CHARACTERISTIC CYCLE AND WILD RAMIFICATION FOR NEARBY CYCLES OF ÉTALE SHEAVES

by

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Abstract. — In this article, we give a bound for the wild ramification of the monodromy action on the nearby cycles complex of a locally constant étale sheaf on the generic fiber of a smooth scheme over an equal characteristic trait in terms of Abbes and Saito's logarithmic ramification filtration. This provides a positive answer to the main conjecture in [**Lea16**] for smooth morphisms in equal characteristic. As an application, we derive various properties for the wild ramification of étale sheaves on abelian schemes over curves in positive characteristic.

The main topic of this article is the wild ramification of the monodromy action on the nearby cycles complex of an étale locally constant sheaf defined on the generic fibre of a semi-stable family of schemes over a trait.

Let S be an henselian trait with a perfect residue field of characteristic $p > 0$. Let s be the closed point of S. Let $\bar{s}$ be a geometric closed point of S. Let η be the generic point of S. Let $\bar{\eta}$ be a geometric generic point of S. Let G be the Galois group of $\bar{\eta}$ over η. Let X be a scheme and let $f : X \longrightarrow S$ be a morphism of finite type. Let $\ell \neq p$ be a prime number. Let Λ be a finite field of characteristic ℓ. Let $\mathscr{K}$ be a bounded below complex of étale sheaves of Λ-modules on X. Introduced by Grothendieck in the 1960s, the nearby cycles complex $R\Psi(\mathscr{K}, f)$ and the vanishing cycles complex $R\Phi(\mathscr{K}, f)$ of $\mathscr{K}$ relative to f are complexes on the geometric special fibre $X_{\bar{s}}$ of f endowed with a G-action, called the *monodromy action*.

The monodromy action on cohomological objects (such as nearby cycles complex, vanishing cycles complex as well as étale cohomological groups of the geometric generic fiber) is a central theme in arithmetic geometry. Major contributions include Grothendieck's proof of the local monodromy theorem [**SGA7**, XI], Deligne's Milnor formula computing the total dimension of the vanishing cycles of the constant sheaf at an isolated critical point of a morphism to a curve [**SGA7**, XVI], good and semi-stable reduction criteria for abelian varieties [**ST68**] [**SGA7**, IX], and Saito's proof of the semi-stable reduction theorem for curves [**Sai87**]. Under the semi-stable condition, the tameness of the monodromy action on the nearby cycles complex of the constant sheaf was investigated in [**RZ82**] and [**Ill04**]. See also [**Ill88**] and [**Ill15**] for a survey. For arbitrary schemes over a trait, the wild ramification of the monodromy action is involved. Bloch's conductor formula [Bl87] provides a geometric interpretation for the alternating sum of the Swan conductors of the cohomology groups of the constant sheaf. In this direction, progress to prove this formula were made (among many other works) in [Ab00] and [KS05], and generalizations to arbitrary étale sheaves were obtained for exemple in [KS13] and [Sai17c].

However, very little is known on the wild ramification for the monodromy action on each *individual* cohomology group of an *arbitrary* ℓ-adic sheaf. In this direction, the following conjecture was formulated in [Lea16].

Conjecture 0.1. — *Let (X, Z) be a proper semi-stable pair*[(1)] *over S. Let $\mathscr{F}$ be a locally constant and constructible sheaf of Λ-modules on $U := X\backslash Z$. Suppose that $\mathscr{F}$ is tamely ramified along the horizontal part of Z. Let $r_{\log}$ be the maximum of the set of Abbes and Saito's logarithmic slopes of $\mathscr{F}$ at the generic points of the special fibre of X.*

Then, for every $r > r_{\log}$, the r-th upper numbering ramification subgroup of G acts trivially on $H^i_c(U_{\overline{\eta}}, \mathscr{F}|_{U_{\overline{\eta}}})$ for every $i \in \mathbb{Z}_{\geqslant 0}$.

This conjecture was proved by Leal in [Lea16] under the assumption that S is equal characteristic, that $f : X \longrightarrow S$ has relative dimension 1 and that $\mathscr{F}$ has rank 1. The main goal of this article is to prove a local version of this conjecture for arbitration relative dimension and arbitrary rank sheaf in the good reduction case. The main result is the following (Theorem 5.7):

Theorem 0.2. — *Suppose that S is the henselization at a closed point of a smooth curve over a perfect field of characteristic $p > 0$. Let (X, Z) be a semi-stable pair over S such that $f : X \longrightarrow S$ is smooth. Let $U := X\backslash Z$ and $j : U \longrightarrow X$ the canonical injection. Let $\mathscr{F}$ be a locally constant and constructible sheaf of Λ-modules on U. Suppose that $\mathscr{F}$ is tamely ramified along the horizontal part of Z. Let $r_{\log}$ be the maximum of the set of Abbes and Saito's logarithmic slopes of $\mathscr{F}$ at the generic points of the special fibre of X.*

Then, for every $r > r_{\log}$, the r-th upper numbering ramification subgroup of G acts trivially on $R^i\Psi(j_!\mathscr{F}, f)$ for every $i \in \mathbb{Z}_{\geqslant 0}$.

Theorem 0.2 says in particular that the slopes of the nearby cycles complex can be bounded in a way depending only on $\mathscr{F}$ but not on the smooth morphism f. This is precisely the content of the boundedness question asked in Question 2 from [Tey15]. Due to the G-equivariant spectral sequence

$$E_2^{ab} = H^a(X_s, R^b\Psi(j_!\mathscr{F}, f)) \Longrightarrow H_c^{a+b}(U_{\bar{\eta}}, \mathscr{F}|_{U_{\bar{\eta}}}).$$

Theorem 0.2 implies that Conjecture 0.1 is valid when $f : X \longrightarrow S$ is smooth and S is equal characteristic.

Let us now comment on our approach to Conjecture 0.1. Leal's proof is global. It relies on Kato and Saito's conductor formula [KS13] and Kato's formula [Ka89] for the Swan class of a clean rank 1 sheaf. By contrast, our proof is local. The tools are the *singular support* and the *characteristic cycle* for constructible étale

(1) See definition 5.5 for details.

sheaves constructed by Beilinson [**Bei16**] and Saito [**Sai17b**], respectively, as well as the semi-continuity properties for Abbes and Saito's ramification invariants following [**HY17, Hu17**]. The strategy to prove Theorem 0.2 has three steps. The first one is to reduce the question to the vanishing of the tame nearby cycles of $j_!(\mathscr{F}\otimes_\Lambda f^*\mathscr{N})$ for every isoclinic sheaf of Λ-modules $\mathscr{N}$ on η with slope strictly bigger than $r_{\log}$. From this point on, we descend to the case where S is a smooth curve. In the second step, when Z is the special fiber of f, we prove that for $\mathscr{N}$ as above, the singular support of $j_!(\mathscr{F}\otimes_\Lambda f^*\mathscr{N})$ is supported on the zero section of $\mathbb{T}^*X$ and the conormal bundle of Z in X. In particular, any smooth curve on X transverse to Z is $SS(j_!(\mathscr{F}\otimes_\Lambda f^*\mathscr{N}))$-transversal, which allows us to reduce the vanishing of the above tame nearby cycles to the curve case, where it is obvious. The semi-continuity of the largest slopes is the key point in this step. In the last step, we apply the semi-continuity of the largest logarithmic slopes to deal with the general case where Z has some horizontal components.

Let us give some applications of Theorem 0.2. In [**Tsu17**], Tsuzuki showed the constancy of Newton polygons for convergent F-isocrystals on abelian varieties over a finite field. For a subsequent proof relying on Tannakian techniques, let us refer to [**D'Ad17**]. Tsuzuki's theorem is striking since the input is a global geometric condition whereas the output is a local property for arbitrary convergent F-isocrystals. Inspired by [**Tsu17**], the following theorem was proved in [**Tey18**]. Let $\mathcal{A}\longrightarrow C$ be an abelian scheme over a smooth complex curve C with a closed point c. Let $\mathcal{M}$ be a meromorphic connection on $\mathcal{A}$ with poles along the special fibre $\mathcal{A}_c$. Then, the function $T\longrightarrow \mathrm{NP}(\mathcal{M}|_T)$ associating to every smooth curve T in $\mathcal{A}$ transverse to $\mathcal{A}_c$ the Newton polygon of $\mathcal{M}|_T$ is constant. Equivalently, the connection $\mathcal{M}$ has good formal decomposition along $\mathcal{A}_c$. We expect an ℓ-adic analogue of this theorem for abelian schemes in characteristic $p>0$. Using Theorem 5.7, we prove the following consequences of the cleanliness on the ramification of étale sheaves ramified along a fibre of a relative elliptic curve (Theorem 6.6):

Theorem 0.3. — *Let C be a smooth connected curve over a perfect field of characteristic $p>0$. Let $f:\mathcal{A}\longrightarrow C$ be a relative elliptic curve over C. Let c be a closed point of C. Put $V:=C\backslash\{c\}$ and let f_V be the restriction of f on $U:=f^{-1}(V)$. Let $\mathscr{F}$ be a locally constant and constructible sheaf of Λ-modules on U. Then*

(1) *We have*

$$\sum_{i=0}^{2}(-1)^i\mathrm{sw}_c(R^if_{V*}\mathscr{F})=0.$$

(2) *Let $j:U\longrightarrow\mathcal{A}$ be the canonical inclusion. For any closed point $x\in\mathcal{A}_c$, the fibre of the singular support $SS(j_!\mathscr{F})\longrightarrow\mathcal{A}$ above x is a union of lines in $\mathbb{T}^*_x\mathcal{A}$.*

The article is organized as follows. In section 1 and 2, we briefly recall Abbes and Saito's ramification theory and introduce the singular support and the characteristic cycle for étale sheaves. In section 3, we prove semi-continuity properties for the largest slope and the largest logarithmic slopes for locally constant sheaves ramified along smooth divisors. Relying on these semi-continuity properties, we compute in section

4 the characteristic cycle of a locally constant sheaf on the complement of a smooth divisor after a wild enough twist by a locally constant sheaf coming from the base curve. Based on results in section 3 and section 4, the main Theorem 0.2 is proved in section 5. In section 6, we prove Theorem 0.3.

Acknowledgement. — We would like to express our gratitude to A. Abbes, A. Beilinson, L. Fu, O. Gabber, F. Orgogozo, T. Saito and Y. Tian for inspiring discussions on this topic and valuable suggestions. The first author would like to thank Y. Cao and T. Deng for helpful discussions on abelian varieties. This manuscript was mainly written while the first author was visiting the Max-Planck Institute for Mathematics in Bonn and while the second author was visiting the Catholic University in Leuven and received support from the long term structural funding-Methusalem grant of the Flemish Government. We would like to thank both Institutes for their hospitality and for providing outstanding working conditions.

1. Recollection on the ramification theory of local fields

1.1. — Let K be a henselian discrete valuation field, $\mathscr{O}_K$ the ring of integer of K, $\mathfrak{m}_K$ the maximal ideal of $\mathscr{O}_K$, F the residue field of $\mathscr{O}_K$, $\overline{K}$ a separable closure of K, and let G_K be the Galois group of $\overline{K}$ over K. We assume that the characteristic of F is $p>0$.

1.2. — Suppose that F is perfect. Let $\{G^r_{K,\mathrm{cl}}\}_{r\in\mathbb{Q}_{\geqslant 0}}$ be the classical upper numbering ramification filtration on G_K [Ser68]. For $r\in\mathbb{Q}_{\geqslant 0}$, put

$$G^{r+}_{K,\mathrm{cl}}=\overline{\bigcup_{s>r}G^s_{K,\mathrm{cl}}}$$

The subgroup $G^0_{K,\mathrm{cl}}$ is the inertia subgroup I_K of G_K and $G^{0+}_{K,\mathrm{cl}}$ is the wild inertia subgroup P_K of G_K. For any $r\in\mathbb{Q}_{>0}$, the graded piece $G^r_{K,\mathrm{cl}}/G^{r+}_{K,\mathrm{cl}}$ is a non-trivial, abelian and p-torsion group [Sai95, Th. 1].

1.3. — When F is not assumed to be perfect, Abbes and Saito defined in [AS02] two decreasing filtrations $\{G^r_K\}_{r\in\mathbb{Q}_{>0}}$ and $\{G^r_{K,\log}\}_{r\in\mathbb{Q}_{\geqslant 0}}$ on G_K by closed normal subgroups. These filtrations are called respectively *the ramification filtration* and *the logarithmic ramification filtration*. For $r\in\mathbb{Q}_{\geqslant 0}$, put

$$G^{r+}_K=\overline{\bigcup_{s>r}G^s_K}\quad\text{and}\quad G^{r+}_{K,\log}=\overline{\bigcup_{s>r}G^s_{K,\log}}.$$

We denote by G^0_K the group G_K. The ramification filtrations satisfy the following properties:

Proposition 1.4 ([AS02, AS03, Sai08, Sai17a]). — (i) *For any $0<r\leqslant 1$, we have*

$$G^r_K=G^0_{K,\log}=I_K\ \text{and}\ G^{+1}_K=G^{0+}_{K,\log}=P_K.$$

(ii) *For any* $r \in \mathbb{Q}_{\geqslant 0}$*, we have*

$$G_K^{r+1} \subseteq G_{K,\log}^r \subseteq G_K^r.$$

If F *is perfect, then for any* $r \in \mathbb{Q}_{\geqslant 0}$*, we have*

$$G_{K,\mathrm{cl}}^r = G_{K,\log}^r = G_K^{r+1}.$$

(iii) *Let* K' *be a finite extension of* K *contained in* $\overline{K}$*. Let* e *be the ramification index of* K'/K*. Then, for any* $r \in \mathbb{Q}_{>1}$*, we have* $G_{K'}^{er} \subseteq G_K^r$ *with equality if* K'/K *is unramified. For any* $r \in \mathbb{Q}_{>0}$*, we have* $G_{K',\log}^{er} \subseteq G_{K,\log}^r$ *with equality if* K'/K *is tamely ramified.*

(iv) *For any* $r \in \mathbb{Q}_{>0}$*, the graded piece* $G_{K,\log}^r/G_{K,\log}^{r+}$ *is abelian,* p*-torsion and contained in the center of* $P_K/G_{K,\log}^{r+}$*. For any* $r \in \mathbb{Q}_{>1}$*, the graded piece* G_K^r/G_K^{r+} *is abelian and contained in the center of* P_K/G_K^{r+}*. If we further assume that the characteristic of* K *is* $p > 0$*, then, for any* $r \in \mathbb{Q}_{>1}$*, the graded piece* G_K^r/G_K^{r+} *is* p*-torsion.*

1.5. — Let Λ be a finite field of characteristic $\ell \neq p$. Let M be a finitely generated Λ-module with a continuous P_K-action. The module M has decompositions

