TOPOLOGICAL EXODROMY WITH COEFFICIENTS

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ABSTRACT. We improve the exodromy equivalence of MacPherson, Treumann and Lurie in several ways: first, we allow stratified spaces that have locally weakly contractible strata, rather than being locally of singular shape, we remove all noetherianity assumptions and we consider more general coefficients (e.g. compactly assembled or stable presentable ∞-categories). Furthermore, our approach shows that the exodromy equivalence is functorial for every morphism of stratified spaces. As an application, we construct, under suitable finiteness assumptions, a higher Artin derived stack of hyperconstructible hypersheaves and prove that every perversity function gives rise to an open substack of perverse hypersheaves. Using the derived structure, we provide the construction of a new cohomological Hall algebra, generalizing the ones of character varieties studied in the past by Schiffmann-Vasserot, Davison and Mistry.

CONTENTS

1. INTRODUCTION

Let X be a locally contractible topological space. In classical terms, the monodromy correspondence is an equivalence between the abelian category $\mathrm{LC}(X;\mathrm{Ab})$ of locally constant sheaves of abelian groups on X and the abelian category $\mathrm{Fun}(\Pi_1(X),\mathrm{Ab})$ of representations of the first fundamental groupoid of X. This equivalence can be seen as a concrete way of bridging the topological world, incarnated by the locally constant sheaves, and the algebraic world, incarnated by the representations of the first fundamental groupoid of X. The advent of higher categorical techniques [HTT] allowed to improve this correspondence by replacing $\Pi_1(X)$ with the *homotopy type* $\Pi_\infty(X)$ of X, which in many senses is still an object of algebraic nature. There is a natural map $\Pi_\infty(X) \to \Pi_1(X)$, which induces an equivalence

$$\mathrm{Fun}(\Pi_1(X),\mathrm{Ab}) \longrightarrow \mathrm{Fun}(\Pi_\infty(X),\mathrm{Ab}) \, ,$$

but interestingly this is no longer true if we replace Ab by its derived ∞-category $\mathcal{D}(\mathrm{Ab})$[1]. On the other hand, what is still true is that there is a canonical equivalence

$$\mathrm{Fun}(\Pi_\infty(X), \mathcal{D}(\mathrm{Ab})) \simeq \mathrm{LC}^{\mathrm{hyp}}(X; \mathcal{D}(\mathrm{Ab})) \,,$$

where the right hand side denotes the ∞-category of hypersheaves that are locally the hypersheafification of a constant presheaf. Notice that these two categories are sensitive to the whole homotopy type of X, rather than just to its first fundamental groupoid.

There is a second natural generalization of the monodromy correspondence, whose fundamental idea is due to MacPherson. When X is equipped with a stratification P, that is a continuous morphism $X \to P$ where P is a poset endowed with the Alexandroff topology (cf. Section 2.1), one can introduce a stratified variant of $\Pi_1(X)$, noted $\Pi_1^\Sigma(X, P)$. This is a non-full subcategory of $\Pi_1(X)$ which is no longer a groupoid; its objects are the same (that is, the points of X), but the only morphisms that are allowed are those that *exit* from lower strata to go to upper ones. For this reason, the category $\Pi_1^\Sigma(X, P)$ is called the *category of exit paths of* (X, P). In [T, Theorem 5.7], D. Treumann studied this category and established an equivalence

$$\mathrm{Fun}\left(\Pi_1^\Sigma(X, P), \mathrm{Ab}\right) \simeq \mathrm{Cons}_P(X; \mathrm{Ab}) \,,$$

where the right hand side denotes the abelian category of Ab-valued constructible sheaves. Following the influential work [BGH], this equivalence is nowadays referred to as the *exodromy equivalence*. It was generalized to the higher categorical world by J. Lurie in [HA, Appendix A]. His first step is to define a version $\Pi_\infty^\Sigma(X, P)$ of $\Pi_1^\Sigma(X, P)$ and establish a major result (Theorem A.6.4 in *loc. cit.*) asserting that if the stratification (X, P) is *conical*, then $\Pi_\infty^\Sigma(X, P)$ is indeed an ∞-category. Then, he proves:

Theorem 1.1 (J. Lurie, [HA, Theorem A.9.3])**.** *Let X be a paracompact topological space which is locally of singular shape and equipped with a conical P-stratification, where P is a partially ordered set satisfying the ascending chain condition. Then there is a natural equivalence*

$$\Psi_{X,P} \colon \mathrm{Fun}\left(\Pi_\infty^\Sigma(X, P), \mathcal{S}\right) \longrightarrow \mathrm{Cons}_P(X; \mathcal{S}) \,,$$

where $\mathcal{S}$ denotes the ∞-category of homotopy types (a.k.a. ∞-groupoids or animated sets).

The goal of this paper is to improve this result in several ways:

(1) replace the condition that X is locally of singular shape with the weaker condition that the strata of (X, P) are locally weakly contractible;

(2) remove the ascending chain condition on the poset P;

(3) allow more general coefficients than $\mathcal{S}$; in particular every presentable ∞-category that is either stable or a retract of a compactly generated ∞-category is permitted.

(4) provide an explicit formula for the inverse of $\Psi_{X,P}$.

We refer to Theorem 5.17 for the precise statement. Let us observe that improvements (1) and (2) are related, and the latter has already been addressed by D. Lejay in [L]. The key idea is to replace the ∞-category $\mathrm{Cons}_P(X; \mathcal{S})$ with the ∞-category $\mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{S})$ of P-*hyper*constructible *hyper*sheaves. Indeed, the singular shape assumption guarantees that every locally constant sheaf is automatically a hypersheaf. Working with hypersheaves from the very beginning, we can relax this assumption and replace it with the weaker one of having locally weakly contractible strata.

Improvements (3) and (4) are also related. The proof contained in [HA] explicitly relies on model categories, and is heavily built on the fact that functors from $\Pi_\infty^\Sigma(X, P)$ to $\mathcal{S}$ can be realized as left fibrations over $\Pi_\infty^\Sigma(X, P)$. These two facts together make it difficult to adapt the

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[1]Notice that throughout the main body of the paper, we will rather write $\mathrm{Mod}_\mathbb{Z}$ instead of $\mathcal{D}(\mathrm{Ab})$.

proof *verbatim* to different coefficients than $\mathcal{S}$. On the other hand, it is also difficult to bootstrap on Lurie's result to deduce the same result for arbitrary presentable ∞-categories, although it would be possible to deal with the case of compactly generated ∞-categories in this way.

