigma,\mathrm{ft}}^{\leq 4r^2}(X, \mathcal{E}, \Lambda)$ by Lemma 5.4 and the fact that $j : U \hookrightarrow X$ is affine, the existence of B follows from the property (2). □

The proof of Theorem 5.7 suggests to introduction the following

Definition 5.8. Let X be a scheme of finite type over k. Let $j : U \hookrightarrow X$ be an affine open immersion with U smooth over k of pure dimension n. For $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Loc}_{\mathrm{ft}}(U, \Lambda)$, we put

$$\mathcal{P}(\mathcal{L}_1, \mathcal{L}_2) := \bigoplus_{a,b \in \{1,2\}} j_! \mathcal{H}om(\mathcal{L}_a, \mathcal{L}_b)[n] \,.$$

Recollection 5.9. Let X be a scheme of finite type over a field k of characteristic $p > 0$ and let $\ell \neq p$. For every $\mathcal{L} \in \mathrm{Loc}(X, \overline{\mathbb{Q}}_\ell)$, there is a finite extension $L/\mathbb{Q}_\ell$ and an integral representative $\mathcal{L}_\bullet = (\mathcal{L}_m)_{m \geq 0}$ for $\mathcal{L}$. If we put $\Lambda_m := \mathcal{O}_L/\mathfrak{m}_L^m$, the sheaf $\mathcal{L}_m$ is an object of $\mathrm{Loc}_{\mathrm{ft}}(X, \Lambda_m)$ and $\Lambda_m \otimes_{\Lambda_{m+1}} \mathcal{L}_{m+1} \simeq \mathcal{L}_m$.

The next lemma upgrades Lemma 5.3 to $\overline{\mathbb{Q}}_\ell$-coefficients.

Lemma 5.10. *Let X be a projective scheme of finite type of pure dimension $n \geq 2$ over k algebraically closed. Let $j : U \hookrightarrow X$ be an affine open immersion with U smooth over k. Let $\mathcal{L}, \mathcal{M} \in \mathrm{Loc}(U, \overline{\mathbb{Q}}_\ell)$ represented by $\mathcal{L}_\bullet, \mathcal{M}_\bullet \in D_c^b(X, L)$ respectively, where $L/\mathbb{Q}_\ell$ is a finite extension. Let $i : X \hookrightarrow \mathbb{P}_k(E)$ be a closed immersion. Let Tr be a tree of depth at most $n-1$ and let $F_\bullet \in \mathrm{Fl}_k(E, \mathrm{Tr})$ transverse to the smooth locus of X and admitting a $j_! \mathcal{H}om(\mathcal{L}_0, \mathcal{M}_0)$-transversal branch B. Then, the canonical morphism*

$$\mathrm{Hom}(\mathcal{L}, \mathcal{M}) \to \mathrm{Hom}(\mathcal{L}|_{U \cap F_v}, \mathcal{M}|_{U \cap F_v})$$

is an isomorphism for every $v \in B$.

Proof. Let $v \in B$. We use the notations from Recollection 5.9. By definition of the morphisms in $D^b_c(X, \overline{\mathbb{Q}}_\ell)$, we have to show that for every finite extension L'/L, the canonical morphism

$$\mathrm{Hom}(\mathcal{L}_\bullet \otimes^L_{\mathcal{O}_L} \mathcal{O}_{L'}, \mathcal{M}_\bullet \otimes^L_{\mathcal{O}_L} \mathcal{O}_{L'}) \to \mathrm{Hom}(\mathcal{L}_\bullet|_{U\cap F_v} \otimes^L_{\mathcal{O}_L} \mathcal{O}_{L'}, \mathcal{M}_\bullet|_{U\cap F_v} \otimes^L_{\mathcal{O}_L} \mathcal{O}_{L'})$$

is an isomorphism. Here the morphisms are taken in $D^b_c(U, \mathcal{O}_{L'})$ and $D^b_c(U \cap F_v, \mathcal{O}_{L'})$ respectively. Let e be the ramification index of L'/L. For $m \geq 0$, choose ι such that $\iota \cdot e < m+1 \leq (\iota+1) \cdot e$. By definition of the morphisms in $D^b_c(U, \mathcal{O}_{L'})$ as limits (see [Fu11, p591]), we have to show with the notations from Recollection 5.9 that the canonical morphism

$$\mathrm{Hom}(\mathcal{L}_\iota \otimes^L_{\Lambda_\iota} \Lambda'_m, \mathcal{M}_\iota \otimes^L_{\Lambda_\iota} \Lambda'_m) \to \mathrm{Hom}(\mathcal{L}_\iota|_{U\cap F_v} \otimes^L_{\Lambda_\iota} \Lambda'_m, \mathcal{M}_\iota|_{U\cap F_v} \otimes^L_{\Lambda_\iota} \Lambda'_m)$$

is an isomorphism for every $m \geq 0$. By Lemma 5.3, it is enough to show that the branch B is $j_! \mathcal{H}om(\mathcal{L}_\iota \otimes^L_{\Lambda_\iota} \Lambda'_m, \mathcal{M}_\iota \otimes^L_{\Lambda_\iota} \Lambda'_m)$-transversal for every $m \geq 0$. By Lemma 4.14, we have

$$SS(i_* j_! \mathcal{H}om(\mathcal{L}_\iota \otimes^L_{\Lambda_\iota} \Lambda'_m, \mathcal{M}_\iota \otimes^L_{\Lambda_\iota} \Lambda'_m)) = SS(i_* j_! \mathcal{H}om(\mathcal{L}_0, \mathcal{M}_0)) .$$

The conclusion thus follows by definition of the branch B. □

Theorem 5.11. *Let X be a projective scheme of pure dimension $n \geq 2$ over k algebraically closed. Let $j : U \hookrightarrow X$ be an affine open immersion with U smooth over k. Then, there is a closed immersion $i : X \hookrightarrow \mathbb{P}_k(E)$, there is an admissible function $\mu : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q}$ and $P \in \mathbb{N}[x]$ depending only on (X, U, i) such that for every $r \geq 1$, every $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$ and every tree Tr of depth $\leq n-1$ ramified enough with respect to $(\mathrm{rk}\, E, P(\mu(\mathcal{E})) \cdot 4r^2)$, there is a dense open subset $V \subset \mathrm{Fl}_S(E, \mathrm{Tr})$ satisfying the following property : every $F_\bullet \in V(k)$ is in good position with respect to X and realizes the Lefschetz recognition principle for $\mathrm{Loc}^{\leq r}(U, \mathcal{E}, \overline{\mathbb{Q}}_\ell)$.*

Proof. Put $\Sigma := \{U, D\}$, let $\mu : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q}$ and $P \in \mathbb{N}[x]$ as given by Theorem 4.34 applied to (X, Σ). Let $r \geq 1$ and $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$. Let Tr be a tree of depth smaller than $n-1$ ramifying enough with respect to $(\mathrm{rk}\, E, P(\mu(\mathcal{E})) \cdot 4r^2)$. Then, there is a dense open subset $V \subset \mathrm{Fl}_k(E, \mathrm{Tr})$ such that for every finite local ring Λ with residue field of residue characteristic $\ell \neq p$, for every $F_\bullet \in V(k)$, we have

(1) $F_\bullet$ is in good position with respect to X.