(1.5.1) $$M = \bigoplus_{r \geqslant 1} M^{(r)} \quad \text{and} \quad M = \bigoplus_{r \geqslant 0} M_{\log}^{(r)}$$

into P_K-stable submodules, where $M^{(1)} = M_{\log}^{(0)} = M^{P_K}$, and such that for every $r \in \mathbb{Q}_{>0}$,

$$(M^{(r+1)})^{G_K^{r+1}} = 0 \quad \text{and} \quad (M^{(r+1)})^{G_K^{(r+1)+}} = M^{(r+1)};$$
$$(M_{\log}^{(r)})^{G_{K,\log}^r} = 0 \quad \text{and} \quad (M_{\log}^{(r)})^{G_{K,\log}^{r+}} = M_{\log}^r.$$

The decompositions (1.5.1) are called respectively the *slope decomposition* and the *logarithmic slope decomposition* of M. The values r for which $M^{(r)} \neq 0$ (resp. $M_{\log}^{(r)} \neq 0$) are the *slopes* (resp. the *logarithmic slopes*) of M. Let $\mathrm{Sl}_K(M)$ be the set of slopes of M. Let $\mathrm{Sl}_{K,\log}(M)$ be the set of logarithmic slopes of M. Let $r_K(M)$ be the largest slope of M. Let $r_{K,\log}(M)$ be the largest logarithmic slope of M. We say that M is *isoclinic* (resp. *logarithmic isoclinic*) if M has only one slope (resp. only one logarithmic slope). We say that M is *tame* if the action of P_K on M is trivial, that is $M = M^{(1)} = M_{\log}^{(0)}$. By proposition 1.4 (ii), we have

(1.5.2) $$r_{K,\log}(M) + 1 \geqslant r_K(M) \geqslant r_{K,\log}(M).$$

The *total dimension* of M is defined by

(1.5.3) $$\mathrm{dimtot}_K(M) := \sum_{r \geqslant 1} r \cdot \dim_\Lambda M^{(r)}.$$

The *Swan conductor* of M is defined by

$$\mathrm{sw}_K(M) := \sum_{r \geqslant 0} r \cdot \dim_\Lambda M_{\log}^{(r)}$$

We have

$$\mathrm{sw}_K(M) + \dim_\Lambda M \geqslant \mathrm{dimtot}_K(M) \geqslant \mathrm{sw}_K(M).$$

If the residue field F is perfect, we have

$$r_{K,\log}(M) + 1 = r_K(M) \tag{1.5.4}$$

$$\mathrm{sw}_K(M) + \dim_\Lambda M = \mathrm{dimtot}_K(M) \tag{1.5.5}$$

and $\mathrm{sw}_K(M)$ is the classical Swan conductor of M.

Let K' be a finite separable extension of K contained in $\overline{K}$. Let e be the ramification index of K'/K. By proposition 1.4 (iii), we have

$$e \cdot r_K(M) \geqslant r_{K'}(M) \quad \text{and} \quad e \cdot r_{K,\log}(M) \geqslant r_{K',\log}(M);$$
$$e \cdot \mathrm{dimtot}_K(M) \geqslant \mathrm{dimtot}_{K'}(M) \quad \text{and} \quad e \cdot \mathrm{sw}_K(M) \geqslant \mathrm{sw}_{K'}(M).$$

If K'/K is tamely ramified, we have

$$e \cdot r_{K,\log}(M) = r_{K',\log}(M) \quad \text{and} \quad e \cdot \mathrm{sw}_K(M) = \mathrm{sw}_{K'}(M).$$

If K'/K is unramified, we have

$$r_K(M) = r_{K'}(M) \quad \text{and} \quad \mathrm{dimtot}_K(M) = \mathrm{dimtot}_{K'}(M)$$

1.6. — We assume that the characteristic of K is $p > 0$. Let M be a finitely generated Λ-module with a continuous P_K-action. Suppose that M is isoclinic of slope $r \in \mathbb{Q}_{>1}$. Then, M has a unique direct sum decomposition

$$M = \bigoplus_{\chi \in X(r)} M_\chi$$

into P_K-stable submodules M_χ, where $X(r)$ is the set of isomorphism classes of non-trivial finite characters $\chi : G^r_K/G^{r+}_K \longrightarrow \Lambda^\times$ and where M_χ is a direct sum of finitely many copies of χ.

Assume moreover that F is of finite type over a perfect field. We denote by $\mathscr{O}_{\overline{K}}$ the integral closure of $\mathscr{O}_K$ in $\overline{K}$, by $\overline{F}$ the residue field of $\mathscr{O}_{\overline{K}}$ and by $\mathrm{ord} : \overline{K} \longrightarrow \mathbb{Q} \bigcup \{\infty\}$ a valuation normalized by $\mathrm{ord}(K^\times) = \mathbb{Z}$. For any $r \in \mathbb{Q}_{\geqslant 0}$, we put

$$\mathfrak{m}^r_{\overline{K}} = \{x \in \overline{K}^\times ; \mathrm{ord}(x) \geqslant r\} \quad \text{and} \quad \mathfrak{m}^{r+}_{\overline{K}} = \{x \in \overline{K}^\times ; \mathrm{ord}(x) > r\}.$$

The quotient $\mathfrak{m}^r_{\overline{K}}/\mathfrak{m}^{r+}_{\overline{K}}$ is a 1-dimensional $\overline{F}$-vector space. For $r \in \mathbb{Q}_{>1}$, there exists an injective homomorphism called the *characteristic form*

$$\mathrm{char} : \mathrm{Hom}_{\mathbb{F}_p}(G^r_K/G^{r+}_K, \mathbb{F}_p) \longrightarrow \mathrm{Hom}_{\overline{F}}(\mathfrak{m}^r_{\overline{K}}/\mathfrak{m}^{r+}_{\overline{K}}, \Omega^1_{\mathscr{O}_K} \otimes_{\mathscr{O}_K} \overline{F}). \tag{1.6.1}$$

If F is perfect, the characteristic form is a bijection [Sai95].

1.7. — We present two elementary lemmas on Galois modules that will be used in the sequel of the article.

Lemma 1.8. — *Assume that the residue field F is perfect and that Λ contains a primitive p-th root of unity. Then, for any $r \in \mathbb{Q}_{\geqslant 0}$, there exists a non-trivial Λ-module N with continuous G_K-action such that N is logarithmic isoclinic of slope r.*

Proof. — When $r = 0$, we take N a tame module. Let $r \in \mathbb{Q}_{>0}$. By [**Sai95**, Th. 1], the quotient $G^r_{K,\log}/G^{r+}_{K,\log}$ is a non-trivial abelian p-torsion closed normal subgroup of $G_K/G^{r+}_{K,\log}$. Hence, $G_K/G^{r+}_{K,\log}$ admits a finite quotient $G_K/G^{r+}_{K,\log} \longrightarrow G$ such that the image $H := \mathrm{Im}(G^r_{K,\log}/G^{r+}_{K,\log} \longrightarrow G)$ is a non trivial normal subgroup of G. Notice that H is a product of a finite number of copies of $\mathbb{Z}/p\mathbb{Z}$. Since Λ contains a primitive p-th root of unity, there exists a non trivial character $\chi : H \longrightarrow \Lambda^\times$. The H-representation $\mathrm{Res}^G_H(\mathrm{Ind}^G_H(\chi))$ is a direct summand of conjugates of χ. Hence $N = \mathrm{Ind}^G_H(\chi)$ is logarithmic isoclinic of logarithmic slope r. □

Lemma 1.9. — *Let M and N be finitely generated Λ-modules with continuous G_K-actions.*

(i) *Assume that M and N are isoclinic of slope r and r' respectively. If $r \geqslant r'$, then we have $r_K(M \otimes_\Lambda N) \leqslant r$ and*

$$\mathrm{dimtot}_K(M \otimes_\Lambda N) \leqslant \dim_\Lambda N \cdot \dim_\Lambda M \cdot r \tag{1.9.1}$$

If $r > r'$, then the Λ-module $M \otimes_\Lambda N$ is isoclinic of slope r and

$$\mathrm{dimtot}_K(M \otimes_\Lambda N) = \dim_\Lambda N \cdot \mathrm{dimtot}_K(M) = \dim_\Lambda N \cdot \dim_\Lambda M \cdot r \tag{1.9.2}$$

(ii) *Assume that M and N are logarithmic isoclinic of logarithmic slope $r_{\log}$ and $r'_{\log}$ respectively. If $r_{\log} \geqslant r'_{\log}$, then we have $r_{K,\log}(M \otimes_\Lambda N) \leqslant r_{\log}$ and*

$$\mathrm{sw}_K(M \otimes_\Lambda N) \leqslant \dim_\Lambda N \cdot \dim_\Lambda M \cdot r_{\log}.$$

If $r_{\log} > r'_{\log}$, then the Λ-module $M \otimes_\Lambda N$ is logarithmic isoclinic of logarithmic slope $r_{\log}$ and

$$\mathrm{sw}_K(M \otimes_\Lambda N) = \dim_\Lambda N \cdot \mathrm{sw}_K(M) = \dim_\Lambda N \cdot \dim_\Lambda M \cdot r_{\log}.$$

Proof. — The proofs of (i) and (ii) are the same. We prove (i). Since the action of G^{r+}_K on M and N is trivial, its action on $M \otimes_\Lambda N$ is trivial too. Hence, $r_K(M \otimes_\Lambda N) \leqslant r$ and we get (1.9.1). If $r > r'$, we have $M \otimes_\Lambda N \cong M^{\oplus \dim_\Lambda N}$ as G^r_K-representation. Hence $(M \otimes_\Lambda N)^{G^r_K} = (M^{G^r_K})^{\oplus \dim_\Lambda N} = \{0\}$. Hence $M \otimes_\Lambda N$ is isoclinic of slope r and we get (1.9.2). □

2. Singular support and characteristic cycle for ℓ-adic sheaves

2.1. — Let k be a field of characteristic $p > 0$. Let X be a smooth connected k-scheme. Let C be a closed conical subset of the cotangent bundle $\mathbb{T}^*X$. Let x be a point of X and $\bar{x} \longrightarrow X$ a geometric point of X above x. Set $\mathbb{T}^*_x X = \mathbb{T}^*X \times_X x$ and

$C_x = C \times_X x$. Set $\mathbb{T}^*_{\bar{x}}X = \mathbb{T}^*X \times_X \bar{x}$ and $C_{\bar{x}} = C \times_X \bar{x}$. Let Λ be a finite field of characteristic $\ell \neq p$.

2.2. — Let $f : U \longrightarrow X$ be a morphism of smooth k-schemes. Let $u \in U$ and let $\bar{u} \longrightarrow U$ be a geometric point of U lying over u. We say that $f : U \longrightarrow X$ is *C-transversal at u* if $\ker df_{\bar{u}} \bigcap C_{f(\bar{u})} \subseteq \{0\} \subseteq \mathbb{T}^*_{f(\bar{u})}X$, where $df_{\bar{u}} : \mathbb{T}^*_{f(\bar{u})}X \longrightarrow \mathbb{T}^*_{\bar{u}}U$ is the cotangent map of f at $\bar{u}$. We say that $f : U \longrightarrow Y$ is C-transversal if it is C-transversal at every point of U. For a C-transversal morphism $f : U \longrightarrow X$, let $f^\circ C$ be the scheme theoretic image of $C \times_X U$ in $\mathbb{T}^*U$ by $df : \mathbb{T}^*X \times_X U \longrightarrow \mathbb{T}^*U$.

Let $g : X \longrightarrow Y$ be a morphism of smooth k-schemes. Let $x \in X$ and let $\bar{x} \longrightarrow X$ be a geometric point of X lying over x. We say that $g : X \longrightarrow Y$ is *C-transversal at x* if $dg_{\bar{x}}^{-1}(C_{\bar{x}}) = \{0\} \subset \mathbb{T}^*_{\bar{x}}Y$. We say that $g : X \longrightarrow Y$ is *C-transversal* if it is C-transversal at every point of X.

Let $(g, f) : Y \longleftarrow U \longrightarrow X$ be a pair of morphisms of smooth k-schemes. We say that (g, f) is *C-transversal* if $f : U \longrightarrow X$ is C-transversal and if $g : U \longrightarrow Y$ is $f^\circ C$-transversal.

Let $\mathscr{F}$ be an object in $D^b_c(X, \Lambda)$. We say that $\mathscr{F}$ is *micro-supported on C* if for any C-transversal pair of morphisms of smooth k-schemes $(g, f) : Y \longleftarrow U \longrightarrow X$, the map $g : U \longrightarrow Y$ is universally locally acyclic with respect to $f^*\mathscr{F}$.

The following theorem is due to Beilinson [**Bei16**, Th. 1.3]

Theorem 2.3. — *Let $\mathscr{F}$ be an object in $D^b_c(X, \Lambda)$. Then, there exists a smallest closed conical subset $SS(\mathscr{F})$ in $\mathbb{T}^*X$ on which $\mathscr{F}$ is micro-supported. Furthermore, $SS(\mathscr{F})$ is equidimensional of dimension $\dim_k X$.*

The closed conical subset $SS(\mathscr{F})$ is called *the singular support of $\mathscr{F}$*.

2.4. — We assume that k is perfect. Let S be a smooth k-curve. Let $f : X \longrightarrow S$ be a morphism. Let x be a closed point of X and put $s = f(x)$. We say that x is an *at most C-isolated characteristic point* for $f : X \longrightarrow S$ if $f : X \backslash \{x\} \longrightarrow S$ is C-transversal. Any local trivialization of $\mathbb{T}^*S$ in a neighborhood of s gives rise to a local section of $\mathbb{T}^*X$ by applying $df : \mathbb{T}^*S \times_S X \longrightarrow \mathbb{T}^*X$. We abusively denote by df this section. Let A be a cycle in $\mathbb{T}^*X$ supported on C. Under the condition that x is an at most C-isolated characteristic point of $f : X \longrightarrow S$, the intersection of A and the cycle $[df]$ is supported at most at a single point in $\mathbb{T}^*_x X$. Since C is conical, the intersection number $(A, [df])_{\mathbb{T}^*X,x}$ is independent of the chosen local trivialization for $\mathbb{T}^*S$ in a neighborhood of s.

The following theorem is due to Saito [**Sai17b**, Th. 5.9].

Theorem 2.5. — *Let $\mathscr{F}$ be an object in $D^b_c(X, \Lambda)$. There exists a unique cycle $CC(\mathscr{F})$ in $\mathbb{T}^*X$ supported on $SS(\mathscr{F})$ such that for every étale morphism $h : U \longrightarrow X$, for every morphism $f : U \longrightarrow S$ with S a smooth k-curve, for every at most $h^\circ(SS(\mathscr{F}))$-isolated characteristic point $u \in U$ for $f : U \longrightarrow S$, we have the following Milnor type formula*

$$-\sum_i (-1)^i \mathrm{dimtot}(R^i\Phi_{\bar{u}}(h^*\mathscr{F}, f)) = (h^*(CC(\mathscr{F})), [df])_{T^*U,u}, \tag{2.5.1}$$

where $R\Phi_{\bar{u}}(h^*\mathscr{F}, f)$ denotes the stalk of the vanishing cycle of $h^*\mathscr{F}$ with respect to $f : U \longrightarrow S$ at the geometric point $\bar{u} \longrightarrow U$ above u.