We propose therefore a new proof which is purely ∞-categorical and that we briefly summarize below. Our approach has also several pleasant consequences. First and foremost, we obtain a *better functoriality* of the exodromy equivalence. Notably, if $f\colon (X, P) \to (Y, Q)$ is a morphism between conically stratified spaces, then the square

$$
\begin{array}{ccc}
\mathrm{Fun}\left(\Pi_\infty^\Sigma(Y, Q), \mathcal{S}\right) & \xrightarrow{\ \Psi_{Y,Q}\ } & \mathrm{Cons}_Q^{\mathrm{hyp}}(Y; \mathcal{S}) \\
\Big\downarrow{\scriptstyle \Pi_\infty^\Sigma(f)^*} & & \Big\downarrow{\scriptstyle f^{*,\mathrm{hyp}}} \\
\mathrm{Fun}\left(\Pi_\infty^\Sigma(X, P), \mathcal{S}\right) & \xrightarrow{\ \Psi_{X,P}\ } & \mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{S})
\end{array}
$$

is made 2-commutative by a natural transformation ψ_f that is a priori not invertible. We say that f is *exodromic* if ψ_f is an equivalence. In [HA, Proposition A.9.6] it is shown that if the stratification on X coincides with the stratification $f^{-1}(Q)$ induced by f, then f is exodromic. Our approach allows to improve this result:

Corollary 1.2 (See Proposition 6.8). *If $f\colon (X, P) \to (Y, Q)$ is a morphism between conically stratified spaces with locally weakly contractible strata, then f is exodromic.*

As a consequence of this improved functoriality, we obtain the following:

Corollary 1.3 (See Corollary 6.12). *Let (X, P) be a conically stratified space and let (X, Q) be a conical refinement of P. Assume that the strata of (X, P) and (X, Q) are locally weakly contractible. Then the natural map*

$$
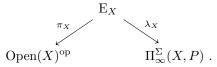f\colon \Pi_\infty^\Sigma(X, Q) \longrightarrow \Pi_\infty^\Sigma(X, P)
$$

is a localization.

Taking P to be the trivial stratification, we deduce that $\Pi_\infty(X)$ is a localization of $\Pi_\infty^\Sigma(X, Q)$. This result, which is of course extremely intuitive at the geometric level, was only known to the best of our knowledge in the conically smooth situation [AFTb, Proposition 1.2.13].

The key new ingredient in our proof is to interpolate between $\mathrm{Open}(X)^{\mathrm{op}}$ and $\Pi_\infty^\Sigma(X, P)$ as follows. Given a conically stratified space (X, P), let E_X be the ∞-category informally defined as:

- its objects are pairs (U, x), where U is an open subset of X and x is an object in $\Pi_\infty^\Sigma(U, P)$, where U is seen as a P-stratified space in the natural way;

- given two objects (U, x) and (V, y), the mapping space $\mathrm{Map}_{\mathrm{E}_X}((U, x), (V, y))$ is non-empty if and only if $V \subset U$, and in that case it coincides with $\mathrm{Map}_{\Pi_\infty^\Sigma(U, P)}(x, y)$.

Put otherwise, E_X is the Grothendieck construction of the functor $\Pi_\infty^\Sigma\colon \mathrm{Open}(X) \to \mathrm{Cat}_\infty$. We refer to Section 3 and to Notation 5.3 for the precise definition of E_X. This ∞-category comes with natural functors

$$
\begin{array}{ccc}
 & \mathrm{E}_X & \\
{\scriptstyle \pi_X}\swarrow & & \searrow{\scriptstyle \lambda_X} \\
\mathrm{Open}(X)^{\mathrm{op}} & & \Pi_\infty^\Sigma(X, P) \ .
\end{array}
$$

Then the functor $\Psi_{X,P}$ of Lurie can be identified with $\pi_{X,*} \circ \lambda_X^*$ where $\pi_{X,*}$ denotes the right Kan extension along $\pi_{X,*}$, and λ_X^* denotes the restriction along λ_X. The main bulk of the paper is dedicated to show, for more general coefficients than just $\mathcal{S}$ and independently from Lurie's result, that this functor and its left adjoint $\Phi_{X,P} \coloneqq \lambda_{X,!} \circ \pi_X^*$ realize the exodromy equivalence.

Let $F \in \mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E})$ be a hypersheaf. A simple inspection reveals that $\Phi_{X,P}(F)$ evaluated at one point $x \in X$ seen as an object in $\Pi_\infty^\Sigma(X, P)$ can be written as

$$\Phi_{X,P}(F)(x) \simeq \operatorname*{colim}_{(U,y,\gamma)} F(U) \ ,$$

where the colimit ranges over all morphisms $\gamma \colon y \to x$ in $\Pi_\infty^\Sigma(X, P)$ and over all possible open neighborhood U of y. If we could simply limit ourselves to the case where γ is the identity of x, this would simply produce the stalk F_x of F at the point x. However, this is typically false and $\Phi_{X,P}(F)(x)$ should be rather thought of as an "average over all the possible stalks at points close to x". Remarkably, when F is P-hyperconstructible, we can prove that there is indeed an equivalence

$$\Phi_{X,P}(F)(x) \simeq F_x \ .$$

See Corollary 6.10. Although this is formally obtained as a consequence of our results, in many ways one should think of this equivalence as the key technical ingredient needed in the proof of our main theorem. Interestingly, the left hand side only depends on the equivalence class of x inside $\Pi_\infty^\Sigma(X, P)$, a statement that neatly encodes the *parallel transport* on the right hand side.

In the rest of the paper, we explore the consequences of the exodromy correspondence. First, we obtain a couple of structural results for constructible hypersheaves:

Theorem 1.4 (See Corollaries 5.20, 5.22, 6.3 and 6.5). *Let (X, P) be a conically stratified space with locally weakly contractible strata. Let $\mathcal{E}$ be a presentable ∞-category satisfying the assumptions of Theorem 5.17. Then:*

(1) *Stability under limits and colimits: the ∞-category $\mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E})$ of P-hyperconstructible hypersheaves is presentable and closed under limits and colimits inside $\mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E})$;*

(2) *Recognition criterion: a hypersheaf $F \in \mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E})$ is P-hyperconstructible if and only if for every pair of open subsets $U \subseteq V$ of X for which the induced morphism $\Pi_\infty^\Sigma(U, P) \to \Pi_\infty^\Sigma(V, P)$ is a categorical equivalence, the restriction map*

$$F(V) \longrightarrow F(U)$$

is an equivalence in $\mathcal{E}$.