(2) $F_\bullet$ is $\mathrm{Perv}^{\leq 4r^2}_{\Sigma, ft}(X, \mathcal{E}, \Lambda)$-transversal.

Let $F_\bullet \in V(k)$ and $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Loc}^{\leq r}(U, \mathcal{E}, \overline{\mathbb{Q}}_\ell)$. By Lemma 5.10, it is enough to find a branch B which is $j_! \mathcal{H}om(\mathcal{L}_{a,0}, \mathcal{L}_{b,0})$-transversal for every $a, b \in \{1, 2\}$, where $\mathcal{L}_{1,\bullet}$ and $\mathcal{L}_{2,\bullet}$ represent $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively in the sense of Recollection 5.9. Since

$$\mathcal{P}(\mathcal{L}_{1,0}, \mathcal{L}_{2,0}) := \bigoplus_{a,b \in \{1,2\}} j_! \mathcal{H}om(\mathcal{L}_{a,0}, \mathcal{L}_{b,0})[n]$$

is an object of $\mathrm{Perv}^{\leq 4r^2}_{\Sigma}(X, \mathcal{E}, \Lambda_0)$ by Lemma 5.4 and the fact that $j : U \hookrightarrow X$ is affine, the existence of B follows from the property (2). □

Remark 5.12. An analogue of Theorem 5.11 for flat bundles in characteristic 0 was obtained in [HT22].

6. Wild Lefschetz theorem

Definition 6.1. Let U be a connected scheme of finite type over k and let $\overline{u} \to U$ be a geometric point. Let Λ be a finite extension of $\mathbb{Q}_\ell$ or a finite local ring of residue characteristic $\ell \neq p$. Let $\mathcal{L} \in \mathrm{Loc}_{\mathrm{tf}}(U, \Lambda)$ and let $\rho : \pi_1(U, \overline{u}) \to \mathrm{GL}(\mathcal{L}_{\overline{u}})$ be the corresponding representation. We define the *monodromy group of $\mathcal{L}$ at $\overline{u}$* as the image of ρ.

Definition 6.2. In the setting of Definition 6.1, let $f : V \to U$ be a morphism between connected schemes of finite type over k. Let $\overline{v} \to V$ be a geometric point and put $\overline{u} = f(\overline{v})$. We say that $\mathcal{L}$ and $f^*\mathcal{L}$ *have the same monodromy group* if the monodromy group of $\mathcal{L}$ at $\overline{u}$ is the monodromy group of $f^*\mathcal{L}$ at $\overline{v}$.

Remark 6.3. Since two geometric points of V are connected by a path, having the same monodromy group does not depend on the choice of a base point of V.

The following lemma is elementary :

Lemma 6.4. *Let $G_1 \to G_2$ be a morphism of groups. Let $r \geq 0$, let A be a commutative ring and let $\rho : G_2 \to \mathrm{GL}_r(A)$ be a representation. Assume that ρ and $\rho|_{G_1}$ have the same image. Then, if ρ is simple (resp. semi-simple), the representation $\rho|_{G_1}$ is simple (resp. semi-simple).*

Corollary 6.5. *In the setting of Definition 6.2, assume that $\mathcal{L}$ and $f^*\mathcal{L}$ have the same monodromy group. Then, if $\mathcal{L}$ is simple (resp. semi-simple), the pullback $f^*\mathcal{L}$ is simple (resp. semi-simple).*

Construction 6.6. Let U be a connected scheme of finite type over k and let $\overline{u} \to U$ be a geometric point. Let Λ be a finite extension of $\mathbb{Q}_\ell$ or a finite local ring of residue characteristic $\ell \neq p$. Let $\mathcal{L} \in \mathrm{Loc}_{\mathrm{tf}}(U, \Lambda)$ and let $\rho : \pi_1(U, \overline{u}) \to \mathrm{GL}(\mathcal{L}_{\overline{u}})$ be the corresponding representation. Assume that the monodromy group G of $\mathcal{L}$ at $\overline{u}$ is finite. We define $\Lambda[\mathcal{L}, \overline{u}]$ as the object of $\mathrm{Loc}(U, \Lambda)$ corresponding to the representation

$$\pi_1(U, \overline{u}) \to G \to \mathcal{L}_{\overline{u}}[G]$$

where the first arrow is induced by ρ and where the second arrow is the regular representation of G.

Remark 6.7. The isomorphism class of $\Lambda[\mathcal{L}, \overline{u}]$ does not depend on $\overline{u}$.

Lemma 6.8. *In the setting of Construction 6.6, let $j : U \to X$ be an open immersion, let $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$ and assume that $j_!\mathcal{L}$ has log conductor bounded by $\mathcal{E}$. Then, so does $j_!\Lambda[\mathcal{L}, \overline{u}]$.*

Proof. Immediate from the fact that the action of $\pi_1(U, \overline{x})$ on the fibres of $\Lambda[\mathcal{L}, \overline{u}]$ at a geometric point $\overline{x} \to U$ factors through that on $\mathcal{L}_{\overline{x}}$. □

Definition 6.9. Let U be a connected scheme of finite type of pure dimension $n \geq 2$ over k. Let Λ be a finite extension of $\mathbb{Q}_\ell$ or a finite local ring of residue characteristic $\ell \neq p$. Let $i : U \hookrightarrow \mathbb{P}_k(E)$ be an immersion, let Tr be a tree of depth smaller than $n-1$ and let $\mathcal{C} \subset \mathrm{Loc}_{\mathrm{ft}}(U, \Lambda)$ be a full subcategory. We say that $F_\bullet \in \mathrm{Fl}_k(E, \mathrm{Tr})(k)$ *realizes the wild Lefschetz theorem for $\mathcal{C}$* if $F_\bullet$ is in good position with respect to U and there is a branch $B \subset \mathrm{Tr}$ such that for every $\nu \in B$, the sheaves $\mathcal{L}$ and $\mathcal{L}|_{U \cap F_\nu}$ have the same monodromy group. We say that such branch B *preserves the monodromy group of $\mathcal{L}$.*