The cycle $CC(\mathscr{F})$ is called the *characteristic cycle* of $\mathscr{F}$.

Example 2.6. — Assume that X is a smooth k-curve. Let U be an open dense subscheme of X and $j : U \to X$ the canonical injection. Let $\mathscr{F}$ be a locally constant and constructible sheaf of Λ-modules on U. Then we have

$$CC(j_!\mathscr{F}) = -\mathrm{rank}_\Lambda \mathscr{F} \cdot [\mathbb{T}^*_X X] - \sum_{x\in X\setminus U} \mathrm{dimtot}_x(\mathscr{F}) \cdot [\mathbb{T}^*_x X]$$

The following facts are due to Saito [Sai17b, 5.13, 5.14].

Proposition 2.7. — (i) *Let $\mathscr{F}$ be an object in $D^b_c(X, \Lambda)$. If $\mathscr{F}$ is perverse, then, the support of $CC(\mathscr{F})$ is $SS(\mathscr{F})$ and the coefficients of $CC(\mathscr{F})$ are positive.*

(ii) *Let $f : Y \longrightarrow X$ be a separated morphism of smooth k-schemes. For every object $\mathscr{G}$ in $D^b_c(Y, \Lambda)$, we have $CC(Rf_*\mathscr{G}) = CC(Rf_!\mathscr{G})$.*

2.8. — Assume that C has pure dimension $\dim_k X$. Let $f : Y \longrightarrow X$ be a C-transversal morphism of smooth connected k-schemes. We say that $f : Y \longrightarrow X$ is *properly C-transversal* if every irreducible component of $Y \times_X C$ has dimension $\dim_k Y$. Assume that $f : Y \longrightarrow X$ is properly C-transversal. Consider the diagram

$$\mathbb{T}^*Y \xleftarrow{df} Y \times_X \mathbb{T}^*X \xrightarrow{\mathrm{pr}_2} \mathbb{T}^*X$$

Let A be a cycle in $\mathbb{T}^*X$ supported on C. We put

$$f^! A := (-1)^{\dim_k X - \dim_k Y} df_* \mathrm{pr}_2^* A$$

The following compatibility of the characteristic cycle with properly transversal morphism is due to Saito [Sai17b, Th. 7.6]

Theorem 2.9. — *Let $\mathscr{F}$ be an object in $D^b_c(X, \Lambda)$. Let $f : Y \longrightarrow X$ be a morphism of smooth connected k-schemes such that $f : Y \longrightarrow X$ is properly $SS(\mathscr{F})$-transversal. Then, we have*

$$f^!(CC(\mathscr{F})) = CC(f^*\mathscr{F}).$$

Assume that C has pure dimension $\dim_k X$. Let $f : X \longrightarrow Y$ be a C-transversal morphism of smooth connected k-schemes with $\dim_k Y \leqslant \dim_k X$. We say that $f : X \longrightarrow Y$ is *properly C-transversal* if for every closed point y in Y, the fibre $y \times_Y C$ has dimension $\dim_k X - \dim_k Y$. The following theorem is due to Saito [Sai17c, Th. 2.2.3].

Theorem 2.10. — *Let $f : X \longrightarrow S$ be a quasi-projective morphism with S a smooth connected k-curve. Let $\bar{\eta}$ be a geometric generic point of S. Let $s \in S$ be a closed point. Let $\mathscr{F}$ be an object in $D^b_c(X, \Lambda)$. Assume that $f : X \longrightarrow S$ is proper on the support of $\mathscr{F}$ and properly $SS(\mathscr{F})$-transversal over a dense open subset in S. Then, we have*

$$(CC(\mathscr{F}), df)_{\mathbb{T}^*X, X_s} = -\sum_{i\in\mathbb{Z}} (-1)^i (\dim_\Lambda (R^i f_*\mathscr{F})_{\bar{\eta}} - \dim_\Lambda (R^i f_*\mathscr{F})_s + \mathrm{sw}_s(R^i f_*\mathscr{F}))$$

2.11. — Let $f : Y \longrightarrow X$ be a separated morphism of smooth k-schemes. Let $\mathscr{F}$ be an object in $D_c^b(X, \Lambda)$. There is a canonical morphism

$$c_{f,\mathscr{F}} : f^*\mathscr{F} \otimes_\Lambda^L Rf^!\Lambda \longrightarrow Rf^!\mathscr{F} \tag{2.11.1}$$

obtained by adjunction from the composition

$$Rf_!(f^*\mathscr{F} \otimes_\Lambda^L Rf^!\Lambda) \xrightarrow{\sim} \mathscr{F} \otimes_\Lambda^L Rf_!Rf^!\Lambda \longrightarrow \mathscr{F}$$

where the first arrow is the projection formula and where the second arrow is induced by the adjunction $Rf_!Rf^!\Lambda \longrightarrow \Lambda$.

The following proposition is due to Saito [Sai17b, Prop. 8.13].

Proposition 2.12. — *Let $\mathscr{F}$ be an object in $D_c^b(X, \Lambda)$. Let $f : Y \longrightarrow X$ be a separated morphism of smooth k-schemes. If $f : Y \longrightarrow X$ is $SS(\mathscr{F})$-transversal, then the canonical morphism (2.11.1) is an isomorphism.*

Corollary 2.13. — *Let D be an effective Cartier divisor in X. Put $U := X \backslash D$. Let Z be a smooth connected closed subscheme in X. Let $i : Z \longrightarrow X$ be the canonical injection. Assume that $i^*D = D \times_X Z$ is a Cartier divisor in Z. Consider the following cartesian diagram*

$$\begin{array}{ccc} U_Z & \xrightarrow{i'} & U \\ {\scriptstyle j'}\downarrow & \square & \downarrow{\scriptstyle j} \\ Z & \xrightarrow[i]{} & X \end{array}$$

Then, for every locally constant and constructible sheaf of Λ-modules $\mathscr{F}$ on U such that $i : Z \longrightarrow X$ is $SS(j_!\mathscr{F})$-transversal, there is an isomorphism in $D_c^b(X, \Lambda)$

$$i^*Rj_*\mathscr{F} \xrightarrow{\sim} Rj'_*i'^*\mathscr{F}. \tag{2.13.1}$$

Proof. — From proposition 2.7, we have $SS(j_!\mathscr{F}) = SS(Rj_*\mathscr{F})$. In particular, $i : Z \longrightarrow X$ is $SS(Rj_*\mathscr{F})$-transversal. Hence, Proposition 2.12 implies that the canonical morphism

$$c_{i,\mathscr{F}} : i^*Rj_*\mathscr{F} \otimes_\Lambda^L Ri^!\Lambda \longrightarrow Ri^!Rj_*\mathscr{F} \tag{2.13.2}$$

is an isomorphism. By proper base change theorem [SGA4III, XVIII 3.1.13], the following canonical morphism

$$Ri^!Rj_*\mathscr{F} \longrightarrow Rj'_*Ri'^!\mathscr{F} \tag{2.13.3}$$

is an isomorphism. Hence,

$$i^*Rj_*\mathscr{F} \otimes_\Lambda^L Ri^!\Lambda \cong Rj'_*Ri'^!\mathscr{F}$$

By the relative purity [Fu11, Cor. 8.5.6], we have

$$Ri^!\Lambda \cong \Lambda(-c)[-2c] \quad \text{and} \quad Ri'^!\mathscr{F} \cong i'^*\mathscr{F}(-c)[-2c]$$

where $c = \dim_k X - \dim_k Z$. Hence, we deduce that

$$i^*Rj_*\mathscr{F}(-c)[-2c] \cong Rj'_*i'^*\mathscr{F}(-c)[-2c],$$

and proposition 2.13 follows. □

2.14. — Let D be a smooth connected divisor in X. Put $U := X\backslash D$. Let $j : U \longrightarrow X$ be the canonical injection. Let ξ be the generic point of D, K the fraction field of the henselization $X_{(\xi)}$ of X at ξ, $\eta = \mathrm{Spec}(K)$ the generic point of $X_{(\xi)}$, $\overline{K}$ a separable closure of K, F the residue field of the ring of integer $\mathscr{O}_K$ of K, $\overline{F}$ the residue field of the ring of integer $\mathscr{O}_{\overline{K}}$ of $\overline{K}$, and let G_K be the Galois group of $\overline{K}$ over K. Let $\mathscr{F}$ be a locally constant and constructible sheaf of Λ-modules on U.

We suppose that the ramification of $\mathscr{F}$ along D is non-degenerate. In a nutshell, this means that étale locally along D, the sheaf $\mathscr{F}$ is a direct sum of sheaves of Λ-modules which are isoclinic at η and whose ramification at a closed point of D is controlled by the ramification at η. See [**Sai17a**, Def. 3.1] for details. Under the non-degenerate condition, the characteristic cycle of $j_!\mathscr{F}$ can be expressed in terms of ramification theory. Let us explain how.

Let M be the finitely generated Λ-module with continuous G_K-action corresponding to $\mathscr{F}|_\eta$. Let

$$M = \bigoplus_{r \in \mathbb{Q}_{\geqslant 1}} M^{(r)}$$

be the slope decomposition of M. For $r \in \mathbb{Q}_{>1}$, let

$$M^{(r)} = \bigoplus_{\chi} M^{(r)}_{\chi}$$

be the central character decomposition of $M^{(r)}$, for $r > 1$. Let ψ be a non-trivial character $\psi : \mathbb{F}_p \longrightarrow \Lambda^\times$. For $r \in \mathbb{Q}_{>1}$, the graded piece G^r_K/G^{r+}_K is abelian and p-torsion. Hence, each central character χ uniquely factors as $\chi : G^r_K/G^{r+}_K \longrightarrow \mathbb{F}_p \xrightarrow{\psi} \Lambda^\times$. We also denote by χ the induced character $G^r_K/G^{r+}_K \longrightarrow \mathbb{F}_p$ and by

$$\mathrm{char}(\chi) : \mathfrak{m}^r_{\overline{K}}/\mathfrak{m}^{r+}_{\overline{K}} \longrightarrow \Omega^1_{\mathscr{O}_K} \otimes_{\mathscr{O}_K} \overline{F}$$

the characteristic form of χ. Let F_χ be a finite extension of F contained in $\overline{F}$ such that $\mathrm{char}(\chi)$ is defined over F_χ. The characteristic form $\mathrm{char}(\chi)$ defines a line in $\mathbb{T}^*X \times_X \mathrm{Spec} F_\chi$. Let $\overline{L}_\chi$ be the closure of the image of L_χ in $\mathbb{T}^*X$. If the ramification of $j_!\mathscr{F}$ is non degenerate along D, we have [**Sai17b**, Th. 7.14]:

$$CC(j_!\mathscr{F}) = (-1)^{\dim_k X}\Bigg(\mathrm{rank}_\Lambda \mathscr{F} \cdot [\mathbb{T}^*_X X] + \dim_\Lambda M^{(1)} \cdot [\mathbb{T}^*_D X] + \sum_{r>1}\sum_{\chi} \frac{r \cdot \dim_\Lambda M^{(r)}_\chi}{[F_\chi : F]}[\overline{L}_\chi]\Bigg). \quad (2.14.1)$$

When $\mathscr{F}$ is not trivial, the non-degeneracy of the ramification of $\mathscr{F}$ along D implies that for every geometric point $\bar{x} \longrightarrow D$, the fibre $SS(j_!\mathscr{F}) \times_X \bar{x}$ is a finite union of 1-dimensional vector spaces in $\mathbb{T}^*_{\bar{x}}X$.

In general, the ramification of $\mathscr{F}$ along D becomes non degenerate after removing a codimension one closed subset in D.

3. Semi-continuity for the largest slope

3.1. — Let k be a perfect field of characteristic $p > 0$. Let Λ be a finite field of characteristic $\ell \neq p$. Let X be a smooth k-scheme. Let D be a smooth connected divisor in X. Put $U := X\backslash D$. Let $j : U \longrightarrow X$ be the canonical injection. Let ξ the generic point of D, K the fraction field of the henselization $X_{(\xi)}$ of X at ξ, $\eta = \mathrm{Spec}(K)$ the generic point of $X_{(\xi)}$, $\overline{K}$ an algebraic closure of K, and let G_K be the Galois group of $\overline{K}$ over K.

3.2. — Let $\mathscr{F}$ be a locally constant and constructible sheaf of Λ-modules on U. The restriction $\mathscr{F}|_\eta$ corresponds to a finitely generated Λ-module with continuous G_K-action. Let $r(D, \mathscr{F})$ be the largest slope of $\mathscr{F}|_\eta$. Set $R_D(\mathscr{F}) := r(D, \mathscr{F}) \cdot D$ and we call it the *largest slope divisor* of $\mathscr{F}$. Let $r_{\log}(D, \mathscr{F})$ be the largest logarithmic slope of $\mathscr{F}|_\eta$. Set $R_{D,\log}(\mathscr{F}) = r_{\log}(D, \mathscr{F}) \cdot D$ and we call it the *largest logarithmic slope divisor* of $\mathscr{F}$. Similarly, we denote by $\mathrm{dimtot}_D(\mathscr{F})$ the total dimension of $\mathscr{F}|_\eta$, by

$$\mathrm{DT}_D(\mathscr{F}) := \mathrm{dimtot}_D(\mathscr{F}) \cdot D$$

the *total dimension divisor* of $\mathscr{F}$, by $\mathrm{sw}_D(\mathscr{F})$ the Swan conductor of $\mathscr{F}|_\eta$ and by

$$\mathrm{SW}_D(\mathscr{F}) := \mathrm{sw}_D(\mathscr{F}) \cdot D$$

the *Swan divisor* of $\mathscr{F}$.

In [**HY17, Hu17**], the authors studied the behavior of the total dimension divisors and the Swan divisors by pull-back, and deduced several semi-continuity properties. In this section, we pursue a similar study for the largest slope divisor and the largest logarithmic slope divisor. For an analogue for differential systems in characteristic 0, we refer to [**And07**].