(3) *Tensor decomposition: the canonical equivalence $\mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E}) \simeq \mathrm{Sh}^{\mathrm{hyp}}(X) \otimes \mathcal{E}$ restricts to an equivalence*

$$\mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E}) \simeq \mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{S}) \otimes \mathcal{E} \ ;$$

(4) *Categorical Künneth formula: given a second conically stratified space (Y, Q) with locally weakly contractible strata, there is a canonical equivalence*

$$\mathrm{Cons}_{P \times Q}^{\mathrm{hyp}}(X \times Y; \mathcal{S}) \simeq \mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{S}) \otimes \mathrm{Cons}_Q^{\mathrm{hyp}}(Y; \mathcal{S}) \ .$$

Observe that point (1) implies that the inclusion $\mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E}) \hookrightarrow \mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E})$ admits both a left and a right adjoint. Combined with Corollary 1.2, it follows that for every morphism $f \colon (X, P) \to (Y, Q)$ of conically stratified spaces with locally weakly contractible strata, the pullback functor

$$f^{*,\mathrm{hyp}} \colon \mathrm{Cons}_Q^{\mathrm{hyp}}(Y; \mathcal{E}) \longrightarrow \mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E})$$

admits a left adjoint $f_\sharp^{\mathrm{hyp}}$, a statement that is generically false for the ∞-category $\mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E})$ (with the due exceptions, see [HPT, §2.2]). Points (2), (3) and (4) are generalizations to the setting of hyperconstructible hypersheaves of the analogous results already obtained in [HPT] for locally hyperconstant hypersheaves. As an immediate consequence of these results, we recover and generalize some of the results on Morita cohomology of J. V. Holstein [Ha, Hb]:

Corollary 1.5. *Let X be a locally weakly contractible topological space and let A be a $\mathbb{E}_\infty$-ring spectrum. Assume that X is connected and let $x \in X$ be a point. Then there exists a canonical equivalence*

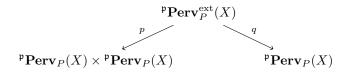$$\mathrm{LC}^{\mathrm{hyp}}(X; \mathrm{Mod}_A) \simeq \mathrm{Mod}_{\mathrm{C}_*(\Omega_x(X); A)} \ ,$$

where $\mathrm{C}_*(\Omega_x(X); A) \coloneqq \Pi_\infty(\Omega_x(X)) \otimes A$ are the A-valued chains on the based loop space of X at x.

Last but not least, we combine the results of this paper together with the recent finiteness result for $\Pi_\infty^\Sigma(X, P)$ of M. Volpe [V, Corollary 2.13], to prove the following result:

Theorem 1.6 (cf. Theorems 7.8 & 7.32). *Let X be a complex analytic space equipped with a Whitney stratification (X, P). Assume that for every $p \in P$, the stratum X_p is homotopically compact.[2] Then there exists a higher Artin derived stack $\mathbf{Cons}_P(X)$ parametrizing hyperconstructible hypersheaves on (X, P). Furthermore, for every perversity function $\mathfrak{p} \colon P \to \mathbb{Z}$, there exists an open substack $^{\mathfrak{p}}\mathbf{Perv}_P(X)$ of $\mathbf{Cons}_P(X)$ parametrizing $\mathfrak{p}$-perverse hypersheaves on X.*

In the main body of the paper we actually prove a finer result which does not require neither a complex analytic structure nor the stratification to be Whitney. The analysis carried out suggests that Volpe's finiteness result can be strengthened and we provide a precise statement of the expected result in Conjecture 7.10. Recall also that a similar result was obtained by Nitsure and Sabbah in [NS]. The novelty in our case is the derived structure and the fact that $\mathbf{Cons}_P(X)$ and $^{\mathfrak{p}}\mathbf{Perv}_P(X)$ are sensitive to the higher homotopical information of (X, P). As an example of application of the derived structure, we obtain the following result:

Theorem 1.7. *Let X be a Riemann surface and let (X, P) be a stratification on X made by a finite number of points. Let $\mathfrak{p}$ be the middle perversity function. Then the stable ∞-category $\mathrm{Coh}^{\mathrm{b}}(^{\mathfrak{p}}\mathbf{Perv}_P(X))$ carries an $\mathbb{E}_1$-monoidal structure à la Hall, whose underlying tensor product is given by pull-push along the correspondence*

$$
\begin{array}{ccc}
& ^{\mathfrak{p}}\mathbf{Perv}_P^{\mathrm{ext}}(X) & \\
{}^{p}\swarrow & & \searrow{}^{q} \\
^{\mathfrak{p}}\mathbf{Perv}_P(X) \times {}^{\mathfrak{p}}\mathbf{Perv}_P(X) & & ^{\mathfrak{p}}\mathbf{Perv}_P(X)
\end{array}
$$

Similarly, the Borel-Moore homology and the G-theory of $^{\mathfrak{p}}\mathbf{Perv}_P(X)$ carry a canonical Hall multiplication.

Implicit in the above theorem is the fact that the map p is derived locally complete intersection and that q is representable by proper algebraic spaces (see Lemma 7.34). This result generalizes [PS, Theorem 1.4], where the case of local systems was dealt with, and generalizes the cohomological Hall algebras of character varieties studied in [M]. This raises new interesting questions in geometric representation theory, that are outlined in Remark 7.36 and to which we will come back in a future work. Finally, we expect $\mathbf{Cons}_P(X)$ to carry a canonical shifted symplectic form, at least under some orientation condition on (X, P) that should conjecturally be related to Poincaré duality for the intersection homology of (X, P) [AFTa], and tightly related to the paper in preparation [APT].

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[2]This means that $\Pi_\infty(X)$ is a compact object in $\mathcal{S}$. This is a very mild condition that includes many cases of interest, see Example 7.3.

2. Review

2.1. **Stratified spaces.** The main reference for the definitions below is [HA, Appendix A].

If P be a poset, we endow P with the topology whose open subsets are the closed upward subsets $Q \subset P$. That is for every $a \in Q$ and $b \in P$ such that $b \geqslant a$, we have $b \in Q$.