Lemma 6.10 ([ES21]). *Let $f : V \to U$ be a morphism between connected schemes of finite type over k. Let $\overline{u} \to U$ be a geometric point and put $\overline{v} = f(\overline{u})$. Let $L/\mathbb{Q}_\ell$ be a finite extension and let $r \geq 0$. Let*

$$\rho : \pi_1^{\acute{e}t}(U, \overline{u}) \to \mathrm{GL}_r(\mathcal{O}_L)$$

be a continuous representation and denote by $\overline{\rho} : \pi_1^{\acute{e}t}(U, \overline{u}) \to \mathrm{GL}_r(\mathcal{O}_L/\mathfrak{m}_L^2)$ the induced representation. Then, if $\overline{\rho}$ and $\overline{\rho}|_{\pi_1^{\acute{e}t}(V,\overline{v})}$ have the same image, so do ρ and $\rho|_{\pi_1^{\acute{e}t}(V,\overline{v})}$.

Lemma 6.11. *Let $G_1 \to G_2$ be a morphism of groups. Let $r \geq 0$, let A be a commutative ring and let $\rho : G_2 \to \mathrm{GL}_r(A)$ be a representation with finite image G. Let $\phi : G \to \mathrm{GL}(A[G])$ be the regular representation of G, so that G_1 and G_2 acts on $A[G]$. Then if $A[G]^{G_1} = A[G]^{G_2}$, the representations ρ and $\rho|_{G_1}$ have the same image.*

Proof. Let H be the image of $\rho|_{G_1} : G_1 \to \mathrm{GL}_r(A)$. We have $H \subset G$ and we have to show that $H = G$. By assumption, we have

$$\sum_{h \in H} h \in A[G]^{G_1} = A[G]^{G_2} = A \cdot \sum_{g \in G} g \, .$$

This is possible only if $H = G$. □

Lemma 6.12. *Let $f : V \to U$ be a morphism between connected schemes of finite type over k. Let $\overline{u} \to U$ be a geometric point. Let $\mathcal{L} \in \mathrm{Loc}(U, \overline{\mathbb{Q}}_\ell)$ represented by $\mathcal{L}_\bullet$ in the sense of Recollection 5.9. The following conditions are equivalent :*

(1) The sheaves $\mathcal{L}$ and $f^*\mathcal{L}$ have the same monodromy group.

(2) The sheaves $\mathcal{L}_1$ and $f^*\mathcal{L}_1$ have the same monodromy group.

(3) The canonical morphism

$$H^0(U, \Lambda_1[\mathcal{L}_1, \overline{u}]) \to H^0(V, f^*\Lambda_1[\mathcal{L}_1, \overline{u}])$$

is an isomorphism.

Proof. The equivalence between (1) and (2) follows from Lemma 6.10. The equivalence between (2) and (3) follows from Lemma 6.11. □

Lemma 6.13. *Let X be a projective scheme of pure dimension $n \geq 2$ over k algebraically closed. Let $j : U \hookrightarrow X$ be an affine open immersion with U smooth connected over k and let $\overline{u} \to U$ be a geometric point. Let $i : X \hookrightarrow \mathbb{P}_k(E)$ be a closed immersion, let Tr be a tree of depth smaller than $n-1$ and let $F_\bullet \in \mathrm{Fl}_k(E, \mathrm{Tr})(k)$ in good position with respect to U. Let $\mathcal{L} \in \mathrm{Loc}(U, \overline{\mathbb{Q}}_\ell)$ represented by $\mathcal{L}_\bullet$ in the sense of Recollection 5.9. Assume that $F_\bullet$ admits a $j_!\Lambda_0[\mathcal{L}_0, \overline{u}]$-transversal branch B. Then, B preserves the monodromy group of $\mathcal{L}$.*

Proof. Let $v \in B$. We have to show that $\mathcal{L}$ and $\mathcal{L}|_{U \cap F_v}$ have the same monodromy group. By Lemma 6.12, we have to show that the canonical morphism

$$H^0(U, \Lambda_1[\mathcal{L}_1, \overline{u}]) \to H^0(U \cap F_v, \Lambda_1[\mathcal{L}_1, \overline{u}]|_{U \cap F_v})$$

is an isomorphism. By Lemma 5.3, it is enough to show that B is $j_!\Lambda_1[\mathcal{L}_1, \overline{u}]$-transversal. For every consecutive vertices $v \leq w$, we have to show that the map $F_w \hookrightarrow F_v$ is $SS((i_*j_!\Lambda_1[\mathcal{L}_1, \overline{u}])|_{F_v})$-transversal. On the other hand, we have

$$i_*j_!\Lambda_1[\mathcal{L}_1, \overline{u}] \otimes^L_{\Lambda_1} \Lambda_0 \simeq i_*j_!\Lambda_0[\mathcal{L}_0, \overline{u}] \, .$$

Hence, Lemma 4.14 implies that

$$SS((i_*j_!\Lambda_1[\mathcal{L}_1,\overline{u}])|_{F_\nu}) = SS((i_*j_!\Lambda_0[\mathcal{L}_0,\overline{u}])|_{F_\nu})$$

and the conclusion follows by definition of the branch B. □

Theorem 6.14. *Let X be a projective scheme of pure dimension $n \geq 2$ over k algebraically closed. Let $j : U \hookrightarrow X$ be an affine open immersion with U smooth connected over k. Then, there is a closed immersion $i : X \hookrightarrow \mathbb{P}_k(E)$, there is an admissible function $\mu : \mathbb{Q}[\mathrm{Coh}(X)] \to \mathbb{Q}$ and $P \in \mathbb{N}[x]$ depending only on (X, U, i) such that for every $r \geq 1$, every $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$, every finite extension $L/\mathbb{Q}_\ell$ and every tree Tr of depth smaller than $n-1$ ramifying enough with respect to $(\mathrm{rk}\, E, P(\mu(\mathcal{E})) \cdot |\mathrm{GL}_r(\mathcal{O}_L/\mathfrak{m}_L^2)|)$, there is a dense open subset $V \subset \mathrm{Fl}_k(E, \mathrm{Tr})$ with the property that every $F_\bullet \in V(k)$ realizes the wild Lefschetz theorem for $\mathrm{Loc}^{\leq r}(U, \mathcal{E}, L)$.*

Proof. Put $\Lambda_1 := \mathcal{O}_L/\mathfrak{m}_L^2$ and let Λ_0 be the residue field of L. Let $i : X \hookrightarrow \mathbb{P}_k(E)$, let $\mu : \mathrm{Coh}(X) \to \mathbb{N}$ and $P \in \mathbb{N}[x]$ as given by Theorem 4.34 applied to (X, Σ) where $\Sigma := \{U, D\}$. Let $r \geq 1$ and $\mathcal{E} \in \mathbb{Q}[\mathrm{Coh}(X)]$. Let Tr be a tree of depth smaller than $n-1$ ramifying enough with respect to $(\mathrm{rk}\, E, P(\mu(\mathcal{E})) \cdot |\mathrm{GL}_r(\Lambda_1)|)$ and let $V \subset \mathrm{Fl}_k(E, \mathrm{Tr})$ be a dense open subset such that for every $F_\bullet \in U(k)$, we have

(1) $F_\bullet$ is in good position with respect to U.