Proposition 3.3. — *Let C be a smooth k-curve. Let $i : C \longrightarrow X$ be an immersion such that C meets D transversely at a closed point x. We put $C_0 = C\backslash\{x\}$. Then,*

(1) *If $i : C \longrightarrow X$ is $SS(j_!\mathscr{F})$-transversal at x and if the ramification of $\mathscr{F}$ along D is non-degenerate at x, then*

$$r(x, \mathscr{F}|_{C_0}) = r(D, \mathscr{F}).$$

(2) *In general, we have*

$$r(x, \mathscr{F}|_{C_0}) \leqslant r(D, \mathscr{F}).$$

Proof. — This is an étale local question. We can thus assume that k is algebraically closed. Since the slopes are unchanged by scalar extensions, we can assume that Λ contains a primitive p-th root of unity.

(1) Since the ramification of $\mathscr{F}$ along D is non-degenerate at x, at the cost of replacing X by an étale neighborhood of x, we can suppose that $\mathscr{F}$ is a direct sum of locally constant and constructible sheaves $\{\mathscr{F}_\alpha\}_{\alpha\in I}$ of Λ-modules on U such that for every $\alpha \in I$, the ramification of $\mathscr{F}_\alpha$ along D is non-degenerate and $\mathscr{F}_\alpha|_\eta$ is isoclinic. Since $i : C \longrightarrow X$ is $SS(j_!\mathscr{F})$-transversal at x, it is $SS(j_!\mathscr{F}_\alpha)$-transversal at x for every $\alpha \in I$. Hence, proposition 3.3 (1) is a direct consequence of [**Sai17a**, Prop. 2.22.2].

(2) We argue by contradiction. Suppose that

$$r(x, \mathscr{F}|_{C_0}) = r(D, \mathscr{F}) + \epsilon$$

for some $\epsilon \in \mathbb{Q}_{>0}$. At the cost of replacing X by an étale neighborhood of x, we can suppose that there exists a smooth morphism $f : X \longrightarrow T$ with T a smooth k-curve, and a closed point $t \in T$ such that $D = f^{-1}(t)$. Let $0 < \epsilon' < \epsilon$ be a rational number such that $r(D, \mathscr{F}) + \epsilon'$ is not a slope for $\mathscr{F}|_{C_0}$ at x. From lemma 1.8, there exists a locally constant and constructible sheaf of Λ-modules $\mathscr{N}$ on the generic point of $T_{(t)}$ such that $\mathscr{N}$ is isoclinic with slope $r(D, \mathscr{F}) + \epsilon'$. At the cost of replacing T by an étale neighborhood of t, we can suppose that $\mathscr{N}$ extends to $T \backslash \{t\}$. We still denote by $\mathscr{N}$ this extension. Let N be the rank of $\mathscr{N}$. Let us consider the sheaf $f^*\mathscr{N} \otimes_\Lambda \mathscr{F}$ on $U = X \backslash D$. Since $f : X \longrightarrow T$ is smooth, it is transversal with respect to the extension by 0 of $\mathscr{N}$ to T. From [Sai17a, Prop. 2.22.2], we deduce that $f^*\mathscr{N}|_\eta$ is isoclinic with slope $r(D, \mathscr{F}) + \epsilon'$. Hence, by lemma 1.9 (1), we have

$$\mathrm{dimtot}_D(f^*\mathscr{N} \otimes_\Lambda \mathscr{F}) = N \cdot \mathrm{rank}_\Lambda \mathscr{F} \cdot (r(D, \mathscr{F}) + \epsilon').$$

Notice that $f|_C : C \longrightarrow T$ is étale at x since C and D intersect transversely at x. Hence, the restriction $\mathscr{N}|_{C_0}$ is isoclinic with slope $r(D, \mathscr{F}) + \epsilon'$ at x. Lemma 1.9 (1) gives again

$$\mathrm{dimtot}_x((f^*\mathscr{N} \otimes_\Lambda \mathscr{F})|_{C_0}) > N \cdot \dim_\Lambda \mathscr{F} \cdot (r(D, \mathscr{F}) + \epsilon')$$

Hence, we have

$$\mathrm{dimtot}_x((f^*\mathscr{N} \otimes_\Lambda \mathscr{F})|_{C_0}) > \mathrm{dimtot}_D(f^*\mathscr{N} \otimes_\Lambda \mathscr{F})$$

This contradicts [HY17, Prop. 4.1]. □

Corollary 3.4. — *Let C be a smooth k-curve. Let $i : C \longrightarrow X$ be an immersion such that C meets D transversely at a single point x. We put $C_0 = C \backslash \{x\}$. Then, we have*

$$r_{\log}(x, \mathscr{F}|_{C_0}) \leqslant r_{\log}(D, \mathscr{F}).$$

Proof. — From (1.5.2), we have $r_{\log}(D, \mathscr{F}) + 1 \geqslant r(D, \mathscr{F})$. Since C is a curve, we have furthermore $r_{\log}(x, \mathscr{F}) + 1 = r(x, \mathscr{F}|_{C_0})$. Hence, corollary 3.4 is an immediate consequence of proposition 3.3. □

Proposition 3.5. — *Let Y be a smooth connected k-scheme. Let $g : Y \longrightarrow X$ be a separated morphism such that $E := D \times_X Y$ is an irreducible smooth divisor in Y. We put $V = Y \backslash E$ and let $j' : V \longrightarrow Y$ be the canonical injection. Then, we have*

$$r(E, \mathscr{F}|_V) \leqslant r(D, \mathscr{F})$$

Proof. — It is enough to treat the case where k is algebraically closed. The morphism $g : Y \longrightarrow X$ is a composition of the graph embedding $\Gamma_g : Y \longrightarrow Y \times_k X$ and the projection $\mathrm{pr}_2 : Y \times_k X \longrightarrow X$. Since pr_2 is smooth, proposition 2.22.2 from [Sai17a] implies

$$r(D \times_k Y, \mathscr{F}|_{U \times_k Y}) = r(D, \mathscr{F}).$$

We are thus reduced to treat the case where $g : Y \longrightarrow X$ is a closed immersion. If Y is a curve, the condition that E is smooth implies that Y is transverse to D. Hence, proposition 3.3 (2) allows to conclude. Suppose that $\dim_k Y > 1$. Let x be a closed point in E such that the ramification of $\mathscr{F}|_V$ is non-degenerate at x. In particular, the fibre of $SS(j'_!\mathscr{F}|_V)$ above x is a finite number of lines $L_1, \dots, L_d \subset \mathbb{T}^*_x Y$. Since E is smooth, $\mathbb{T}_x E = \mathbb{T}_x Y \cap \mathbb{T}_x D \subset \mathbb{T}_x X$ is a hyperplane in $\mathbb{T}_x Y$. Since $\dim_k Y > 1$ and since k is infinite, the union of $\mathbb{T}_x E$ with the $\operatorname{Ker} L_i, i = 1 \dots d$ is a strict closed subset in $\mathbb{T}_x Y$. Pick $L \in \mathbb{T}_x Y$ in its complement. Let C be a smooth curve in Y passing through x such that $T_x C = k \cdot L$. Then, the curve C is transverse to D and the immersion $i : C \longrightarrow Y$ is $SS(j'_!\mathscr{F}|_Y)$-transversal at x. We put $C_0 = C \backslash \{x\}$. By Proposition 3.3 (1), we have $r(x, \mathscr{F}|_{C_0}) = r(E, \mathscr{F}|_V)$. By Proposition 3.3 (2), we have $r(x, \mathscr{F}|_{C_0}) \leqslant r(D, \mathscr{F})$. Hence, we get $r(E, \mathscr{F}|_V) \leqslant r(D, \mathscr{F})$. □

Proposition 3.6. — *Let Y be a smooth connected k-scheme. Let $g : Y \longrightarrow X$ be a separated morphism such that $E := D \times_X Y$ is an irreducible smooth divisor in Y. We put $V = Y \backslash E$ and let $j' : V \longrightarrow Y$ be the canonical injection. Then, we have*

$$r_{\log}(E, \mathscr{F}|_V) \leqslant r_{\log}(D, \mathscr{F})$$

Proof. — It is enough to treat the case where k is algebraically closed. The morphism $g : Y \longrightarrow X$ is a composition of the graph embedding $\Gamma_g : Y \longrightarrow Y \times_k X$ and the projection $\mathrm{pr}_2 : Y \times_k X \longrightarrow X$. Since pr_2 is smooth, it is $SS(j_!\mathscr{F})$-transversal. Hence, proposition 1.22 from [Sai08] implies

$$r_{\log}(D \times_k Y, \mathscr{F}|_{U \times_k Y}) = r_{\log}(D, \mathscr{F}).$$

We are thus reduced to treat the case where $g : Y \longrightarrow X$ is a closed immersion.

We argue by contradiction. Suppose that

$$r_{\log}(x, \mathscr{F}|_{C_0}) = r_{\log}(D, \mathscr{F}) + \epsilon$$

for some $\epsilon \in \mathbb{Q}_{>0}$. At the cost of replacing X by an étale neighborhood of the generic point of E, we can suppose that there exists a smooth morphism $f : X \longrightarrow T$ with T a smooth k-curve, and a closed point $t \in T$ such that $D = f^{-1}(t)$. Let x be a closed point in E. Since E is smooth, $\mathbb{T}_x E = \mathbb{T}_x Y \cap \mathbb{T}_x D \subset \mathbb{T}_x X$ is a hyperplane in $\mathbb{T}_x Y$. Hence, the tangent map of $f|_Y$ at x is surjective. Thus, the restriction $f|_Y : Y \longrightarrow T$ is smooth in a neighborhood of E. At the cost of replacing X by a Zariski neighborhood of E, we can suppose that $f|_Y : Y \longrightarrow T$ is smooth.

Let $0 < \epsilon' < \epsilon$ be a rational number such that $r_{\log}(D, \mathscr{F}) + \epsilon'$ is not a logarithmic slope for $\mathscr{F}|_{C_0}$ at x. From lemma 1.8, there exists a locally constant and constructible sheaf of Λ-modules $\mathscr{N}$ on the generic point of $T_{(t)}$ such that $\mathscr{N}$ is isoclinic with logarithmic slope $r_{\log}(D, \mathscr{F}) + \epsilon'$. At the cost of replacing T by an étale neighborhood of t, we can suppose that $\mathscr{N}$ extends to $T \backslash \{t\}$. We still denote by $\mathscr{N}$ this extension. Let N be the rank of $\mathscr{N}$. Let ξ' be the generic point of E and let η' be the generic point of the henselianization of E at ξ'. Since $f : X \longrightarrow T$ and $f|_Y : Y \longrightarrow T$ are smooth, $f^*\mathscr{N}|_\eta$ and $f|_Y^*\mathscr{N}|_{\eta'}$ are isoclinic with slope $r_{\log}(D, \mathscr{F}) + \epsilon'$. By Lemma 1.9 (2), we deduce

$$\mathrm{sw}_D(f^*\mathscr{N} \otimes_\Lambda \mathscr{F}) = N \cdot \mathrm{rank}_\Lambda \mathscr{F} \cdot (r_{\log}(D, \mathscr{F}) + \epsilon') \tag{3.6.1}$$

$$\mathrm{sw}_E(f|_Y^*\mathscr{N} \otimes_\Lambda \mathscr{F}|_V) > N \cdot \mathrm{rank}_\Lambda \mathscr{F} \cdot (r_{\log}(D, \mathscr{F}) + \epsilon') \tag{3.6.2}$$

Thus,

$$\mathrm{sw}_E(f|_Y^*\mathscr{N} \otimes_\Lambda \mathscr{F}|_V) > \mathrm{sw}_D(f^*\mathscr{N} \otimes_\Lambda \mathscr{F}). \tag{3.6.3}$$

Let $\mathcal{PC}(Y)$ be the set of triples $(C, i : C \longrightarrow Y, x)$, where C is a smooth k-curve, $i : C \longrightarrow Y$ an immersion and where $x = C \cap E$ is a closed point of E. By [**Hu17**, Th. 6.5], we have

$$\mathrm{sw}_E(f|_Y^*\mathscr{N} \otimes_\Lambda \mathscr{F}|_V) = \sup_{\mathcal{PC}(Y)} \frac{\mathrm{sw}_x((f|_Y^*\mathscr{N} \otimes_\Lambda \mathscr{F}|_V)|_C)}{(E,C)_x} \tag{3.6.4}$$

Since $(f|_Y^*\mathscr{N} \otimes_\Lambda \mathscr{F}|_V)|_C) = (f^*\mathscr{N} \otimes_\Lambda \mathscr{F})|_C$ and since $(E,C)_x = (D,C)_x$, proposition 6.2 from [**Hu17**] gives for every $(C, i : C \longrightarrow Y, x) \in \mathcal{PC}(Y)$,

$$\frac{\mathrm{sw}_x((f|_Y^*\mathscr{N} \otimes_\Lambda \mathscr{F}|_V)|_C)}{(E,C)_x} \leqslant \mathrm{sw}_D(f^*\mathscr{N} \otimes_\Lambda \mathscr{F})$$

Thus

$$\mathrm{sw}_E(f|_Y^*\mathscr{N} \otimes_\Lambda \mathscr{F}|_V) \leqslant \mathrm{sw}_D(f^*\mathscr{N} \otimes_\Lambda \mathscr{F})$$

This contradicts (3.6.3), which finishes the proof of proposition 3.6. □

4. Characteristic cycle of a sheaf after twist

Lemma 4.1. — *Let k be an algebraically closed field of characteristic $p > 0$. Let X be a smooth connected k-scheme of dimension $d \geqslant 2$. Let D be a smooth connected divisor in X. Let x be a closed point in D. Let $g : X \longrightarrow \mathbb{A}^1_k$ be a smooth morphism such that $g|_D : D \longrightarrow \mathbb{A}^1_k$ is smooth. Then, at the cost of replacing X by a suitable Zariski neighborhood of x, there exists a map $h : X \longrightarrow \mathbb{A}^{d-1}_k$ such that*

(1) *g is the composition of h and the projection $\mathrm{pr}_1 : \mathbb{A}^{d-1}_k \longrightarrow \mathbb{A}^1_k$;*
(2) *$h : X \longrightarrow \mathbb{A}^{d-1}_k$ is smooth and the restriction $h|_D : D \longrightarrow \mathbb{A}^{d-1}_k$ is étale.*

Proof. — Suppose that the image of x in $\mathbb{A}^1_k$ is the origin $o \in \mathbb{A}^1_k$. Consider the canonical maps

$$\mathbb{T}^*_o\mathbb{A}^1_k \longrightarrow \mathbb{T}^*_x X \longrightarrow \mathbb{T}^*_x D \tag{4.1.1}$$

We denote by $t_1, \dots, t_d$ a basis of $\mathbb{T}^*_x X$, by $s_1, \dots, s_{d-1}$ a basis of $\mathbb{T}^*_x D$ such that t_i maps to s_i for $1 \leqslant i \leqslant d-1$ and such that t_d maps to 0. Let l be a basis of the line $\mathbb{T}^*_o\mathbb{A}^1_k$. Let $\sum_{1 \leqslant i \leqslant d} \lambda_i t_i$ $(\lambda_i \in k)$ be the image of l by $\mathbb{T}^*_o\mathbb{A}^1_k \longrightarrow \mathbb{T}^*_x X$. Since $g|_D : D \longrightarrow \mathbb{A}^1_k$ is smooth, the composition (4.1.1) is injective. Hence, $(\lambda_1, \dots \lambda_{d-1}) \neq 0$. We may assume that $\lambda_1 \neq 0$. Let $\widetilde{t}_2, \dots \widetilde{t}_{d-1}$ be regular functions on X defined around x and lifting $t_2, \dots, t_{d-1}$ respectively. We have the following morphism of k-algebras

$$k[y, x_2, \dots, x_{d-1}] \longrightarrow \mathscr{O}_{X,x} \quad y \mapsto g^*(y), \quad x_i \mapsto \widetilde{t}_i. \tag{4.1.2}$$

In a neighbourhood of x, the map (4.1.2) induces a map

$$h : X \longrightarrow \mathbb{A}^{d-1}_k.$$

It induces an injection $dh_x : \mathbb{T}^*_o\mathbb{A}^{d-1} \longrightarrow \mathbb{T}^*_x X$ and an isomorphism $(dh|_D)_x : \mathbb{T}^*_o\mathbb{A}^{d-1}_k \longrightarrow \mathbb{T}^*_x D$. Hence, $h : X \longrightarrow \mathbb{A}^{d-1}_k$ is smooth at x and $h|_D : \mathbb{A}^{d-1}_k \longrightarrow D$ is étale at x. By construction, the composition of h and pr_1 is g. □

4.2. — In the rest of this section, for a smooth scheme X over a field and a closed conical subscheme C of $\mathbb{T}^*X$, the *base locus* $B(C)$ of C in X denotes the scheme theoretic image of C in X by the canonical projection $\pi : \mathbb{T}^*X \to X$.