Definition 2.1. Let X be a topological space. Let P be a poset. A *stratification of X by P* is a continuous morphism $f \colon X \to P$. For a subset $S \subset P$, we let $f|_S \colon X_S \to S$ be the induced stratification. For $a \in P$, the subset X_a is the stratum of (X, P) over a.

Remark 2.2. We abuse notations by denoting a stratification of X by P as (X, P) instead of $f \colon X \to P$ and refer to (X, P) as a stratified space. The collection of stratified spaces organize into a category in an obvious manner.

Example 2.3. Let $f \colon Y \to Q$ be a stratified space. Put $C(Y) := * \sqcup (Y \times \mathbb{R}_{>0})$. The set $C(Y)$ is endowed with the topology whose open subsets are the subsets $U \subset C(Y)$ such that $U \cap (Y \times \mathbb{R}_{>0})$ is open and if $* \in U$, then

$$C_\varepsilon(Y) := * \sqcup (Y \times (0, \varepsilon)) \subset U$$

for some $\varepsilon > 0$. Let $Q^\triangleleft$ be the poset obtained from Q by adding a smallest element $-\infty$. We define a continuous map $g \colon C(Y) \to Q^\triangleleft$ by sending $*$ to $-\infty$ and $(y, t) \in Y \times \mathbb{R}_{>0}$ to $f(y)$. We refer to $(C(Y), Q^\triangleleft)$ as the cone of (Y, Q).

Following [HA, A.6.2], we introduce the

Definition 2.4. Let (X, P) be a stratified space. We define $\mathrm{Exit}(X, P)$ as the simplicial subset of $\mathrm{Sing}\, X$ formed by the simplices $\sigma \colon |\Delta^n| \to X$ such that there exists a chain $a_1 \leqslant \cdots \leqslant a_n$ of elements of P such that for every $(t_0, \ldots, t_i, 0, \ldots, 0) \in |\Delta^n|$ with $t_i > 0$, we have $\sigma(t_0, \ldots, t_i, 0, \ldots, 0) \in X_{a_i}$.

The simplicial set $\mathrm{Exit}(X, P)$ is the *exit-path simplicial set of (X, P)*.

Notation 2.5. If X is trivially stratified $\Pi_\infty^\Sigma(X)$ is the homotopy type $\Pi_\infty(X)$ of X. In virtue of this observation, we think of $\Pi_\infty^\Sigma(X)$ as the *stratified homotopy type of X* [D, H], and for this reason if $\mathrm{Exit}(X, P)$ is an ∞-category we write $\Pi_\infty^\Sigma(X, P)$ instead of $\mathrm{Exit}(X, P)$.

The following lemma follows immediately from the definition of the exit-paths.

Lemma 2.6. *Let (X, P) be a stratified space. Let $S \subset P$ be a subset. Assume that $\mathrm{Exit}(X, P)$ and $\mathrm{Exit}(X_S, S)$ are ∞-categories. Then, the natural inclusion $\mathrm{Exit}(X_S, S) \to \mathrm{Exit}(X, P)$ is fully-faithful.*

Lemma 2.7. *Let (Y, Q) be a stratified space such that $\mathrm{Exit}(C(Y), Q^\triangleleft)$ is an ∞-category. Then, $*$ is an initial object in $\mathrm{Exit}(C(Y), Q^\triangleleft)$.*

Proof. Since $*$ is the only point of its stratum, $\mathrm{Map}_{\mathrm{Exit}(C(Y),Q^\triangleleft)}(*,*)$ is contractible. Let $(y,\varepsilon) \in Y \times \mathbb{R}_{>0}$. We are going to construct a deformation retract of

$$\mathrm{Map}_{\mathrm{Exit}(C(Y),Q^\triangleleft)}(*,(y,\varepsilon))$$

on the simplicial subset spanned by the exit path $\gamma \colon [0,1] \to C(Y)$ sending 0 to $*$ and t to $(y,t\varepsilon)$. To do this, we represent the above mapping space as $\mathrm{Hom}^R(*,(y,\varepsilon))$ and construct an homotopy

$$H \colon \mathrm{Hom}^R(*,(y,\varepsilon)) \times \Delta^1 \longrightarrow \mathrm{Hom}^R(*,(y,\varepsilon))$$

between the map constant to γ and the identity of $\mathrm{Hom}^R(*,(y,\varepsilon))$. At the cost of writing $\mathrm{Hom}^R(*,(y,\varepsilon))$ as a colimit of its simplices $\sigma \colon \Delta^n \longrightarrow \mathrm{Hom}^R(*,(y,\varepsilon))$, it is enough to construct an homotopy H between the n-simplex constant to γ and σ. That is, we have to construct

$$H \colon \Delta^n \times \Delta^1 \longrightarrow \mathrm{Hom}^R(*,(y,\varepsilon))$$

such that $H|_{\Delta^n \times \Delta^{\{0\}}}$ is constantly equal to γ and $H|_{\Delta^n \times \Delta^{\{1\}}} = \sigma$. Set $Z := |\Delta^n| \times [0,1]$ and $Z_\lambda := |\Delta^n| \times \{\lambda\}$ for every $\lambda \in [0,1]$. By definition, the above construction amounts to the construction of a continuous map

$$H \colon Z \star \Delta^0 \longrightarrow C(Y)$$

satisfying the following conditions :

(1) $H|_{Z_0 \star \Delta^0}$ is constantly equal to γ and $H|_{Z_1 \star \Delta^0} = \sigma$.

(2) H sends the base of $Z \star \Delta^0$ to $*$ and the tip of $Z \star \Delta^0$ to (y,ε).

(3) H sends $Z \times]0,1]$ in the stratum of (y,ε).