(2) $F_\bullet$ is $\mathrm{Perv}^{\leq|\mathrm{GL}_r(\Lambda_1)|}_{\Sigma,ft}(X, \mathcal{E}, \Lambda_0)$-transversal.

Let $F_\bullet \in V(k)$ and let $\mathcal{L} \in \mathrm{Loc}^{\leq r}(U, \mathcal{E}, L)$. We have to show the existence of a branch preserving the monodromy group of $\mathcal{L}$. Let $\mathcal{L}_\bullet$ be a representative of $\mathcal{L}$ as in Recollection 5.9. Given a geometric point $\overline{u} \to U$, we have to show by Lemma 6.13 that $F_\bullet$ admits a $j_!\Lambda_0[\mathcal{L}_0, \overline{u}]$-transversal branch. Since $j_!\mathcal{L}_0$ has log conductors bounded by $\mathcal{E}$, Lemma 6.8 implies that $j_!\Lambda_0[\mathcal{L}_0, \overline{u}]$ has log conductors bounded by $\mathcal{E}$ as well. Since $j : U \hookrightarrow X$ is affine, $j_!\Lambda_0[\mathcal{L}_0, \overline{u}][n]$ is thus an object of $\mathrm{Perv}_\Sigma(X, \mathcal{E}, \Lambda_0)$. Since $j_!\Lambda_0[\mathcal{L}_0, \overline{u}][n]$ has rank smaller than $|\mathrm{GL}_r(\Lambda_1)|$, the existence of the sought-after branch follows by construction of V.

□

Appendix : Moduli of multi-flags

The goal of this appendix 14 is to provide the basic language to formulate generic statements instead of merely existence statements for the Lefschetz recognition principle (Theorem 5.11) and the wild Lefschetz theorem (Theorem 6.14), thus giving some flexibility for future applications.

Recollection 6.15. Let S be a scheme of finite type over k and let E be a locally free sheaf of $\mathcal{O}_S$-modules on S. For integers $1 \leq n_1 < \cdots < n_r \leq \mathrm{rk}\, E$, we denote by $\mathrm{Fl}_S(E, n_1, \ldots, n_r)$ the scheme over S of $(n_1, \ldots, n_r)$-flags in E. Recall that $\mathrm{Fl}_S(E, n_1, \ldots, n_r)$ is smooth over S. By definition, $\mathrm{Fl}_S(E, n_1, \ldots, n_r)$ represents the functor sending $f : T \to S$ to the set of isomorphism classes of epimorphisms

$$f^*E \to E_r \to \cdots \to E_1$$

where E_i is a locally free sheaf of rank n_i on T for every $1 \leq i \leq r$. If $r = 1$, we note $\mathbb{G}_S(E, n)$ for $\mathrm{Fl}_S(E, n)$ and if $r = n = 1$, we note $\mathbb{P}_S(E)$ for $\mathbb{G}_S(E, 1)$.

Construction 6.16. Let S be a scheme of finite type over k and let E be a locally free sheaf of $\mathcal{O}_S$-modules on S. Let $\pi : \mathbb{P}_S(E) \to S$ be the structural morphism. For $d \geq 1$, the universal locally free rank one quotient

$$\pi^* E \to \mathcal{O}_{\mathbb{P}_S(E)}(1)$$

induces a short exact sequence of locally free sheaves

$$0 \to K \to \pi^* \operatorname{Sym}^d E \to \mathcal{O}_{\mathbb{P}_S(E)}(d) \to 0 \tag{6.16.1}$$

Intuitively, the fiber K_x of K at a point $x \in \mathbb{P}_S(E)$ over $s \in S$ is the hyperplane of homogeneous polynomials of degree d of $\mathbb{P}_s(E_s)$ vanishing at x.

Lemma 6.17. *Assume that* $S = \operatorname{Spec} k$ *and let* $m \geq \binom{\operatorname{rk} E - 1 + d}{d}$. *Then, the set of points* $(x_1, \dots, x_m) \in \mathbb{P}_k(E)^m$ *such that no hypersurface of degree* d *in* $\mathbb{P}_k(E)$ *contains all the* x_i *is dense open.*

Proof. We can suppose that $m = \binom{\operatorname{rk} E - 1 + d}{d}$. Let $i_d : \mathbb{P}_k(E) \hookrightarrow \mathbb{P}_k^{m-1}$ be the d-Veronese embedding and let $V_d \subset \mathbb{P}_k^{m-1}$ be its image. Let $U \subset (\mathbb{P}_k^{m-1})^m$ be the dense open set of m ordered points in generic position in $\mathbb{P}_k^{m-1}$. That is, any $(x_1, \dots, x_m) \in U$ spans $\mathbb{P}_k^{m-1}$. Then, for every $(x_1, \dots, x_m) \in \mathbb{P}_k(E)^m$, we have

$$\begin{aligned} &\text{no hypersurface of degree } d \text{ in } \mathbb{P}_k(E) \text{ contains all the } x_i \\ \iff &\text{no hyperplane in } \mathbb{P}_k^{m-1} \text{ contains all the } i_d(x_i) \\ \iff &(i_d(x_1), \dots, i_d(x_N)) \in U. \end{aligned}$$

Hence, we are left to show that $U \cap V_d^m$ is dense open in V_d^m. Since V_d^m is irreducible and since U is open, it is enough to show that $U \cap V_d^m$ is not empty. Since V_d is not contained in any hyperplane of $\mathbb{P}_k^{m-1}$, one can recursively construct m points on V_d spanning $\mathbb{P}_k^{m-1}$ and the conclusion follows. $\square$

Lemma 6.18. *In the setting of Construction 6.16, for every* $m \geq \binom{\operatorname{rk} E - 1 + d}{d}$, *there is a dense open subset*

$$U \subset \mathop{\times}_{S}^{m} \mathbb{P}_S(E)$$

of the m*-th self product of* $\mathbb{P}_S(E)$ *in* Sch_S *such that for every* $(x_1, \dots, x_m) \in U(\bar{s})$ *over a geometric algebraic point* $\bar{s} \to S$, *no hypersurface of degree at most* d *in* $\mathbb{P}_{\bar{s}}(E_{\bar{s}})$ *contains all the* x_i.