The next two lemmas are corollaries of [**Bei16**, 4.12]. See also [**Sai17b**, 5.19].

Lemma 4.3. — *Let k be an algebraically closed field of characteristic $p > 2$. Let X be a smooth connected k-scheme of dimension $d \geqslant 2$. Let D be a smooth connected divisor in X. Let C be an integral closed conical subscheme in $\mathbb{T}^*X$ of dimension $\dim_k X$. Assume that $C \neq \mathbb{T}^*_D X$ and that the base locus $B(C)$ is contained in D. Let $(x,\omega) \in C$ such that*

(1) *$(x,\omega) \notin \mathbb{T}^*_D X$.*

(2) *C is smooth in a neighbourhood of (x,ω).*

Then, at the cost of replacing X by a Zariski neighbourhood of x, there exists a smooth morphism $g : X \longrightarrow \mathbb{A}^1_k$ such that

(3) *The induced section $dg : X \longrightarrow \mathbb{T}^*X$ intersects C transversally at (x,ω).*

(4) *The restriction $g|_D : D \longrightarrow \mathbb{A}^1_k$ is smooth.*

Proof. — From [**Sai17b**, Prop 5.19], after replacing X by a Zariski neighbourhood of x, there exists a morphism $g : X \longrightarrow \mathbb{A}^1_k$ such that the induced section $dg : X \longrightarrow \mathbb{T}^*X$ intersects C transversally at (x,ω). Since $\omega \neq 0$, the morphism $g : X \longrightarrow \mathbb{A}^1_k$ is smooth at x. Hence, $g : X \longrightarrow \mathbb{A}^1_k$ satisfies condition (3). From (1), the composition

$$\mathbb{T}^*_{g(x)}\mathbb{A}^1_k \longrightarrow \mathbb{T}^*_x X \longrightarrow \mathbb{T}^*_x D$$

is injective. Hence, the restriction $g|_D : D \longrightarrow \mathbb{A}^1_k$ is smooth at x. □

Lemma 4.4. — *Let k be an algebraically closed field of characteristic 2. Let X be a smooth connected k-scheme of dimension $\geqslant 2$. Let D be a smooth connected divisor in X. Let C be an integral closed conical subscheme in $\mathbb{T}^*X$ of dimension $\dim_k X$. Assume that $C \neq \mathbb{T}^*_D X$ and that the base locus $B(C)$ is contained in D. Let $(x,\omega) \in C$ such that*

(1) *$(x,\omega) \notin \mathbb{T}^*_D X$.*

(2) *C is smooth in a neighbourhood of (x,ω).*

Assume that there are no regular functions defined in a neighbourhood of x as in (3),(4) of Lemma 4.3. Then, at the cost of replacing $X \times_k \mathbb{A}^1_k$ by a Zariski neighbourhood of $(x,0)$, there exists a smooth morphism $g : X \times_k \mathbb{A}^1_k \longrightarrow \mathbb{A}^1_k$ such that

(3) *The induced section $dg : X \times_k \mathbb{A}^1_k \longrightarrow \mathbb{T}^*(X \times_k \mathbb{A}^1_k)$ intersects $\mathrm{pr}_1^\circ C$ transversally at $((x,0), \mathrm{pr}_1^*\omega)$.*

(4) *The restriction $g|_{D\times_k \mathbb{A}^1_k} : D \times_k \mathbb{A}^1_k \longrightarrow \mathbb{A}^1_k$ is smooth.*

Proof. — From the proofs of [**Sai17b**, Th. 5.18] and [**Sai17b**, Prop. 5.19, Lemma 5.20], if there are no regular functions defined in a neighborhood of x as in (3) of lemma 4.3, then at the cost of replacing $X \times_k \mathbb{A}^1_k$ by a Zariski neighbourhood of $(x,0)$, there exists a smooth morphism $g : X \times_k \mathbb{A}^1_k \longrightarrow \mathbb{A}^1_k$ satisfying condition (3). From (1), the composition

$$\mathbb{T}^*_{g(x\times_k 0)}\mathbb{A}^1_k \longrightarrow \mathbb{T}^*_{(x\times_k 0)}(X \times_k \mathbb{A}^1_k) \longrightarrow \mathbb{T}^*_{(x\times_k 0)}(D \times_k \mathbb{A}^1_k)$$

is injective. Hence, the restriction $g|_{D\times_k\mathbb{A}^1_k} : D\times_k \mathbb{A}^1_k \longrightarrow \mathbb{A}^1_k$ is smooth at x. □

Theorem 4.5. — *Let k be a perfect field of characteristic $p > 0$. Let X be a smooth connected k-scheme. Put $d = \dim_k X$. Let D be a smooth connected divisor in X. Put $U := X\backslash D$. Let $j : U \longrightarrow X$ be the canonical injection. Let S be a connected smooth k-curve. Let s be a closed point in S. Put $V := S\backslash\{s\}$. Let $f : X \longrightarrow S$ be a smooth separated morphism such that $D = f^{-1}(s)$. Let Λ be a finite field of characteristic $\ell \neq p$. Let $\mathscr{F}$ be a non-zero locally constant and constructible sheaf of Λ-modules on U. Let $\mathscr{N}$ be a locally constant and constructible sheaf of Λ-modules on V of rank N. Suppose that $\mathscr{N}$ has pure logarithmic slope $r_{\log}$ at s with $r_{\log} > r(D,\mathscr{F}) - 1$. Then, we have*

$$\begin{aligned}
SS(j_!(\mathscr{F}\otimes_\Lambda f^*\mathscr{N})) &= SS(j_!f^*\mathscr{N}) = \mathbb{T}^*_X X \bigcup \mathbb{T}^*_D X\\
CC(j_!(\mathscr{F}\otimes_\Lambda f^*\mathscr{N})) &= \mathrm{rank}_\Lambda\mathscr{F}\cdot CC(j_!f^*\mathscr{N})\\
&= (-1)^d\big(N\cdot \mathrm{rank}_\Lambda\mathscr{F}\cdot[\mathbb{T}^*_X X] + (r_{\log}+1)N\cdot\mathrm{rank}_\Lambda\mathscr{F}\cdot[\mathbb{T}^*_D X]\big)
\end{aligned}$$

Proof. — We may assume that k is algebraically closed. If X is a k-curve, the theorem is nothing but lemma 1.9. We consider the case where $\dim_k X \geqslant 2$.

The complex $j_!(\mathscr{F}\otimes_\Lambda f^*\mathscr{N}))[d]$ is perverse. From proposition 2.7, we deduce that its singular support is the support of its characteristic cycle. Hence, we are left to prove

(4.5.1)
$$CC(j_!(\mathscr{F}\otimes_\Lambda f^*\mathscr{N})) = (-1)^d\big(N\cdot \mathrm{rank}_\Lambda\mathscr{F}\cdot[\mathbb{T}^*_X X] + (r_{\log}+1)N\cdot\mathrm{rank}_\Lambda\mathscr{F}\cdot[\mathbb{T}^*_D X]\big).$$

Let us write

$$CC(j_!(\mathscr{F}\otimes_\Lambda f^*\mathscr{N})) = (-1)^d N\cdot\mathrm{rank}_\Lambda\mathscr{F}\cdot[\mathbb{T}^*_X X] + \sum_{a\in A}(-1)^d n_a[C_a],$$

where A is a finite set, n_a is a positive integer and C_a is an integral closed conical subscheme of dimension d in $\mathbb{T}^*X$ with base locus $B(C_a) \subseteq D$. Let C be one of C_a's and let n be the coefficient of C in $CC(j_!(\mathscr{F}\otimes_\Lambda f^*\mathscr{N}))$. Suppose that C is distinct from $\mathbb{T}^*_D X$. We want to show that n is zero. To do this, we borrow an argument due to Beilinson (see the proof of Theorem 5.18 in [**Sai17b**]). Let $(x,\omega)\in C$ such that

(1) (x,ω) is not contained in any irreducible component of $CC(j_!(\mathscr{F}\otimes_\Lambda f^*\mathscr{N}))$ distinct from C.

(2) C is smooth in a neighbourhood of (x,ω).

Note in particular that $(x,\omega)\notin \mathbb{T}^*_D X$. Assume that $\mathrm{char}(k)\neq 2$. At the cost of replacing X by a Zariski neighbourhood of x, lemma 4.3 ensures that there exists a smooth morphism $g : X \longrightarrow \mathbb{A}^1_k$ such that

(3) The induced section $dg : X \longrightarrow \mathbb{T}^*X$ intersects C transversally at (x,ω).

(4) The restriction $g|_D : D \longrightarrow \mathbb{A}^1_k$ is smooth.

At the cost of replacing X by a Zariski neighborhood of x, lemma 4.1 ensures that $g = \mathrm{pr}_1\circ h$ where $\mathrm{pr}_1 : \mathbb{A}^{d-1}_k \longrightarrow \mathbb{A}^1_k$ is the first projection and $h : X \longrightarrow \mathbb{A}^{d-1}_k$ is a smooth morphism such that $h(x) = 0$ and $h|_D : D \longrightarrow \mathbb{A}^{d-1}_k$ is étale. For any closed point $z\in\mathbb{A}^{d-1}_k$ in a sufficiently small neighbourhood of 0, $T_z = h^{-1}(z)$ is a smooth

k-curve transverse to D. Notice that $f|_{T_z} : T_z \longrightarrow S$ is étale in a neighborhood of $T_z \bigcap D$. From lemma 1.9 and proposition 3.3, for every closed point $w \in T_z \bigcap D$, we have

$$\text{dimtot}_w(j_!(\mathscr{F} \otimes_\Lambda f^*\mathscr{N})|_{T_z}) = (r_{\log} + 1)N \cdot \text{rank}_\Lambda \mathscr{F}. \tag{4.5.2}$$

In particular, the left-hand side of (4.5.2) does not depend on the closed point w. Applying Deligne and Laumon's semi-continuity theorem for Swan conductors [**Lau81**, Th. 2.1.1] to the sheaf $j_!(\mathscr{F} \otimes_\Lambda f^*\mathscr{N})$ on the relative curve $h : X \longrightarrow \mathbb{A}^{d-1}_k$, we obtain that $h : X \longrightarrow \mathbb{A}^{d-1}_k$ is universally locally acyclic with respect to $j_!(\mathscr{F} \otimes_\Lambda f^*\mathscr{N})$. Since $\text{pr}_1 : \mathbb{A}^{d-1}_k \longrightarrow \mathbb{A}^1_k$ is smooth, lemma 7.7.6 from [**Fu11**] ensures that $g = \text{pr}_1 \circ h$ is universally locally acyclic with respect to $j_!(\mathscr{F} \otimes_\Lambda f^*\mathscr{N})$. If $\overline{x} \longrightarrow X$ is a geometric point of X lying over x, we deduce

$$R\Phi_{\overline{x}}(j_!(\mathscr{F} \otimes_\Lambda f^*\mathscr{N}), g) = 0$$

On the other hand, applying Saito's Milnor type formula (2.5.1) to $j_!(\mathscr{F} \otimes_\Lambda f^*\mathscr{N})$ with respect to $g : X \longrightarrow \mathbb{A}^1_k$ gives

(4.5.3)
$$\begin{aligned}\sum_i (-1)^i \text{dimtot}(R^i\Phi_{\overline{x}}(j_!(\mathscr{F} \otimes_\Lambda f^*\mathscr{N})) &= -(CC(j_!(\mathscr{F} \otimes_\Lambda f^*\mathscr{N})), [dg])_{\mathbb{T}^*X,x} \\ &= -(-1)^d n \cdot ([C],[dg])_{\mathbb{T}^*X,x} = (-1)^{d+1} n\end{aligned}$$

Since the left-hand side of (4.5.3) is zero, we deduce $n = 0$.

Let us now suppose that $\text{char}(k) = 2$. By lemma 4.4, either we can apply the argument as above, or after replacing $Y = X \times_k \mathbb{A}^1_k$ by a Zariski neighbourhood of $(x, 0)$, there exists a smooth morphism $g : Y \longrightarrow \mathbb{A}^1_k$ such that

(3) The induced section $dg : Y \longrightarrow \mathbb{T}^*Y$ intersects $\text{pr}_1^\circ C$ transversally at $((x, 0), \text{pr}_1^*\omega)$.

(4) The restriction $g|_E : E \longrightarrow \mathbb{A}^1_k$ is smooth, where $E = D \times_k \mathbb{A}^1_k$.

Since $\text{pr}_1 : Y \longrightarrow X$ is smooth, theorem 2.9 gives

$$\text{pr}_1^! CC(j_!(\mathscr{F} \otimes_\Lambda f^*\mathscr{N})) = CC(\text{pr}_1^* j_!(\mathscr{F} \otimes_\Lambda f^*\mathscr{N})).$$

Hence, $CC(\text{pr}_1^* j_!(\mathscr{F} \otimes_\Lambda f^*\mathscr{N}))$ is equal to

$$(-1)^{d+1} N \cdot \text{rank}_\Lambda \mathscr{F} \cdot [\mathbb{T}^*_Y Y] + \sum_{a \in A} (-1)^{d+1} n_a \text{pr}_1^\circ [C_a]$$

Applying the same argument as in the case where $\text{char}(k) \neq 2$ to the sheaf $\text{pr}_1^* j_!(\mathscr{F} \otimes_\Lambda f^*\mathscr{N})$ and to the morphism g yields again that each n_a is 0 for $C_a \neq \mathbb{T}^*_D X$.