For $(u,t) \in Z_1 \times]0,1]$, put

$$\sigma(u,t) = (y(u,t),\varepsilon(u,t)) \in Y \times \mathbb{R}_{>0}$$

By definition $\sigma(u,1) = (y,\varepsilon)$. For $(u,\lambda,t) \in Z \star \Delta^0$, we define

$$H(u,\lambda,t) = \begin{cases} \big(y(u,t/\lambda),\lambda\varepsilon(u,t/\lambda)\big) & \text{if } 0 < t \leqslant \lambda \ , \\ \gamma(t) & \text{if } \lambda \leqslant t \ , \\ * & \text{if } t = 0 \ . \end{cases}$$

Observe that H satisfies conditions (1), (2), (3). To conclude the proof of Lemma 2.7, we have to show that H is continuous. Away from $t = 0$, the map H is continuous as the glueing of two continuous maps along $t = \lambda$. Let $(u_0,\lambda_0,0) \in Z \star \Delta^0$. We want to check that H is continuous at $(u_0,\lambda_0,0)$. If $\lambda_0 > 0$, we have to check that H is continuous on the open set $\lambda > t$, which is true since σ is continuous. The continuity of H at $(u_0,0,0)$ follows from the definition of the topology of $C(Y)$ and the observation that ε is bounded. $\square$

Definition 2.8. Let (X,P) be a stratified space. We say that (X,P) is *conically stratified* if for every point $x \in X$ lying over $a \in P$, there exists a topological space Z, a stratified space $(Y,P_{>a})$ and a morphism of stratified spaces $(Z \times C(Y), P_{\geqslant a}) \to (X,P)$ inducing an homeomorphism between $Z \times C(Y)$ and an open neighbourhood of x in X.

Remark 2.9. The open sub-stratified space $(Z \times C(Y), P_{\geqslant a})$ of (X,P) in Definition 2.8 is a *conical chart* of (X,P) at x. By definition of the topology of $C(Y)$, the set of conical charts of (X,P) at x form a fundamental system of open neighbourhoods of x in X.

Conically stratified spaces enjoy the following stability property:

Lemma 2.10. *Let (X,P) be a conically stratified space. Let $S \subset P$ be a subset. Then (X_S,S) is a conically stratified space.*

Proof. Let $x \in X_S$ and let $s \in S$ be the stratum of x. We have to show that (X_S, S) is conical at x. The question is local on X_S. At the cost of replacing X by the open set $X_{\geqslant s}$, we can thus suppose that s is the minimum of P. Since X is conical at x, we can suppose further that (X, P) is of the form $(Z \times C(Y), P)$ where Z is a topological space and where $(Y, P_{>s})$ is a stratified space. Then $X_S = Z \times C(Y_{S_{>s}})$ and Lemma 2.10 is proved. $\qquad\square$

The following theorem due to Lurie [HA, A.6.4] provides a wide range of stratified spaces whose exit-paths form an ∞-category:

Theorem 2.11. *Let (X, P) be a conically stratified space. Then $\mathrm{Exit}(X, P)$ is an ∞-category.*

Remark 2.12. In view of Theorem 2.11 and Notation 2.5, the ∞-category of exit-paths of a conically stratified space (X, P) will be denoted as $\Pi_\infty^\Sigma(X, P)$.

Definition 2.13. Let (X, P) be a conically stratified space. We say that (X, P) is *locally weakly contractible* if every point $x \in X$ admits a fundamental system of open neighbourhoods U such that x is an initial object of $\Pi_\infty^\Sigma(U, P)$.

Lemma 2.14. *Let $f \colon X \to X'$ be a weak homotopy equivalence and let (Y, Q) be a stratified space. Then the induced map*

$$\mathrm{Exit}(X \times Y, Q) \longrightarrow \mathrm{Exit}(X' \times Y, Q)$$

is a categorical equivalence of simplicial sets.

Proof. Unraveling the definitions, we find canonical *isomorphisms* of simplicial sets

$$\mathrm{Exit}(X \times Y, Q) \simeq \mathrm{Sing}(X) \times \mathrm{Exit}(Y, Q) \quad \text{and} \quad \mathrm{Exit}(X' \times Y, Q) \simeq \mathrm{Sing}(X') \times \mathrm{Exit}(Y, Q) \ .$$

Since the map f induces a categorical equivalence $\mathrm{Sing}(X) \to \mathrm{Sing}(X')$, the conclusion now follows from [HTT, Corollary 2.2.5.4]. $\qquad\square$

Proposition 2.15. *Let (X, P) be a conically stratified space. The following conditions are equivalent:*

(1) (X, P) is locally weakly contractible.

(2) The strata of (X, P) are locally weakly contractible.

Remark 2.16. Let (X, P) be a conically stratified space whose strata are CW-complexes. Then Proposition 2.15 states that (X, P) is locally weakly contractible if and only if its strata are locally contractible.

Proof of Proposition 2.15. Lemma 2.6 shows that (1) implies (2). Assume now that (2) holds. Let $x \in X$ be a point and let a be its image in P. Since X is conical, we can suppose that $X = Z \times C(Y)$ where (Y, Q) is a stratified space with a the minimum of $P = Q^\triangleleft$. Let $\mathcal{W}_x$ be a fundamental system of weakly contractible open neighborhoods of x in Z. Then $\{W \times C_\varepsilon(Y)\}_{W \in \mathcal{W}_x, \varepsilon > 0}$ is a fundamental system of open neighborhoods of x inside X. For $W \in \mathcal{W}_x$ and $\varepsilon > 0$, we have an isomorphism of simplicial sets

$$\mathrm{Exit}(W \times C_\varepsilon(Y), P) \simeq \mathrm{Sing}(W) \times \mathrm{Exit}(C_\varepsilon(Y), Q^\triangleleft)$$

On the other hand, if K and L are non empty simplicial sets such that $K \times L$ is an ∞-category, so are K and L. Thus, Theorem 2.11 ensures that $\mathrm{Exit}(C_\varepsilon(Y), Q^\triangleleft)$ is an ∞-category. Since W is weakly contractible, we obtain an equivalence of ∞-categories

$$\mathrm{Exit}(W \times C_\varepsilon(Y), P) \simeq \mathrm{Exit}(C_\varepsilon(Y), Q^\triangleleft)$$

sending x to $* \in C(Y)$. We conclude using Lemma 2.7. $\qquad\square$

2.2. Local excellence at strata.

Definition 2.17. Let (X, P) be a conically stratified space and let $S \subset P$ be a subposet. We say that an open neighbourhood U of X_S inside X is *excellent at S* if the functor

$$\text{Exit}(X_S, S) \longrightarrow \text{Exit}(U, P)$$

is a final.

Definition 2.18. Let (X, P) be a conically stratified space and let $S \subset P$ be a subposet. We say that (X, P) is *excellent at S* if the collection of excellent at S open neighbourhoods of X_S inside X forms a fundamental system of neighbourhoods of X_S inside X.