Proof. It is enough to find a dense open subset $U \subset \times_S^m \mathbb{P}_S(E)$ such that for every $(x_1, \dots, x_m) \in U(\bar{s})$ over a geometric algebraic point $\bar{s} \to S$, no hypersurface of degree exactly d in $\mathbb{P}_{\bar{s}}(E_{\bar{s}})$ contains all the x_i. For $1 \leq i \leq m$, let

$$\pi_i : \mathop{\times}_{S}^{m} \mathbb{P}_S(E) \to \mathbb{P}_S(E)$$

be the projection on the i-th factor and let

$$p : \mathop{\times}_{S}^{m} \mathbb{P}_S(E) \to S$$

be the structural morphism. Consider the exact sequence of coherent sheaves

$$0 \to \bigcap_{i=1}^{m} \pi_i^* K \to p^* \operatorname{Sym}^d E \to \bigoplus_{i=1}^{m} \pi_i^* \mathcal{O}_{\mathbb{P}_S(E)}(d)$$

on $\times_S^m \mathbb{P}_S(E)$. Let F be the image of the last arrow and consider the short exact sequence

$$0 \to F \to \bigoplus_{i=1}^{m} \pi_i^* \mathcal{O}_{\mathbb{P}_S(E)}(d) \to G \to 0\,.$$

Choose a dense open subset $V \subset \times_S^m \mathbb{P}_S(E)$ such that F and G are locally free on V. Then, for every $x = (x_1, \ldots, x_m) \in V(\bar{s})$ over a geometric algebraic point $\bar{s} \to S$, there is a canonical isomorphism

$$x^* \bigcap_{i=1}^{m} \pi_i^* K \simeq \bigcap_{i=1}^{m} x_i^* K$$

where the right-hand side is the subspace of $\operatorname{Sym}^d E_{\bar{s}}$ of homogeneous polynomials of degree d vanishing at $x_1, \ldots, x_m$. To prove Lemma 6.18, we are thus left to show the existence of a dense open subset $U \subset V$ on which $\bigcap_{i=1}^m \pi_i^* K$ vanishes. Since the vanishing locus of a coherent sheaf is an open subset, we are left to show the existence of a dense set of points in V on which $\bigcap_{i=1}^m \pi_i^* K$ vanishes. This follows from Lemma 6.17. □

Definition 6.19. A *tree* is a finite poset Tr with an initial object 0 such that for every $v \in \mathrm{Tr}$, the subset $\mathrm{Tr}^{\leq v} := \{w \in \mathrm{Tr} \text{ with } w \leq v\}$ is totally ordered. For a vertex $v \in \mathrm{Tr}$, the natural number $d(v) := |\mathrm{Tr}^{\leq v}| - 1$ is the *depth* of v. The *depth* of Tr is the maximal depth of its vertices. A *branch* of Tr is a maximal totally ordered subset of Tr. A *subtree* of Tr is a subposet $\mathrm{Tr}' \subset \mathrm{Tr}$ which is closed downwards.

Construction 6.20. Let S be a scheme of finite type over k and let E be a non zero locally free sheaf of $\mathcal{O}_S$-modules on S. Let Tr be a tree and let $W : \mathrm{Tr}^{\mathrm{op}} \to \mathbb{N}^*$ be a strictly decreasing morphism of posets sending 0 to $\operatorname{rk} E$. Define

$$\mathrm{Fl}_S(E, \mathrm{Tr}, W) : \mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{Set}$$

as the functor sending $f : T \to S$ to the set of isomorphism classes of diagrams

$$E_\bullet : \mathrm{Tr} \to \mathrm{Coh}(T)$$

where

(1) $E_0 = f^* E$ and for every $v \in \mathrm{Tr}$, the sheaf E_v is locally free of rank $W(v)$.

(2) The arrows of Tr are sent to epimorphisms of $\mathrm{Coh}(T)$.

Remark 6.21. The diagram $E_\bullet : \mathrm{Tr} \to \mathrm{Coh}(T)$ is equivalent to a collection of projective subspaces over T

$$F_v \subset \mathbb{P}_T(f^* E),\ v \in \mathrm{Tr}$$

where F_v has relative dimension $W(v) - 1$ over T. We will tacitly use both descriptions.

Remark 6.22. If $W = \operatorname{rk} E - d(-)$ where $d(-)$ is the depth function, we simply note $\mathrm{Fl}_S(E, \mathrm{Tr})$ for $\mathrm{Fl}_S(E, \mathrm{Tr}, W)$.

Example 6.23. If Tr has a single branch, then $\mathrm{Fl}_S(E, \mathrm{Tr})$ is a partial flag functor. In particular, $\mathrm{Fl}_S(E, \mathrm{Tr})$ is a smooth scheme of finite type over S.

Example 6.24. If every element of Tr distinct from 0 has depth 1, then

$$\mathrm{Fl}_S(E, \mathrm{Tr}, W) = \prod_{\nu \in \mathrm{Tr} \setminus \{0\}} \mathbb{G}_S(E, W(\nu))$$

where the product is performed in the category of schemes over S.

Lemma 6.25. *The functor* $\mathrm{Fl}_S(E, \mathrm{Tr}, W)$ *is a scheme of finite type over* S.