To prove that (4.5.1) holds, we are thus left to compute the coefficient of $\mathbb{T}^*_D X$ in $CC(j_!(\mathscr{F} \otimes_\Lambda f^*\mathscr{N}))$. Let $i : C \longrightarrow X$ be a smooth curve in X transverse to D. Since $SS(j_!(\mathscr{F} \otimes_\Lambda f^*\mathscr{N}))$ is contained in $\mathbb{T}^*_X X \bigcup \mathbb{T}^*_D X$, the curve $i : C \longrightarrow X$ is properly $SS(j_!(\mathscr{F} \otimes_\Lambda f^*\mathscr{N}))$-transversal. By Saito's compatibility of the characteristic cycle with properly transversal restriction Theorem 2.9, we have

$$i^! CC(j_!(\mathscr{F} \otimes_\Lambda f^*\mathscr{N})) = CC(i^* j_!(\mathscr{F} \otimes_\Lambda f^*\mathscr{N})). \tag{4.5.4}$$

Hence, example 2.6 and proposition 3.3 imply that the sought-after coefficient is

$$(-1)^d(r_{\log}+1)N\cdot \mathrm{rank}_\Lambda \mathscr{F},$$

which finishes the proof of Theorem 4.5. □

Corollary 4.6. — *Under the conditions of Theorem 4.5, we have*

$$Rj_*(\mathscr{F}\otimes_\Lambda f^*\mathscr{N}) = j_!(\mathscr{F}\otimes_\Lambda f^*\mathscr{N})$$

Proof. — We need to show that for each geometric point $\bar{x}\longrightarrow D$ above a closed point x of D, we have

$$(Rj_*(\mathscr{F}\otimes_\Lambda f^*\mathscr{N}))_{\bar{x}} = 0$$

Let $i: C\longrightarrow X$ be a locally closed immersion from a smooth k-curve C to X such that C is transverse to D and meets D only at x. Consider the following cartesian diagram

$$\begin{array}{ccc} V & \xrightarrow{j'} & C \\ {\scriptstyle i'}\downarrow & \square & \downarrow{\scriptstyle i} \\ U & \xrightarrow[j]{} & X \end{array}$$

By proposition 4.5, the morphism $i: C\longrightarrow X$ is $SS(Rj_*(\mathscr{F}\otimes_\Lambda f^*\mathscr{N}))$-transversal. From corollary 2.13, we deduce

(4.6.1) $$i^*Rj_*(\mathscr{F}\otimes_\Lambda f^*\mathscr{N})\cong Rj'_*((\mathscr{F}\otimes_\Lambda f^*\mathscr{N})|_V).$$

Notice that $f^*\mathscr{N}|_V$ has pure slope $r_{\log}+1$ at x since $f|_C: C\longrightarrow S$ is étale in a neighborhood of x. From proposition 3.3, the largest slope of $\mathscr{F}|_V$ at x is no greater than $r(D,\mathscr{F})$. Hence $(\mathscr{F}\otimes_\Lambda f^*\mathscr{N})|_V$ is totally wild ramified at x of pure slope $r_{\log}+1$. By [**Fu11**, Prop. 8.1.4], we deduce

(4.6.2) $$Rj'_*((\mathscr{F}\otimes_\Lambda f^*\mathscr{N})|_V) = j'_!((\mathscr{F}\otimes_\Lambda f^*\mathscr{N})|_V).$$

Hence, by (4.6.1) and (4.6.2), we have

$$(Rj_*(\mathscr{F}\otimes_\Lambda f^*\mathscr{N}))_{\bar{x}} = (j'_!((\mathscr{F}\otimes_\Lambda f^*\mathscr{N})|_V))_{\bar{x}} = 0,$$

which finishes the proof of corollary 4.6. □

5. Bound for the wild ramification of nearby cycles

5.1. — In this section, let R be a henselian discrete valuation ring. Let K be the fraction field of R and let F be the residue field of R. Suppose that F is perfect of characteristic $p>0$. Put $S=\mathrm{Spec}(R)$. Let s be the closed point of S. Let $\bar{s}$ be a geometric point of S above s. Let η be the generic point of S, η^{t} a maximal tame cover of η, $\bar{\eta}$ a geometric point above η^{t}. Let G be the Galois group of $\bar{\eta}$ over η. Let P be the wild inertia subgroup of G.

Let $f: X\longrightarrow S$ be a morphism of schemes, $f_\eta: X_\eta\longrightarrow\eta$ its generic fiber and $f_s: X_s\longrightarrow s$ its closed fiber. Let Λ be a finite field of characteristic $\ell\neq p$. For an object $\mathscr{F}$ in $D^+(X_\eta,\Lambda)$, we denote by $R\Psi(\mathscr{F},f)$ (resp. $R\Psi^{\mathrm{t}}(\mathscr{F},f)$) the nearby cycles complex (resp. the tame nearby cycles complex) of $\mathscr{F}$ with respect to $f: X\longrightarrow S$.

For an object $\mathscr{F}$ in $D^+(X,\Lambda)$, we denote by $R\Phi(\mathscr{F},f)$ the vanishing cycles complex of $\mathscr{F}$ with respect to $f : X \longrightarrow S$. All these complex are objects of $D^+(X_{\bar{s}},\Lambda)$ with G-actions.

Note that $R\Psi$, $R\Psi^{\mathrm{t}}$ and $R\Phi$ also make sense when S is a smooth curve over a perfect field with a closed point s. For nearby cycles and vanishing cycles on a more general base scheme, we refer to [**Or06**].

The following definition was introduced in [**Tey15**].

Definition 5.2. — Let $\mathscr{F}$ be an object in $D^b_c(X_\eta,\Lambda)$. Let $\bar{x} \longrightarrow X_s$ be a geometric point above a closed point $x \in X_s$. We say that $r \in \mathbb{Q}_{\geqslant 0}$ is a *nearby slope for $\mathscr{F}$ with respect to $f : X \longrightarrow S$ at x* if there is a locally constant and constructible sheaf $\mathscr{N}$ of Λ-modules on η with pure logarithmic slope r such that

$$R\Psi^{\mathrm{t}}_{\bar{x}}(\mathscr{F} \otimes_\Lambda f_\eta^*\mathscr{N}, f) \neq 0.$$

We denote by $\mathrm{Sl}(\mathscr{F},f,x)$ the set of nearby slopes of $\mathscr{F}$ with respect to $f : X \longrightarrow S$ at x.

Lemma 5.3. — *Suppose that $f : X \longrightarrow S$ is of finite type. Let $\mathscr{F}$ be an object in $D^b_c(X_\eta,\Lambda)$. Let $\mathscr{N}$ be an object in $D^b_c(\eta,\Lambda)$. Then, for any geometric point $\bar{x} \longrightarrow X_s$ above a closed point x in X_s, there is a canonical G/P-equivariant isomorphism*

$$R\Psi^{\mathrm{t}}_{\bar{x}}(\mathscr{F} \otimes^L_\Lambda f_\eta^*\mathscr{N}, f) \cong (R\Psi_{\bar{x}}(\mathscr{F},f) \otimes^L_\Lambda \mathscr{N}|_{\bar{\eta}})^P. \tag{5.3.1}$$

Proof. — Consider the following cartesian diagram

$$\begin{array}{ccccccccc}
X_{(\bar{x})} \times_S \bar{\eta} & \longrightarrow & X_{(\bar{x})} \times_S \eta^{\mathrm{t}} & \longrightarrow & X_{(\bar{x})} \times_S \eta & \longrightarrow & X_{(\bar{x})} & \longleftarrow & X_{(\bar{x})} \times_S s \\
\downarrow{\scriptstyle f_{\bar{\eta}}} & \square & \downarrow{\scriptstyle f_{\eta^{\mathrm{t}}}} & \square & \downarrow{\scriptstyle f_\eta} & \square & \downarrow{\scriptstyle f} & \square & \downarrow{\scriptstyle f_s} \\
\bar{\eta} & \longrightarrow & \eta^{\mathrm{t}} & \longrightarrow & \eta & \longrightarrow & S & \longleftarrow & s
\end{array}$$

Then we have

$$\begin{aligned}
R\Psi^{\mathrm{t}}_x(\mathscr{F} \otimes^L_\Lambda f_\eta^*\mathscr{N}) &\cong R\Gamma(X_{(\bar{x})} \times_S \eta^{\mathrm{t}}, (\mathscr{F} \otimes^L_\Lambda f_\eta^*\mathscr{N})|_{X_{(\bar{x})}\times\eta^{\mathrm{t}}}) \\
&= R\Gamma(\eta^{\mathrm{t}}, Rf_{\eta^{\mathrm{t}}*}(\mathscr{F}|_{X_{(\bar{x})}\times_S\eta^{\mathrm{t}}} \otimes^L_\Lambda f^*_{\eta^{\mathrm{t}}}(\mathscr{N}|_{\eta^{\mathrm{t}}}))) \\
&\cong R\Gamma(\eta^{\mathrm{t}}, Rf_{\eta^{\mathrm{t}}*}(\mathscr{F}|_{X_{(\bar{x})}\times_S\eta^{\mathrm{t}}}) \otimes^L_\Lambda \mathscr{N}|_{\eta^{\mathrm{t}}}) \\
&= R\Gamma(P, Rf_{\bar{\eta}*}(\mathscr{F}|_{X_{(\bar{x})}\times_S\bar{\eta}}) \otimes^L_\Lambda \mathscr{N}|_{\bar{\eta}}) \\
&= (Rf_{\bar{\eta}*}(\mathscr{F}|_{X_{(\bar{x})}\times_S\bar{\eta}}) \otimes^L_\Lambda \mathscr{N}|_{\bar{\eta}})^P \\
&= (R\Gamma(X_{(\bar{x})} \times_S \bar{\eta}, \mathscr{F}|_{X_{(\bar{x})}\times_S\bar{\eta}}) \otimes^L_\Lambda \mathscr{N}|_{\bar{\eta}})^P \\
&= (R\Psi_{\bar{x}}(\mathscr{F},f) \otimes^L_\Lambda \mathscr{N}|_{\bar{\eta}})^P,
\end{aligned} \tag{5.3.2}$$

where the third identification comes from the projection formula

$$Rf_{\eta^{\mathrm{t}}*}(\mathscr{F}|_{X_{(x)}\times_S\eta^{\mathrm{t}}} \otimes^L_\Lambda f^*_{\eta^{\mathrm{t}}}(\mathscr{N}|_{\eta^{\mathrm{t}}})) \cong Rf_{\eta^{\mathrm{t}}*}(\mathscr{F}|_{X_{(x)}\times_S\eta^{\mathrm{t}}}) \otimes^L_\Lambda \mathscr{N}|_{\eta^{\mathrm{t}}},$$

which is valid since $Rf_{\eta^{\mathrm{t}}*}$ has finite cohomological dimension and each cohomology sheaf of $\mathscr{N}$ is locally constant. $\square$

Corollary 5.4. — *Suppose that $f : X \longrightarrow S$ is of finite type. Let $\mathscr{F}$ be an object in $D^b_c(X_\eta, \Lambda)$. Let $\bar{x} \longrightarrow X_s$ be a geometric point above a closed point x in X_s. Then*

$$\mathrm{Sl}(\mathscr{F}, f, x) = \bigcup_{i\in\mathbb{Z}} \mathrm{Sl}_{\log,K}(R^i\Psi_{\bar{x}}(\mathscr{F}, f))$$

Proof. — Let i be an integer such that the Galois module $M = R^i\Psi_{\bar{x}}(\mathscr{F}, f)$ admits r as logarithmic slope. Set $N := M^{(r)}_{\log}$. It is a P-stable Λ-module. Since each $G^b_{\log}$ ($b \in \mathbb{Q}_{\geqslant 0}$) is a normal subgroup of G, the module N is also G-stable. We denote by $\mathscr{N}$ the locally constant and constructible sheaf of Λ-modules on η associated to N. By lemma 5.3, we have

$$R^i\Psi^{\mathrm{t}}_{\bar{x}}(\mathscr{F} \otimes^L_\Lambda f^*_\eta\mathscr{N}^\vee, f) \cong (R^i\Psi_{\bar{x}}(\mathscr{F}, f) \otimes_\Lambda N^\vee)^P.$$

Note that N is a direct summand of $R^i\Psi_{\bar{x}}(\mathscr{F}, f)$. Hence, $(N \otimes_\Lambda N^\vee)^P$ is a direct summand of $R^i\Psi^{\mathrm{t}}_{\bar{x}}(\mathscr{F} \otimes^L_\Lambda f^*_\eta\mathscr{N}^\vee, f)$. The evaluation map $\mathrm{ev} : N \otimes_\Lambda N^\vee \longrightarrow \Lambda$ is surjective and P-equivariant. Since taking the P invariants is exact, we deduce that $(N \otimes_\Lambda N^\vee)^P \neq 0$. Hence, r is a nearby slope for $\mathscr{F}$ with respect to f at x.

Conversely, suppose that r is a nearby slope for $\mathscr{F}$ with respect to f at x. Then, there exists an integer i and a locally constant and constructible sheaf of Λ-modules $\mathscr{N}$ on η with pure logarithmic slope r such that

$$R^i\Psi^{\mathrm{t}}_{\bar{x}}(\mathscr{F} \otimes^L_\Lambda f^*_\eta\mathscr{N}, f) \cong (R^i\Psi_{\bar{x}}(\mathscr{F}, f) \otimes_\Lambda \mathscr{N}|_{\bar{\eta}})^P \neq 0.$$

By lemma 1.9, the module $R^i\Psi_{\bar{x}}(\mathscr{F}, f)$ admits r as a logarithmic slope. □

5.5. — Let Z be a reduced subscheme of X containing X_s. We say that (X, Z) is a *semi-stable pair over* S if $f : X \longrightarrow S$ is of finite type and if, étale locally, X is étale over $\mathrm{Spec}(R[t_1, \ldots, t_d]/(t_{r+1}\cdots t_d - \pi))$, where $r < d$ and $Z = X_s \cup Z_f$ with Z_f defined by $t_1\cdots t_m = 0$, where $m \leqslant r$.

5.6. — We now prove the main theorem of this article.