Definition 2.19. Let (X, P) be a conically stratified space and let $S \subset P$ be a subposet. We say that (X, P) is *locally excellent at S* if every point $x \in X_S$ admits a fundamental system of open neighbourhoods U such that the stratified space (U, P) is excellent at S.

Proposition 2.20. *Let (X, P) be a conically stratified space. Then (X, P) is locally excellent at every stratum.*

Proof. Let $a \in P$ and let $x \in X$ be a point lying over a. Since (X, P) is conically stratified, the point x admits a fundamental system of open neighbourhoods of the form $Z \times C(Y)$ where (Y, Q) is a stratified space, where Z is an open set of X_a containing x and where a is the minimum of $P_{\geqslant a} = Q^{\triangleleft}$. To prove Proposition 2.20, it is thus enough to prove that the conically stratified space $(Z \times C(Y), Q^{\triangleleft})$ is excellent at a. In that case, $\{Z \times C_\varepsilon(Y)\}_{\varepsilon > 0}$ is a fundamental system of open neighborhoods of Z. On the other hand, there is an isomorphism of simplicial sets

$$\text{Exit}(Z \times C_\varepsilon(Y), Q^{\triangleleft}) \simeq \text{Sing}(Z) \times \text{Exit}(C_\varepsilon(Y), Q^{\triangleleft})$$

Hence, the functor

$$\text{Exit}(Z \times *, \{a\}) \longrightarrow \text{Exit}(Z \times C_\varepsilon(Y), Q^{\triangleleft})$$

reads as a product

$$(2.21) \qquad \text{Sing}(Z) \times * \longrightarrow \text{Sing}(Z) \times \text{Exit}(C_\varepsilon(Y), Q^{\triangleleft})$$

From Lemma 2.7, the functor $* \to \text{Exit}(C_\varepsilon(Y), Q^{\triangleleft})$ is final. Thus, Eq. (2.21) is final as a consequence of [HTT, 4.1.1.13]. $\qquad\square$

2.3. Hypersheaves, hyperconstancy and hyperconstructibility.

Definition 2.22. Let X be a topological space. Let $\mathcal{E}$ be a presentable ∞-category. We denote by $\text{Sh}(X; \mathcal{E})$ the full subcategory of $\text{Fun}(\text{Open}(X)^{\text{op}}, \mathcal{E})$ spanned by $\mathcal{E}$-valued presheaves satisfying descent.

Remark 2.23. If $\mathcal{E} = \mathcal{S}$, we simply denote $\text{Sh}(X; \mathcal{E})$ by $\text{Sh}(X)$. In general, we have $\text{Sh}(X; \mathcal{E}) \simeq \text{Sh}(X) \otimes \mathcal{E}$, where the tensor product denotes the tensor product of presentable ∞-categories introduced in [HA, §4.8].

Definition 2.24. The category of *hypersheaves on X* is the subcategory $\text{Sh}^{\text{hyp}}(X) \subseteq \text{Sh}(X)$ spanned by objects which are local with respect to ∞-connective morphisms in $\text{Sh}(X)$. For a presentable ∞-category $\mathcal{E}$, we put

$$\text{Sh}^{\text{hyp}}(X; \mathcal{E}) := \text{Sh}^{\text{hyp}}(X) \otimes \mathcal{E} \simeq \text{Fun}^{\text{R}}(\text{Sh}^{\text{hyp}}(X)^{\text{op}}, \mathcal{E})$$

Remark 2.25. The canonical inclusion $\text{Sh}^{\text{hyp}}(X) \subset \text{Sh}(X)$ admits a left adjoint $(-)^{\text{hyp}} : \text{Sh}(X) \to \text{Sh}^{\text{hyp}}(X)$ referred to as the hypersheafification functor. Tensoring with $\mathcal{E}$ thus induces a functor

$$(-)^{\text{hyp}} : \text{Sh}(X; \mathcal{E}) \to \text{Sh}^{\text{hyp}}(X; \mathcal{E})$$

admitting a fully-faithful right adjoint. Hence, we can think of objects of $\mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E})$ as $\mathcal{E}$-valued presheaves satisfying hyperdescent.

Let $f\colon X \to Y$ be a morphism of topological spaces. Let $\mathcal{E}$ be a presentable ∞-category. Let

$$f^*\colon \mathrm{Sh}(Y;\mathcal{E}) \leftrightarrows \mathrm{Sh}(X;\mathcal{E})\colon f_*$$

be the canonical adjunction. The functor f_* preserve hypersheaves and thus restricts to a functor

$$f_*\colon \mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E}) \longrightarrow \mathrm{Sh}^{\mathrm{hyp}}(Y;\mathcal{E})$$

Furthermore, the functor f_* admits

$$f^{*,\mathrm{hyp}} \coloneqq (-)^{\mathrm{hyp}} \circ f^* \colon \mathrm{Sh}^{\mathrm{hyp}}(Y;\mathcal{E}) \longrightarrow \mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E}).$$

as left adjoint.

In the next definition, $\Gamma_X : X \to *$ denotes the tautological morphism.

Definition 2.26. Fix $F \in \mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E})$. We say that F is *hyperconstant* if F lies in the essential image of $\Gamma_X^{*,\mathrm{hyp}}\colon \mathcal{E} \to \mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E})$.

We say that F is *locally hyperconstant* if there exists a cover of X by open subsets $i\colon U \to X$ such that $i^{*,\mathrm{hyp}}(F)$ is hyperconstant. We denote by $\mathrm{LC}(X;\mathcal{E})$ the full subcategory of $\mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E})$ spanned by locally hyperconstant hypersheaves.

Following [L], we now introduce the main player of this paper:

Definition 2.27. Let (X,P) be a stratified space. Let $\mathcal{E}$ be a presentable ∞-category. An hypersheaf $F\colon \mathrm{Open}(X)^{\mathrm{op}} \to \mathcal{E}$ with value in $\mathcal{E}$ is *hyperconstructible* if for every $p \in P$, the hypersheaf $j_p^{*,\mathrm{hyp}}(F)$ is locally hyperconstant on X_p, where $j_p\colon X_p \to X$ denotes the canonical inclusion. We denote by $\mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathcal{E})$ the full-subcategory of $\mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E})$ spanned by hyperconstructible hypersheaves on (X,P).