Proof. We argue by recursion on the depth d of Tr. The case of depth 0 follows from Example 6.23. Assume that Tr has depth $d > 0$. Let $\mathrm{Tr}^{\leq d-1} \subset \mathrm{Tr}$ be the subtree of vertices of depth smaller than $d-1$ and $\mathrm{Tr}^d \subset \mathrm{Tr}$ be the set of vertices of depth d. Then, the square

$$\begin{tikzcd}
\coprod_{\nu \in \mathrm{Tr}^d} \mathrm{Tr}^{<\nu} \arrow[r] \arrow[d] & \mathrm{Tr}^{\leq d-1} \arrow[d] \\
\coprod_{\nu \in \mathrm{Tr}^d} \mathrm{Tr}^{\leq\nu} \arrow[r] & \mathrm{Tr}
\end{tikzcd}$$

is a pushout in the category of posets. Thus, the induced square

$$\begin{tikzcd}
\mathrm{Fl}_S(E, \mathrm{Tr}, W) \arrow[r] \arrow[d] & \mathrm{Fl}_S(E, \mathrm{Tr}^{\leq d-1}, W|_{\mathrm{Tr}^{\leq d-1}}) \arrow[d] \\
\prod_{\nu \in \mathrm{Tr}^d} \mathrm{Fl}_S(E, \mathrm{Tr}^{\leq\nu}, W|_{\mathrm{Tr}^{\leq\nu}}) \arrow[r] & \prod_{\nu \in \mathrm{Tr}^d} \mathrm{Fl}_S(E, \mathrm{Tr}^{<\nu}, W|_{\mathrm{Tr}^{<\nu}})
\end{tikzcd}$$

in $\mathrm{Fun}(\mathrm{Sch}_S^{\mathrm{op}}, \mathrm{Set})$ is cartesian. By recursion assumption, the upper right functor is a scheme of finite type over S. By Example 6.23, so are the bottom functors. The conclusion thus follows. $\square$

Lemma 6.26. *In the setting of Construction 6.20, let* $\nu, w \in \mathrm{Tr}$ *where* w *is an immediate successor of* ν *and is maximal in* Tr. *Put* $\mathrm{Tr}^\circ := \mathrm{Tr} \setminus \{w\}$ *and* $W^\circ := W|_{\mathrm{Tr} \setminus \{w\}}$. *Let*

$$\mathcal{E}_\bullet : \mathrm{Tr}^\circ \to \mathrm{Coh}(\mathrm{Fl}_S(E, \mathrm{Tr}^\circ, W^\circ))$$

be the universal object of $\mathrm{Fl}_S(E, \mathrm{Tr}^\circ, W^\circ)$. *Then the restriction*

$$\mathrm{Fl}_S(E, \mathrm{Tr}, W) \to \mathrm{Fl}_S(E, \mathrm{Tr}^\circ, W^\circ)$$

exhibits $\mathrm{Fl}_S(E, \mathrm{Tr}, W)$ *as the Grassmannian of* $W(w)$*-plans in* $\mathcal{E}_\nu$ *over* $\mathrm{Fl}_S(E, \mathrm{Tr}^\circ, W^\circ)$.

Proof. For $f : T \to \mathrm{Fl}_S(E, \mathrm{Tr}^\circ, W^\circ)$, we need to construct a bijection

$$\alpha_T : \mathrm{Gr}_{\mathrm{Fl}_S(E, \mathrm{Tr}^\circ, W^\circ)}(\mathcal{E}_\nu, W(w))(T) \to \mathrm{Fl}_S(E, \mathrm{Tr}, W)(T)$$

natural in T. An element of the left hand side is the datum of an epimorphism

$$h : f^*\mathcal{E}_\nu \to F \tag{6.26.1}$$

where F is a locally free sheaf of rank $W(w)$ on T. Since w is maximal in Tr, concatenating the pullback diagram

$$f^*\mathcal{E}_\bullet : \mathrm{Tr}^\circ \to \mathrm{Coh}(T)$$

with (6.26.1) gives rise to a diagram

$$\alpha_T(h) : \mathrm{Tr} \to \mathrm{Coh}(T)$$

natural in the choice of $f : T \to \mathrm{Fl}_S(E, \mathrm{Tr}^\circ, W^\circ)$. One readily checks that α is an isomorphism of functors. $\square$

Lemma 6.27. *In the setting of Construction 6.20, let* $\mathrm{Tr}' \subset \mathrm{Tr}$ *be a subtree and put* $W' := W|_{\mathrm{Tr}'}$*. Then, the induced morphism over* S

$$\mathrm{Fl}_S(E, \mathrm{Tr}, W) \to \mathrm{Fl}_S(E, \mathrm{Tr}', W')$$

is smooth projective surjective of relative pure dimension

$$\sum_{v \in \mathrm{Tr}} \sum_{\substack{w \in \mathrm{Tr} \setminus \mathrm{Tr}', v \leq w \\ d(w) = d(v)+1}} W(w)(W(v) - W(w)) \, .$$

In particular $\mathrm{Fl}_S(E, \mathrm{Tr}, W)$ *is smooth proper surjective over* S *of relative pure dimension*

$$\sum_{\substack{v \leq w \\ d(w) = d(v)+1}} W(w)(W(v) - W(w)) \, .$$

Proof. Since Tr' is obtained from Tr by successive deletions of maximal vertices, we can suppose that $\mathrm{Tr}' = \mathrm{Tr} \setminus \{w\}$ where $w \in \mathrm{Tr}$ is maximal. Then Lemma 6.27 follows from Lemma 6.26. $\square$

Lemma 6.28. *In the setting of Construction 6.20, assume that* S *is irreducible. Then so is* $\mathrm{Fl}_S(E, \mathrm{Tr}, W)$*.*

Proof. Recall that by going-down property, for every flat morphism $f : X \to S$ over an irreducible base and with irreducible generic fiber, the source X is irreducible. To prove the irreducibility of $\mathrm{Fl}_S(E, \mathrm{Tr}, W)$, we argue recursively on the cardinality of Tr. If $\mathrm{Tr} = \{0\}$, we have $\mathrm{Fl}_S(E, \mathrm{Tr}, W) \simeq S$ and there is nothing to prove. Let $w \in \mathrm{Tr}$ be a maximal element. Put $\mathrm{Tr}^\circ := \mathrm{Tr} \setminus \{w\}$ and $W^\circ := W|_{\mathrm{Tr} \setminus \{w\}}$. Then, Lemma 6.26 ensures that $\mathrm{Fl}_S(E, \mathrm{Tr}, W)$ identifies canonically with a Grassmannian over $\mathrm{Fl}_S(E, \mathrm{Tr}^\circ, W^\circ)$. By recursion assumption, $\mathrm{Fl}_S(E, \mathrm{Tr}^\circ, W^\circ)$ is irreducible. We thus conclude with the above observation and the fact that the grassmannian over a field is irreducible. $\square$

Lemma 6.29. *In the setting of Construction 6.20, assume that* S *is irreducible and that every non maximal vertex* $v \in \mathrm{Tr}$ *admits at least 2 immediate successors. Consider the map induced by the inclusion* $\mathrm{Tr}^{\max} \subset \mathrm{Tr}$

$$f : \mathrm{Fl}_S(E, \mathrm{Tr}) \to \prod_{v \in \mathrm{Tr}^{\max}} \mathbb{G}_S(E, \mathrm{rk}\, E - d(v)) \, , \tag{6.29.1}$$

where the product is done in Sch_S*. Then,* $\mathrm{Fl}_S(E, \mathrm{Tr})$ *and its image have the same dimension.*