Theorem 5.7. — *Let k be a perfect field of characteristic $p > 0$. Suppose that S is the henselization at a closed point of a smooth curve over k. Let (X, Z) be a semi-stable pair over S such that $f : X \longrightarrow S$ is smooth. Let $\mathscr{F}$ be a locally constant and constructible sheaf of Λ-modules on $U := X\backslash Z$. Let $j : U \longrightarrow X$ be the canonical injection. Suppose that $\mathscr{F}$ is tamely ramified along the horizontal part of Z. Let $r_{\log}(\mathscr{F})$ be the maximum of the set of logarithmic slope of $\mathscr{F}$ at generic points of the special fibre X_s of X.*

Then, for every $r > r_{\log}(\mathscr{F})$, the r-th upper numbering ramification subgroup of G acts trivially on $R^i\Psi(j_!\mathscr{F}, f)$ for every $i \in \mathbb{Z}_{\geqslant 0}$.

Proof. — We may assume that k is algebraically closed and that the special fibre X_s is irreducible. To show that the action of $G^{r_{\log}(\mathscr{F})+}_{K,\mathrm{cl}}$ on each $R^i\Psi(\mathscr{F}, f)$ is trivial is equivalent to show that for every $i \in \mathbb{Z}$, for every closed point x of X_s, we have

$$r_{\log,K}(R^i\Psi_x(\mathscr{F}, f)) \leqslant r_{\log}(\mathscr{F}). \tag{5.7.1}$$

Step 1. We first prove (5.7.1) in the case where $U = X_\eta$, that is $Z_f = \varnothing$. From corollary 5.4, we have to prove that for any locally constant and constructible sheaf $\mathscr{N}$ of Λ-modules on η with pure logarithmic slope $r_{\log} > r_{\log}(\mathscr{F})$, we have

$$R\Psi^{\mathrm{t}}(\mathscr{F} \otimes_\Lambda f_\eta^* \mathscr{N}, f) = 0. \tag{5.7.2}$$

Let $n > 0$ be an integer prime to p. Let ϖ be a uniformizer of R. Put

$$S_n = \mathrm{Spec}(R[t]/(t^n - \varpi)),$$

put $X_n = X \times_S S_n$ and put $U_n = U \times_S S_n$. Let η_n be the generic point of S_n. We thus have the following commutative diagram

$$\begin{array}{ccccc} U_n & \xrightarrow{j_n} & X_n & \xleftarrow{i_n} & X_s \\ {\scriptstyle h_n}\downarrow & \square & \downarrow{\scriptstyle g_n} & & \| \\ U & \xrightarrow[j]{} & X & \xleftarrow[i]{} & X_s \end{array}$$

By [SGA4II, VII 5.11], we have

$$R\Psi^{\mathrm{t}}(\mathscr{F} \otimes_\Lambda f_\eta^* \mathscr{N}) = \varinjlim_{(n,p)=1} i_n^* Rj_{n*}(h_n^* \mathscr{F} \otimes_\Lambda h_n^* f_\eta^* \mathscr{N}). \tag{5.7.3}$$

To show (5.7.2), it is enough to show that for any integer n prime to p, we have

$$Rj_{n*}(h_n^* \mathscr{F} \otimes_\Lambda h_n^* f_\eta^* \mathscr{N}) = j_{n!}(h_n^* \mathscr{F} \otimes_\Lambda h_n^* f_\eta^* \mathscr{N}). \tag{5.7.4}$$

Since S is the henselization of a smooth curve over k, since f is of finite type and since $\mathscr{F}$ and $\mathscr{N}$ are constructible, we may descend to situation where S is a smooth k-curve. From corollary 4.6, to show (5.7.4), it is enough to show that

$$r(X_s, h_n^* \mathscr{F}) - 1 < r_{\log}(s, \mathscr{N}|_{\eta_n})$$

From (ii) and (iii) of proposition 1.4, we have

$$r(X_s, h_n^* \mathscr{F}) - 1 \leqslant r_{\log}(X_s, h_n^* \mathscr{F}) = n \cdot r_{\log}(X_s, \mathscr{F}) = n \cdot r_{\log}(\mathscr{F}) \tag{5.7.5}$$

Furthermore, $\mathscr{N}|_{\eta_n}$ has pure logarithmic slope $n \cdot r_{\log}$ at s. Hence,

$$r(X_s, h_n^* \mathscr{F}) - 1 < n \cdot r_{\log} = r_{\log}(s, \mathscr{N}|_{\eta_n})$$

This concludes the proof of Theorem 5.7 in the case where Z has no horizontal part.

Step 2. We now prove the general case. We need to prove (5.7.1) for a fixed closed point x of X_s. As Step 1, we may descend to the case where S is an affine smooth k-curve with local coordinate ϖ at the closed point s. Since this is an étale local question, we may assume that X is affine and étale over $\mathrm{Spec}(\mathscr{O}_S[t_1, \dots, t_d]/(t_d - \varpi))$ such that the image of x is the point associated to the ideal $(t_1, \dots, t_d)$, and $Z = X_s \cup Z_f$ with Z_f defined by $t_1 \cdots t_m = 0$, where $m < d$. We denote by $\{Z_i\}_{i \in I}$ the set of irreducible components of Z passing through x and, for any subset $J \subseteq I$, we put $Z_J = \cap_{i \in J} Z_i$. Since each Z_J is smooth at x, at the cost of replacing X by a Zariski neighborhood of x, we may assume that the Z_J's are connected and smooth.

From Abhyankar's Lemma [**SGA1**, XIII 5.2], there exist prime to p integers $n_1, \dots n_m$ such that after setting

$$X' = \mathrm{Spec}(\mathscr{O}_X[t'_1, \dots, t'_m]/(t'^{n_1}_1 - t_1, \dots, t'^{n_m}_m - t_m))$$

the locally constant sheaf $\mathscr{F}|_{X' \times_X U}$ is unramified at the generic points of $X' \times_X Z_f$. Consider the following cartesian diagram

$$\begin{array}{ccccc} U' & \xrightarrow{j'} & X' \backslash X'_s & \longrightarrow & X' \\ \downarrow{\scriptstyle \pi_U} & \square & \downarrow & \square & \downarrow{\scriptstyle \pi} \\ U & \xrightarrow{j} & X \backslash X_s & \longrightarrow & X \end{array}$$

where $X'_s = X' \times_S s$. By Zariski-Nagata's purity theorem, $\mathscr{F}|_{X' \times_X U} = \pi_U^* \mathscr{F}$ extends into a locally constant and constructible sheaf of Λ-modules $\mathscr{F}'$ on $X' \backslash X'_s$. Note that $\pi_U : U' \longrightarrow U$ is finite étale. Hence, $\mathscr{F}$ is a direct summand of $\pi_{U*}\pi_U^*\mathscr{F}$. Thus, each $R^i\Psi_x(j_!\mathscr{F}, f)$ is a direct summand of $R^i\Psi_x(j_!\pi_{U*}\pi_U^*\mathscr{F}, f)$. Let x' be the unique point of X'_s above $x \in X_s$. Since the nearby cycles functor commutes with proper push-forward, there is a Galois-equivariant isomorphism

$$R\Psi_x(j_!\pi_{U*}\pi_U^*\mathscr{F}, f) \xrightarrow{\sim} R\Psi_{x'}(j'_!\pi_U^*\mathscr{F}, f \circ \pi)$$

Hence, we are left to show that

$$r_{\log,K}(R^i\Psi_{x'}(j'_!\pi_U^*\mathscr{F}, f \circ \pi)) \leqslant r_{\log}(X_s, \mathscr{F}).$$

Observe that the map induced by π at the level of the henselianization of X and X' at the generic points of X_s and X'_s is unramified. Hence, $r_{\log}(X_s, \mathscr{F}) = r_{\log}(X'_s, \pi_U^*\mathscr{F})$. Thus, we are left to show that

$$r_{\log,K}(R^i\Psi_{x'}(j'_!\pi_U^*\mathscr{F}, f \circ \pi)) \leqslant r_{\log}(X'_s, \pi_U^*\mathscr{F}).$$

Hence, we are left to prove (5.7.1) in the case where $\mathscr{F}$ extends into a locally constant and constructible sheaf of Λ-modules $\mathscr{G}$ on $X \backslash X_s$. We will thus make this assumption from now on.

We put $Z_f^\circ = Z_f \backslash X_s$. Applying the nearby cycles functor to the short exact sequence

$$0 \longrightarrow j_!\mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{G}|_{Z_f^\circ} \longrightarrow 0$$

produces a distinguished triangle of complexes with Galois actions

$$R\Psi_x(j_!\mathscr{F}, f) \longrightarrow R\Psi_x(\mathscr{G}, f) \longrightarrow R\Psi_x(\mathscr{G}|_{Z_f^\circ}, f|_{Z_f}) \longrightarrow$$

Apply Step 1 to the locally constant sheaf $\mathscr{G}$, we are reduced to prove

$$r_{\log,K}(R^i\Psi_x(\mathscr{G}|_{Z_f^\circ}, f|_{Z_f})) \leqslant r_{\log}(X_s, \mathscr{F}). \tag{5.7.6}$$

for every integer i. We proceed by the induction on m. If $m = 0$, there is nothing to do. Suppose that $m > 0$ and suppose that (5.7.6) holds when Z_f has $m - 1$ irreducible components. For $1 \leqslant n \leqslant m$, let Z_n (resp Z_n°) be the component of Z_f (resp. Z_f°) defined by $t_n = 0$. Set $Z_f^{<m} := \bigcup_{n=1}^{m-1} Z_n$ (resp. $Z_f^{\circ<m} := \bigcup_{n=1}^{m-1} Z_n^\circ$). Let

$i_m : Z_m^\circ \longrightarrow Z_f^\circ$, $i_{<m} : Z_f^{\circ<m} \longrightarrow Z_f^\circ$ and $j_m : Z_m^\circ \setminus Z_f^{\circ<m} \longrightarrow Z_f^\circ$ be the canonical injections. Applying the nearby cycles functor to the short exact sequence

$$0 \longrightarrow j_{m!}j_m^*(\mathscr{G}|_{Z_f^\circ}) \longrightarrow \mathscr{G}|_{Z_f^\circ} \longrightarrow i_{<m*}(\mathscr{G}|_{Z_f^{\circ<m}}) \longrightarrow 0$$

produces a distinguished triangle of Galois modules

$$R\Psi_x(j_{m!}j_m^*(\mathscr{G}|_{Z_f^\circ}), f|_{Z_f}) \longrightarrow R\Psi_x(\mathscr{G}|_{Z_f^\circ}, f|_{Z_f}) \longrightarrow R\Psi_x(\mathscr{G}|_{Z_f^{\circ<m}}, f|_{Z_f^{<m}}) \longrightarrow$$

By the induction hypothesis, we have

$$r_{\log,K}(R^i\Psi_x(\mathscr{G}|_{Z_f^{\circ<m}}, f|_{Z_f^{<m}})) \leqslant r_{\log}(X_s, \mathscr{F})$$

for every integer i. We are thus left to prove that

$$R^i\Psi_x(j_{m!}j_m^*(\mathscr{G}|_{Z_f^\circ}), f|_{Z_f}) \leqslant r_{\log}(X_s, \mathscr{F})$$

for every integer i. Let $\jmath_m : Z_m^\circ \setminus Z_f^{\circ<m} \longrightarrow Z_m^\circ$ be the canonical injection. Observe that

$$j_{m!}j_m^*(\mathscr{G}|_{Z_f^\circ}) \simeq i_{m*}\jmath_{m!}\jmath_m^*(\mathscr{G}|_{Z_m^\circ}).$$

Hence,

$$R\Psi(j_{m!}j_m^*(\mathscr{G}|_{Z_f^\circ}), f|_{Z_f}) \simeq R\Psi(\jmath_{m!}\jmath_m^*(\mathscr{G}|_{Z_m^\circ}), f|_{Z_m})$$

We are thus left to prove that

$$r_{\log,K}(R^i\Psi_x(\jmath_{m!}\jmath_m^*(\mathscr{G}|_{Z_m^\circ}), f|_{Z_m})) \leqslant r_{\log}(X_s, \mathscr{F}). \tag{5.7.7}$$

for every integer i. Put $T_f := Z_m \cap Z_f^{<m}$, $T_f^\circ = Z_m^\circ \cap Z_f^{\circ<m}$ and $T := Z_{m,s} \cup T_f$. Note that (Z_m, T) is a semi-stable pair over S and that T_f has $m-1$ irreducible components. The sheaf $\jmath_m^*(\mathscr{G}|_{Z_m^\circ})$ on $Z_m^\circ \setminus Z_f^{\circ<m}$ is extended to a locally constant sheaf $\mathscr{G}|_{Z_m^\circ}$.

Applying the nearby cycle functor to the short exact sequence

$$0 \longrightarrow \jmath_{m!}\jmath_m^*(\mathscr{G}|_{Z_m^\circ}) \longrightarrow \mathscr{G}|_{Z_m^\circ} \longrightarrow \mathscr{G}|_{T_f^\circ} \longrightarrow 0$$

produces a distinguished triangle of Galois modules

$$R\Psi_x(\jmath_{m!}\jmath_m^*(\mathscr{G}|_{Z_m^\circ}), f|_{Z_m}) \longrightarrow R\Psi_x(\mathscr{G}|_{Z_m^\circ}, f|_{Z_m}) \longrightarrow R\Psi_x(\mathscr{G}|_{T_f^\circ}, f|_{T_f}) \longrightarrow$$

From Step 1 applied to the locally constant sheaf $\mathscr{G}|_{Z_m^\circ}$ on the generic fibre of Z_m, we have for every integer i

$$r_{\log,K}(R^i\Psi_x(\mathscr{G}|_{Z_m^\circ}, f|_{Z_m})) \leqslant r_{\log}(Z_{m,s}, \mathscr{G}|_{Z_m^\circ})$$

Applying proposition 3.5 to the sheaf $\mathscr{G}$ on $X \setminus X_s$ and the closed immersion $Z_m \longrightarrow X$, we have $r_{\log}(Z_{m,s}, \mathscr{G}|_{Z_m^\circ}) \leqslant r_{\log}(X_s, \mathscr{G}) = r_{\log}(X_s, \mathscr{F})$. Hence, for each integer i,

$$r_{\log,K}(R^i\Psi_x(\mathscr{G}|_{Z_m^\circ}, f|_{Z_m})) \leqslant r_{\log}(X_s, \mathscr{F})$$

By the induction hypothesis and proposition 3.5, we have for every integer i

$$r_{\log,K}(R^i\Psi_x(\mathscr{G}|_{T_f^\circ}, f|_{T_f})) \leqslant r_{\log}(Z_{m,s}, \mathscr{G}|_{Z_m^\circ}) \leqslant r_{\log}(X_s, \mathscr{F})$$

Hence, the inequality (5.7.7) holds and this finishes the proof of Theorem 5.7. □

In view of Theorem 5.7, it is natural to formulate the following local version of the conjecture from [Lea16].