2.4. **Change of coefficients and hyperconstructibility.** Let X be a topological space. Let $f\colon \mathcal{E} \to \mathcal{D}$ be a cocontinuous functor between presentable ∞-categories with right adjoint $g\colon \mathcal{D} \to \mathcal{E}$. Then, there is an adjunction

$$(2.28) \qquad f \circ -\colon \mathrm{Fun}(\mathrm{Open}(X)^{\mathrm{op}},\mathcal{E}) \leftrightarrows \mathrm{Fun}(\mathrm{Open}(X)^{\mathrm{op}},\mathcal{D})\colon g \circ -$$

whose left adjoint induces a cocontinuous functor

$$f^{\mathrm{hyp}} \coloneqq (-)^{\mathrm{hyp}} \circ f \circ -\colon \mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E}) \leftrightarrows \mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{D})\ .$$

Furthermore, for every continuous morphism of topological spaces $i : Y \to X$, the following diagram

$$
\begin{array}{ccc}
\mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E}) & \xrightarrow{\ i^{*,\mathrm{hyp}}\ } & \mathrm{Sh}^{\mathrm{hyp}}(Y;\mathcal{E}) \\
\Big\downarrow{\scriptstyle f^{\mathrm{hyp}}} & & \Big\downarrow{\scriptstyle f^{\mathrm{hyp}}} \\
\mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{D}) & \xrightarrow{\ i^{*,\mathrm{hyp}}\ } & \mathrm{Sh}^{\mathrm{hyp}}(Y;\mathcal{D})\ .
\end{array}
$$

commutes. In particular, if (X,P) is a stratified space, f^{hyp} restricts to a functor

$$f^{\mathrm{hyp}}\colon \mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathcal{E}) \longrightarrow \mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathcal{D})\ .$$

Remark 2.29. Note that f^{hyp} may not have a right adjoint in general. We will show that when Exodromy holds, a right adjoint exists and admits an explicit description. See Corollary 6.16.

Lemma 2.30. *Let X be a locally weakly contractible topological space. Let $f \colon \mathcal{E} \to \mathcal{D}$ be a cocontinuous functor between presentable ∞-categories with right adjoint $g \colon \mathcal{D} \to \mathcal{E}$. Then, the commutative diagram*

$$
\begin{array}{ccc}
\mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{D}) & \xrightarrow{\ g \circ -\ } & \mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E}) \\
\downarrow{\scriptstyle \Gamma_{X,*}} & & \downarrow{\scriptstyle \Gamma_{X,*}} \\
\mathcal{D} & \xrightarrow{\quad g \quad} & \mathcal{E}
\end{array}
$$

is vertically left adjointable. That is, the Beck-Chevalley transformation

$$
\Gamma_X^{*,\mathrm{hyp}} \circ g \longrightarrow g \circ \Gamma_X^{*,\mathrm{hyp}}
$$

is an equivalence. In particular, $g \circ -$ restricts to a well-defined functor

$$
g \circ - \colon \mathrm{LC}(X,\mathcal{D}) \longrightarrow \mathrm{LC}(X,\mathcal{E}).
$$

Proof. Let $d \in \mathcal{D}$ be an object. We have to show that

$$
\Gamma_X^{*,\mathrm{hyp}} g(d) \longrightarrow g \circ \Gamma_X^{*,\mathrm{hyp}}(d)
$$

is an equivalence of hypersheaves on X. It is enough to check that the above morphism is an equivalence above any weakly contractible open subset U of X. From [HPT, Theorem 2.13], we have

$$
(\Gamma_X^{*,\mathrm{hyp}} \circ g(d))(U) := \left(\Gamma_X^{-1}(g(d))\right)^{\mathrm{hyp}}(U) \simeq (\Gamma_X^{-1} \circ g(d))(U) \simeq g(d)
$$

From [HPT, Theorem 2.13] again, we have

$$
(g \circ \Gamma_X^{*,\mathrm{hyp}}(d))(U) = g\left((\Gamma_X^{*,\mathrm{hyp}}(d))(U)\right) \simeq g(d)
$$

and Lemma 2.30 follows. $\qquad\square$

3. The categorical framework

Let $\mathcal{X}$ be an ∞-category and let $\mathsf{A} \colon \mathcal{X} \to \mathrm{Cat}_\infty$ be a functor with values in small ∞-categories. Throughout this section we either assume $\mathcal{X}$ to be a small ∞-category with a terminal object $\mathbf{1}_{\mathcal{X}}$ or to be presentable. In the latter case, we assume A to be an accessible functor. We let

$$
\pi_{\mathsf{A}} \colon \mathcal{A} \longrightarrow \mathcal{X}^{\mathrm{op}}
$$

be the associated *cartesian* fibration. Writing $\mathbf{1}_{\mathcal{X}}$ for the terminal object of $\mathcal{X}$, [HTT, Corollary 3.3.4.3] provides a canonical *localization* functor

$$
\lambda_{\mathsf{A}} \colon \mathcal{A} \longrightarrow \mathsf{A}(\mathbf{1}_{\mathcal{X}}).
$$

Example 3.1. In this paper we are mainly interested in the following situations:

(1) Let X be a topological space. Take $\mathcal{X} := \mathrm{Open}(X)$ and $\mathsf{A} := \Pi_\infty$ the functor $\mathrm{Open}(X) \to \mathcal{S}$ sending an open set U to its homotopy type $\Pi_\infty(U)$. We will also take $\mathcal{X} := \mathrm{PSh}(X)$ and $\mathsf{A} \colon \mathrm{PSh}(X) \to \mathcal{S}$ the colimit-preserving functor obtained as left Kan extension of Π_∞ along the Yoneda embedding. From [HA, A.3.10], A carries ∞-connective morphisms in $\mathrm{PSh}(X)$ to equivalences in $\mathcal{S}$. If we finally put $\mathcal{X} := \mathrm{Sh}^{\mathrm{hyp}}(X)$, the functor A thus factors as a colimit-preserving functor issued from $\mathrm{Sh}^{\mathrm{hyp}}(X)$.