Proof. By Lemma 6.27, the source and the target of f are proper over S. Hence, f is proper by [SP23, 01W6]. In particular, $f(\mathrm{Fl}_S(E, \mathrm{Tr}))$ is a closed subset of the target of f. Hence, $f(\mathrm{Fl}_S(E, \mathrm{Tr}))$ is proper over S. By [SP23, 01W6] again, the induced morphism $g : \mathrm{Fl}_S(E, \mathrm{Tr}) \to f(\mathrm{Fl}_S(E, \mathrm{Tr}))$ is proper. By Lemma 6.28, note that $\mathrm{Fl}_S(E, \mathrm{Tr})$ and $f(\mathrm{Fl}_S(E, \mathrm{Tr}))$ are irreducible. By [GD61, Corollaire 4.4.11], the subset $U \subset f(\mathrm{Fl}_S(E, \mathrm{Tr}))$

of points y such that $g^{-1}(y)$ is discrete is open in $\mathrm{Fl}_S(E,\mathrm{Tr})$ and the induced morphism $g^{-1}(U) \to U$ is finite. Hence, we are left to show that U is not empty. It is enough to show the existence of $E_\bullet \in \mathrm{Fl}_S(E,\mathrm{Tr})$ such that $g^{-1}(g(E_\bullet)) = \{E_\bullet\}$. The following recursive construction of $E_\bullet$ on the depth of v does the job : if E_v is constructed with v not maximal, then v having at least 2 immediate successors $v_1,\dots,v_n$, take hyperplanes $E_{v_1},\dots,E_{v_n}$ spanning E_v. ☐

Lemma 6.30. *In the setting of Construction 6.20, let $v \in \mathrm{Tr}$ such that every immediate successor $w_1,\dots,w_n$ of v is maximal. Let $\mathrm{Tr}' \subset \mathrm{Tr}$ be the subtree obtained from Tr by removing the immediate successors of v and put $W' := W|_{\mathrm{Tr}'}$. Let*

$$\mathcal{E}_\bullet : \mathrm{Tr}' \to \mathrm{Coh}(\mathrm{Fl}_S(E,\mathrm{Tr}',W'))$$

be the universal object of $\mathrm{Fl}_S(E,\mathrm{Tr}',W')$. Then, the induced morphism over S

$$\mathrm{Fl}_S(E,\mathrm{Tr},W) \to \mathrm{Fl}_S(E,\mathrm{Tr}',W')$$

exhibits $\mathrm{Fl}_S(E,\mathrm{Tr},W)$ as the fiber product over $\mathrm{Fl}_S(E,\mathrm{Tr}',W')$ of the Grassmannian of $W(w_i)$-plans in $\mathcal{E}_v$ over $\mathrm{Fl}_S(E,\mathrm{Tr}',W')$ where $1 \le i \le n$.

Proof. For $1 \le i \le n$, put $\mathrm{Tr}'_i = \mathrm{Tr}' \cup \{w_i\}$ and $W_i := W|_{\mathrm{Tr}'_i}$. Then, the pushout square of posets

$$\begin{array}{ccc} \coprod_{i=1}^n \mathrm{Tr}' & \longrightarrow & \mathrm{Tr}' \\ \downarrow & & \downarrow \\ \coprod_{i=1}^n \mathrm{Tr}'_i & \longrightarrow & \mathrm{Tr} \end{array}$$

gives rise to a pullback of schemes

$$\begin{array}{ccc} \mathrm{Fl}_S(E,\mathrm{Tr},W) & \longrightarrow & \mathrm{Fl}_S(E,\mathrm{Tr}',W') \\ \downarrow & & \downarrow \\ \prod_{i=1}^n \mathrm{Fl}_S(E,\mathrm{Tr}'_i,W_i) & \longrightarrow & \prod_{i=1}^n \mathrm{Fl}_S(E,\mathrm{Tr}',W') \end{array}$$

where the right vertical arrow is the diagonal map. Hence, there is a canonical isomorphism

$$\mathrm{Fl}_S(E,\mathrm{Tr},W) \simeq \mathrm{Fl}_S(E,\mathrm{Tr}'_1,W_1) \times_{\mathrm{Fl}_S(E,\mathrm{Tr}',W')} \cdots \times_{\mathrm{Fl}_S(E,\mathrm{Tr}',W')} \mathrm{Fl}_S(E,\mathrm{Tr}'_n,W_n)$$

Thus, the conclusion follows from Lemma 6.26. ☐

Definition 6.31. Let $r, d \ge 1$. We say that a tree Tr *ramifies enough with respect to* (r,d) if Tr has depth at most $r-2$, has at least two vertices and if every non maximal vertex $v \in \mathrm{Tr}$ has at least

$$\binom{r-1-d(v)+d}{d}$$

immediate successors, where $d(-) : \mathrm{Tr} \to \mathbb{N}$ is the depth function.

Remark 6.32. The above definition would make sense for d rational number by replacing d by $\lfloor d \rfloor$ in the above formula.

Remark 6.33. The smallest tree Tr whose maximal vertices have all depth $\delta \leq r-2$ and ramifying enough with respect to (r, d) has exactly

$$N(r,d,a) := \prod_{i=0}^{a-1} \binom{r-1-i+d}{d}$$

vertices of depth a for every $0 \leq a \leq \delta$. In that case, Lemma 6.27 gives that for every field k and every finite dimensional vector space E over k, we have

$$\dim \mathrm{Fl}_k(E, \mathrm{Tr}) = \sum_{a=1}^{\delta} (\operatorname{rk} E - a) \cdot \prod_{i=0}^{a-1} \binom{r-1-i+d}{d} .$$

We denote by $C(r, d, \delta)$ the above dimension.