Conjecture 5.8. — *Let S be an henselian trait with perfect residue field of characteristic $p > 0$. Let G be an absolute Galois group of the field of function of S. Let (X, Z) be a semi-stable pair over S. Let $U := X \backslash Z$ and $j : U \longrightarrow X$ the canonical injection. Let $\mathscr{F}$ be a locally constant and constructible sheaf of Λ-modules on U. Suppose that $\mathscr{F}$ is tamely ramified along the horizontal part of Z. Let $r_{\log}(\mathscr{F})$ be the maximum of the set of logarithmic slopes of $\mathscr{F}$ at generic points of the special fibre of X.*

Then, for every $r > r_{\log}(\mathscr{F})$, the r-th upper numbering ramification subgroup of G acts trivially on $R^i\Psi(j_!\mathscr{F}, f)$ for every $i \in \mathbb{Z}_{\geqslant 0}$.

6. Ramification along the special fibre of an abelian scheme over a trait

6.1. — In this section, let k be a perfect field of characteristic $p > 0$. Let S be a smooth connected k-curve, s a closed point of S, $\bar{s}$ a geometric point above s, and $\bar{\eta}$ a geometric generic point of the strict henselization $S_{(\bar{s})}$ of S at s. Put $V = S \backslash \{s\}$. Let $f : \mathcal{A} \longrightarrow S$ be an abelian scheme of relative dimension $g \geqslant 1$ over S. Put $V = f^{-1}(U)$. Consider the cartesian diagram

$$\begin{array}{ccccc} U & \xrightarrow{j} & \mathcal{A} & \xleftarrow{i} & D \\ {\scriptstyle f_V}\downarrow & \square & \downarrow{\scriptstyle f} & \square & \downarrow{\scriptstyle f_s} \\ V & \longrightarrow & S & \longleftarrow & s \end{array}$$

Let Λ be a finite field of characteristic $\ell \neq p$. Note that the Swan conductor $\mathrm{sw}_s(-)$ at s is a well-defined function on $D^b_c(V, \Lambda)$ associating to $\mathscr{K}$ the integer $\sum_n (-1)^n \mathrm{sw}_s(\mathcal{H}^n\mathscr{K})$. Inspired by [**Tsu17**, 3.1], we prove the following boundedness result. For a variant valid for complex differential equations, we refer to [**Tey18**, 6.2.1].

Proposition 6.2. — *Let $\mathscr{F}$ be a locally constant and constructible sheaf of Λ-modules on U. Then, there exists a constant $c(g, \mathrm{rank}_\Lambda \mathscr{F}, r_{\log}(D, \mathscr{F}))$ depending only on $g, \mathrm{rank}\, \mathscr{F}$ and $r_{\log}(D, \mathscr{F})$ such that*

$$\sum_{n=0}^{2g} \mathrm{sw}_s(R^n f_{V*}\mathscr{F}) \leqslant c(g, \mathrm{rank}_\Lambda \mathscr{F}, r_{\log}(D, \mathscr{F}))$$

We start with the following general lemma.

Lemma 6.3. — *Let G be the product of n finite cyclic groups. Let M be a finite dimensional Λ-vector space with a G-action. Then, for any $i \geqslant 0$, we have*

$$\dim_\Lambda H^i(G, M) \leqslant (1+i)^{n-1} \dim_\Lambda M. \tag{6.3.1}$$

Proof. — We proceed by induction on n. The case $n = 1$ is standard [**Fu11**, Prop. 4.1.2]. Suppose that $n \geqslant 2$ and assume that (6.3.1) holds for those H which are products of $n-1$ finite cyclic groups. Let $G = H \times \mathbb{Z}/m\mathbb{Z}$ where H is the product of

$n-1$ finite cyclic groups and where $m \geqslant 2$. The Hochschild-Serre spectral sequence associated to the above product decomposition of G reads

$$E_2^{i-j,j} = H^{i-j}(H, H^j(\mathbb{Z}/m\mathbb{Z}, M)) \Longrightarrow H^i(G, M).$$

By induction hypothesis, we deduce

$$\begin{aligned}\dim_\Lambda H^i(G,M) &\leqslant \sum_{j=0}^{i} H^{i-j}(H, H^j(\mathbb{Z}/m\mathbb{Z}, M)) \\ &\leqslant \sum_{j=0}^{i} (1+i-j)^{n-2} \dim_\Lambda H^j(\mathbb{Z}/m\mathbb{Z}, M) \\ &\leqslant \sum_{j=0}^{i} (1+i-j)^{n-2} \dim_\Lambda M \leqslant (1+i)^{n-1} \dim_\Lambda M\end{aligned}$$

□

Lemma 6.4. — *Suppose that k is algebraically closed. Let A be an abelian variety of dimension g over k. Let $\mathscr{F}$ be a locally constant and constructible sheaf of Λ-modules on A. Then, for any $0 \leqslant i \leqslant 2g$, we have*

$$\dim_\Lambda H^i(A, \mathscr{F}) \leqslant (2 + 2\min\{i, 2g-i\})^{2g} \cdot \mathrm{rank}_\Lambda \mathscr{F}$$

In particular, we have

$$\dim_\Lambda H^i(A, \mathscr{F}) \leqslant (2 + 2g)^{2g} \cdot \mathrm{rank}_\Lambda \mathscr{F}$$

Proof. — Let $\pi : A' \longrightarrow A$ be a finite Galois étale cover of A trivializing $\mathscr{F}$. From the chinese reminder theorem and Serre-Lang classification of finite étale cover of A [**SGA1**, XI 2.1], the Galois group $G = \mathrm{Gal}(A'/A)$ is a product of at most $2g$ cyclic groups. From (6.3.1) applied to the E_2-page members of the following spectral sequence

$$E_2^{ij} = H^i(G, H^j(A', \pi^*\mathscr{F})) \Longrightarrow H^{i+j}(A, \mathscr{F}),$$

we deduce that, for every $i = 0, \ldots, 2g$,

$$\begin{aligned}\dim_\Lambda H^i(A, \mathscr{F}) &\leqslant \sum_{j=0}^{i} \dim_\Lambda H^{i-j}(G, H^j(A', \pi^*\mathscr{F})) \\ &\leqslant \sum_{j=0}^{i} (1+i-j)^{2g-1} \dim_\Lambda H^j(A', \pi^*\mathscr{F}) \\ &\leqslant \sum_{j=0}^{i} (1+i-j)^{2g-1} \binom{2g}{j} \mathrm{rank}_\Lambda \mathscr{F} \leqslant (2+2i)^{2g} \mathrm{rank}_\Lambda \mathscr{F}\end{aligned}$$

By Poincaré duality, we have $H^i(A, \mathscr{F}) \cong H^{2g-i}(A, \mathscr{F}^\vee(g))^\vee$. Hence, we have also

$$\dim_\Lambda H^i(A, \mathscr{F}) \leqslant (2 + 2(2g-i))^{2g} \cdot \mathrm{rank}_\Lambda \mathscr{F}$$

This finishes the proof of Lemma 6.4. □

6.5. — We give a proof of proposition 6.2. From Theorem 5.7, we have for every $n = 0, \dots, 2g$,

$$\mathrm{sw}_s(R^n f_{V*}\mathscr{F}) \leqslant \dim_\Lambda H^n(\mathcal{A}_{\bar{\eta}}, \mathscr{F}|_{\mathcal{A}_{\bar{\eta}}}) \cdot r_{\log}(D, \mathscr{F}).$$

By Lemma 6.4, we deduce

$$\mathrm{sw}_s(R^n f_{V*}\mathscr{F}) \leqslant (2+2g)^{2g} \cdot \mathrm{rank}_\Lambda \mathscr{F} \cdot r_{\log}(D, \mathscr{F}).$$

Hence,

$$\sum_{n=0}^{2g} \mathrm{sw}_s(R^n f_{V*}\mathscr{F}) \leqslant (1+2g)(2+2g)^{2g} \cdot \mathrm{rank}_\Lambda \mathscr{F} \cdot r_{\log}(D, \mathscr{F})$$

Setting

$$c(g, \mathrm{rank}_\Lambda \mathscr{F}, r_{\log}(D, \mathscr{F})) = (1+2g)(2+2g)^{2g} \cdot \mathrm{rank}_\Lambda \mathscr{F} \cdot r_{\log}(D, \mathscr{F})$$

finishes the proof of proposition 6.2.

Theorem 6.6. — *Assume that $f : \mathcal{A} \longrightarrow S$ is a relative elliptic curve. Let $\mathscr{F}$ be a locally constant and constructible sheaf of Λ-modules on U. For any irreducible component Z of $SS(j_!\mathscr{F})$ distinct from $\mathbb{T}^*_D\mathcal{A}$ and $\mathbb{T}^*_{\mathcal{A}}\mathcal{A}$, we have $Z \cap df = \varnothing$. In particular,*

(1) *We have*

$$\mathrm{sw}_s(Rf_{V*}\mathscr{F}) = 0.$$

(2) *For any closed point $x \in D$, the fibre of the singular support $SS(j_!\mathscr{F}) \longrightarrow \mathcal{A}$ above x is a union of lines in $\mathbb{T}^*_x\mathcal{A}$.*

Proof. — From a theorem of Laumon [**Lau82**], we have

$$\chi(\mathcal{A}_{\bar{\eta}}, \mathscr{F}|_{\mathcal{A}_{\bar{\eta}}}) = \mathrm{rank}_\Lambda \mathscr{F} \cdot \chi(\mathcal{A}_{\bar{\eta}}, \Lambda) = 0$$

By Saito's conductor formula recalled in Theorem 2.10, we deduce

$$\mathrm{sw}_s(Rf_{V*}\mathscr{F}) = -(CC(j_!\mathscr{F}), df)_{\mathbb{T}^*\mathcal{A}, D} \tag{6.6.1}$$

We are going to analyse how each side of (6.6.1) behaves with respect to pulling-back by the finite étale map $[n] : \mathcal{A} \longrightarrow \mathcal{A}$ for a positive integer n prime to p. Let us write

$$CC(j_!\mathscr{F}) = \mathrm{rank}_\Lambda \mathscr{F} \cdot [\mathbb{T}^*_{\mathcal{A}}\mathcal{A}] + \alpha[\mathbb{T}^*_D\mathcal{A}] + \sum_Z \beta_Z[Z], \tag{6.6.2}$$

where Z runs over the set of integral 2-dimensional closed conical subschemes in $\mathbb{T}^*\mathcal{A}$, distinct from the conormal bundle $\mathbb{T}^*_D\mathcal{A}$ and the zero section $\mathbb{T}^*_{\mathcal{A}}\mathcal{A}$.

The section $df : \mathcal{A} \longrightarrow T^*\mathcal{A}$ is a regular embedding of codimension 2. Since $f : \mathcal{A} \longrightarrow S$ is smooth, we have $df \cap \mathbb{T}^*_{\mathcal{A}}\mathcal{A} = \varnothing$. By applying formula (6.6.1) to the constant sheaf Λ on U, we obtain that

$$([\mathbb{T}^*_D\mathcal{A}], [df])_{\mathbb{T}^*\mathcal{A}, D} = (CC(j_!\Lambda), [df])_{\mathbb{T}^*\mathcal{A}, D} = 0$$

Let Z be an integral 2-dimensional closed conical subscheme in $\mathbb{T}^*\mathcal{A}$, distinct from $\mathbb{T}^*_D\mathcal{A}$ and $\mathbb{T}^*_{\mathcal{A}}\mathcal{A}$. Then, Z meets df at most at a finite number of points. Hence, df

and Z intersect properly. As a general fact from intersection theory [**Ful98**, 7.1], we have

$$([Z],[df])_{T^*\mathcal{A},D} \geqslant \sharp(Z \times_{\mathbb{T}^*\mathcal{A}} df)_{\mathrm{red}} \tag{6.6.3}$$

Let us recall that from proposition 2.7, each β_Z is a positive integer. Hence, we have

$$\begin{aligned} -\mathrm{sw}_s(Rf_{V*}\mathscr{F}) &= (CC(j_!\mathscr{F}),[df])_{\mathbb{T}^*\mathcal{A},D} \\ &= \sum_Z \beta_Z([Z],[df])_{\mathbb{T}^*\mathcal{A},D} \\ &\geqslant \sum_Z \beta_Z \cdot \sharp(Z \times_{\mathbb{T}^*\mathcal{A}} df)_{\mathrm{red}}. \end{aligned}$$

From proposition 6.2, we deduce

$$\sum_Z \beta_Z \cdot \sharp(Z \times_{\mathbb{T}^*\mathcal{A}} df)_{\mathrm{red}} \leqslant c(\mathrm{rank}_\Lambda \mathscr{F}, r_{\log}(D,\mathscr{F})) \tag{6.6.4}$$

where $c(\mathrm{rank}_\Lambda \mathscr{F}, r_{\log}(D,\mathscr{F}))$ depends only on $\mathrm{rank}_\Lambda \mathscr{F}$ and $r_{\log}(D,\mathscr{F})$.

Let Z as above and such that Z meets df above a closed point $x \in D$. Pick a positive integer n prime to p. Since $[n] : \mathcal{A} \longrightarrow \mathcal{A}$ is étale, Saito's Theorem 2.9 implies

$$CC(j_![n]^*\mathscr{F}) = \mathrm{rank}_\Lambda \mathscr{F} \cdot [\mathbb{T}^*_{\mathcal{A}}\mathcal{A}] + \alpha[\mathbb{T}^*_D\mathcal{A}] + \sum_Z \beta_Z [n]^![Z],$$

By definition, $[n]^![Z] := d[n]_* \mathrm{pr}_2^*[Z]$ where pr_2 is the second projection in the diagram

$$\mathbb{T}^*\mathcal{A} \xleftarrow{d[n]} \mathcal{A} \times_{\mathcal{A}} \mathbb{T}^*\mathcal{A} \xrightarrow{\mathrm{pr}_2} \mathbb{T}^*\mathcal{A}$$

Hence, for every point $y \in [n]^{-1}\{x\}$, the support of $[n]^![Z]$ contains

$$d[n]_y \cdot df_x = d(f \circ [n])_y = df_y.$$

Hence, $[n]^![Z]$ intersects $[df]$ at at least n^2 points. Since the generic logarithmic slopes are étale local invariants, formula (6.6.4) applied to $[n]^*\mathscr{F}$ gives

$$0 \leqslant n^2 \beta_Z \cdot \sharp(Z \times_{\mathbb{T}^*\mathcal{A}} df)_{\mathrm{red}} \leqslant c(\mathrm{rank}_\Lambda \mathscr{F}, r_{\log}(D,\mathscr{F}))$$

for every positive integer n prime to p. Hence, $\beta_Z = 0$. This proves the main statement in Theorem 6.6. Then, the singular support $SS(j_!\mathscr{F})$ cannot contain the conormal to a closed point in D, so the claim (2) follows. Finally, claim (1) is an immediate consequence of the main statement in Theorem 6.6 combined with formula (6.6.1). □

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