Lemma 6.34. *Let S be a scheme of finite type over k, let E be a locally free sheaf of $\mathcal{O}_S$-modules on S and let $d \geq 1$. Let Tr be a tree ramifying enough with respect to $(\operatorname{rk} E, d)$. Then, there is a dense open subset $U \subset \mathrm{Fl}_S(E, \mathrm{Tr})$ such that for every geometric algebraic point $\bar{s} \to S$, every $F_\bullet \in U(\bar{s})$ and every non maximal vertex $v \in \mathrm{Tr}$, no hypersurface of $F_v^\vee$ of degree at most d contains all the F_w, for w immediate successor of v.*

Proof. Let $v \in \mathrm{Tr}$ be a non maximal vertex. Since the intersection of a finite number of dense open subsets is again dense open, it is enough to find a dense open subset $U_v \subset \mathrm{Fl}_S(E, \mathrm{Tr})$ such that for every geometric algebraic point $\bar{s} \to S$ and every $F_\bullet \in U_v(\bar{s}\,\bar{s})$, no hypersurface of $F_v^\vee$ of degree at most d contains all the F_w, for w immediate successor of v. Let $\mathrm{Tr}' \subset \mathrm{Tr}$ be the subtree obtained by removing the vertices of depth at least $d(v) + 2$. By Lemma 6.27, the restriction morphism

$$\mathrm{Fl}_S(E, \mathrm{Tr}) \to \mathrm{Fl}_S(E, \mathrm{Tr}')$$

is flat. By going-down property, the pre-image of any dense open subset of $\mathrm{Fl}_S(E, \mathrm{Tr}')$ is dense open in $\mathrm{Fl}_S(E, \mathrm{Tr})$. Hence, at the cost of replacing Tr by Tr', we can suppose that every immediate successor of v is maximal in Tr. In that case, let $\mathrm{Tr}' \subset \mathrm{Tr}$ be the subtree obtained from Tr by removing the immediate successors of v. Let

$$\mathcal{E}_\bullet : \mathrm{Tr}' \to \mathrm{Coh}(\mathrm{Fl}_S(E, \mathrm{Tr}'))$$

be the universal object of $\mathrm{Fl}_S(E, \mathrm{Tr}')$. Then Lemma 6.30 gives a canonical isomorphism

$$\mathrm{Fl}_S(E, \mathrm{Tr}) \simeq \mathbb{G}_{\mathrm{Fl}_S(E,\mathrm{Tr}')}(\mathcal{E}_v, \operatorname{rk} \mathcal{E}_v - 1) \times_{\mathrm{Fl}_S(E,\mathrm{Tr}')} \cdots \times_{\mathrm{Fl}_S(E,\mathrm{Tr}')} \mathbb{G}_{\mathrm{Fl}_S(E,\mathrm{Tr}')}(\mathcal{E}_v, \operatorname{rk} \mathcal{E}_v - 1)$$

where each factor corresponds to an immediate successor of v. Thus,

$$\mathrm{Fl}_S(E, \mathrm{Tr}) \simeq \mathbb{P}_{\mathrm{Fl}_S(E,\mathrm{Tr}')}(\mathcal{E}_v^\vee) \times_{\mathrm{Fl}_S(E,\mathrm{Tr}')} \cdots \times_{\mathrm{Fl}_S(E,\mathrm{Tr}')} \mathbb{P}_{\mathrm{Fl}_S(E,\mathrm{Tr}')}(\mathcal{E}_v^\vee) .$$

Since Tr ramifies enough, v has at least

$$\binom{\operatorname{rk} E - 1 - d(v) + d}{d} = \binom{\operatorname{rk} \mathcal{E}_v - 1 + d}{d}$$

successors. Thus, the existence of U_v follows from Lemma 6.18. □

The following lemma gives some flexibility in the choice of the dense open subset U from Lemma 6.34.

Lemma 6.35. *In the setting of Construction 6.20, assume that* Tr *has at least two vertices. Let*

$$\mathrm{Fl}_S(E, \mathrm{Tr}, W) \to \prod_{v \in \mathrm{Tr} \setminus \{0\}} \mathbb{G}_S(E, W(v)) \tag{6.35.1}$$

be the morphism of schemes over S *induced by the injections of posets* $\{0, v\} \subset \mathrm{Tr}$ *for* $v \in \mathrm{Tr} \setminus \{0\}$. *For every* $1 \leq a \leq \operatorname{rk} E$, *choose a dense open subset* $U_a \subset \mathbb{G}_S(E, a)$. *Then, the inverse image of*

$$\prod_{v \in \mathrm{Tr} \setminus \{0\}} U_{W(v)} \tag{6.35.2}$$

by (6.35.1) *is dense open in* $\mathrm{Fl}_S(E, \mathrm{Tr}, W)$.

Proof. Since $\mathrm{Fl}_S(E, \mathrm{Tr}, W)$ is flat over S by Lemma 6.27, the going-down property implies that the inverse image of a dense open subset of S is dense in $\mathrm{Fl}_S(E, \mathrm{Tr}, S)$. Thus, we can always replace S by a dense open subset. Hence, we can suppose that S is irreducible. By Lemma 6.28, the scheme $\mathrm{Fl}_S(E, \mathrm{Tr}, W)$ is then irreducible. Hence, to prove Lemma 6.35, we are left to show that the inverse image of (6.35.2) by (6.35.1) is not empty. To do this, we argue by recursion on the cardinality of Tr. If $|\mathrm{Tr}| = 1$, the map (6.35.1) is an isomorphism and we are done. Assume that $|\mathrm{Tr}| > 1$. Choose a maximal element $w \in \mathrm{Tr}$ and let v be the its immediate antecedent. Put $\mathrm{Tr}' := \mathrm{Tr} \setminus \{w\}$ and $W' := W|_{\mathrm{Tr} \setminus \{w\}}$ and consider the span

$$\mathbb{G}_S(E, W(v)) \xleftarrow[p]{} \mathrm{Fl}_S(E, W(w), W(v)) \xrightarrow{q} \mathbb{G}_S(E, W(w)) \, .$$

By Lemma 6.27, the map p is flat. Thus, $p(q^{-1}(U_{W(w)}))$ is an open subset of $\mathbb{G}_S(E, W(v))$. Since S is irreducible, so is $\mathbb{G}_S(E, W(v))$. Hence, $p(q^{-1}(U_{W(w)}))$ is a dense open subset of $\mathbb{G}_S(E, W(v))$. For $a \neq W(v)$, put $U'_a = U_a$ and put

$$U'_{W(v)} := U_{W(v)} \cap p(q^{-1}(U_{W(w)})) \, .$$

By recursion assumption applied to Tr' and to the U'_a, there is $F_\bullet \in \mathrm{Fl}_S(E, \mathrm{Tr}', W')$ such that $F_{v'} \in U'_{W(v')} \subset U_{W(v')}$ for every $v' \in \mathrm{Tr}' = \mathrm{Tr} \setminus \{w\}$. By construction, $F_v \in U_{W(v)}$ contains a $W(w) - 1$-dimensional subspace $F_w \in U_{W(w)}$. Then, completing $F_\bullet$ with F_w gives a point of $\mathrm{Fl}_S(E, \mathrm{Tr}, W)$ meeting our requirement. □

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