(2) Let (X, P) be a conically stratified topological space. Take $\mathcal{X} := \mathrm{Open}(X)$ and $\mathsf{A} := \Pi_\infty^{\Sigma}$ the functor $\mathrm{Open}(X) \to \mathrm{Cat}_\infty$ sending an open set U to $\Pi_\infty^{\Sigma}(U, P|_U)$. We will also take $\mathcal{X} := \mathrm{PSh}(X)$ and $\mathsf{A} \colon \mathrm{PSh}(X) \to \mathcal{S}$ the colimit-preserving functor obtained as left Kan extension of Π_∞^{Σ} along the Yoneda embedding. Similarly as in situation (1), A carries ∞-connective morphisms in $\mathrm{PSh}(X)$ to equivalences in Cat_∞. If we finally put $\mathcal{X} := \mathrm{Sh}^{\mathrm{hyp}}(X)$, the functor A thus factors as a colimit-preserving functor issued from

$\mathrm{Sh}^{\mathrm{hyp}}(X)$. We will abuse notations by using the notation Π_∞^Σ to also refer to the functors A and its factorization through $\mathrm{Sh}^{\mathrm{hyp}}(X)$.

Remark 3.2. Situation (1) is a special case of situation (2) in the same way monodromy is a special case of exodromy. Still, the proof of the latter is a reduction to the trivial stratification case. It will therefore be useful to analyze the situation (1) on its own.

3.1. **Size issues.** The main point of this paper is to show that the exodromy adjunction is realized by push-pull along the correspondence $\pi_A \times \lambda_A \colon \mathcal{A} \to \mathcal{X}^{\mathrm{op}} \times \mathsf{A}(\mathbf{1}_\mathcal{X})$. When $\mathcal{X}$ is a presentable ∞-category, there may be size-theoretical issues in defining the right Kan extension along π_A and the left Kan extension along λ_A. The following lemmas show this is not an actual problem:

Lemma 3.3. *For every $x \in \mathcal{X}$, the canonical functor*

$$\mathsf{A}(x) \longrightarrow \mathcal{A} \times_{\mathcal{X}^{\mathrm{op}}} (\mathcal{X}^{\mathrm{op}})_{x/}$$

is limit-final. In particular, for every presentable ∞-category $\mathcal{E}$, the pullback functor

$$\pi_A^* \colon \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{E}) \longrightarrow \mathrm{Fun}(\mathcal{A}, \mathcal{E})$$

admits a right adjoint, given by right Kan extension along π_A.

Proof. The second half of the statement is a direct consequence of the first half and the fact that $\mathsf{A}(x)$ is a small ∞-category, which implies that the right Kan extension along p is indeed well-defined. To prove the first half, we use Quillen's theorem A [HTT, Theorem 4.1.3.1]. Write $\mathcal{D} := \mathcal{A} \times_{\mathcal{X}^{\mathrm{op}}} (\mathcal{X}^{\mathrm{op}})_{x/}$. We can represent an object $\mathbf{a} \in \mathcal{D}$ as a pair $\mathbf{a} = (a, f)$ where a belongs to $\mathcal{A}$ and $f \colon x \to \pi_A(a)$ is a morphism in $\mathcal{X}^{\mathrm{op}}$. We have to prove that for every object $\mathbf{a} \in \mathcal{D}$, the ∞-category

$$\mathsf{A}(x) \times_\mathcal{D} \mathcal{D}_{/\mathbf{a}}$$

is weakly contractible. By definition, an object of $\mathsf{A}(x) \times_\mathcal{D} \mathcal{D}_{/\mathbf{a}}$ is a lift of f with target a. In particular, $\mathsf{A}(x) \times_\mathcal{D} \mathcal{D}_{/\mathbf{a}}$ identifies with $\mathcal{A}_{/a} \times_{\mathcal{X}^{\mathrm{op}}_{/\pi_A(a)}} \{f\}$. From [HTT, 2.4.1.9], the latter category admits a cartesian lift of f as final object. It is thus weakly contractible. $\square$

It is slightly more technical to deal with the left Kan extension along λ_A. It is useful to introduce the following definition:

Definition 3.4. Let κ be a regular cardinal. We say that a functor $f \colon \mathcal{C} \to \mathcal{D}$ is κ-cofiltered if for every essentially κ-small ∞-category I, the functor f has the right lifting property against the inclusion $I \hookrightarrow I^\triangleleft$. When $\kappa = \omega$, we simply say that f is cofiltered instead of ω-cofiltered.

The proof of the following lemma is straightforward and it is left to the reader:

Lemma 3.5. *Let κ be a regular cardinal. Then:*

(1) the collection of κ-filtered functors is closed under composition and pullback in Cat_∞;

(2) the functor $\Gamma_\mathcal{C} \colon \mathcal{C} \to *$ is κ-filtered if and only if $\mathcal{C}$ is κ-filtered.

Lemma 3.6. *Choose a regular cardinal $\kappa \gg 0$ such that $\mathcal{X}$ is κ-presentable and A is κ-accessible. Let*

$$\mathcal{A}^{(\kappa)} := (\mathcal{X}^\kappa)^{\mathrm{op}} \times_{\mathcal{X}^{\mathrm{op}}} \mathcal{A}.$$

Then for every $\alpha \in \mathsf{A}(\mathbf{1}_\mathcal{X})$ the canonical functor

$$\mathcal{A}^{(\kappa)} \times_{\mathsf{A}(\mathbf{1}_\mathcal{X})} \mathsf{A}(\mathbf{1}_\mathcal{X})_{/\alpha} \longrightarrow \mathcal{A} \times_{\mathsf{A}(\mathbf{1}_\mathcal{X})} \mathsf{A}(\mathbf{1}_\mathcal{X})_{/\alpha}$$

is colimit-final. In particular, for every presentable ∞-category $\mathcal{E}$, the pullback functor

$$\lambda_A^* \colon \mathrm{Fun}(\mathsf{A}(\mathbf{1}_\mathcal{X}), \mathcal{E}) \longrightarrow \mathrm{Fun}(\mathcal{A}, \mathcal{E})$$

admits a left adjoint, given by left Kan extension along λ_A.

Proof. The second half of the statement is a direct consequence of the first half and the fact that $\mathcal{A}^{(\kappa)} \times_{\mathsf{A}(\mathbf{1}_\mathcal{X})} \mathsf{A}(\mathbf{1}_\mathcal{X})_{/\alpha}$ is a small ∞-category, which implies that the left Kan extension along λ_A is indeed well-defined. To prove the first half, we use Quillen's theorem A. Write for simplicity

$$\mathcal{A}_{/\alpha} \coloneqq \mathcal{A} \times_{\mathsf{A}(\mathbf{1}_\mathcal{X})} \mathsf{A}(\mathbf{1}_\mathcal{X})_{/\alpha} \quad \text{and} \quad \mathcal{A}_{/\alpha}^{(\kappa)} \coloneqq \mathcal{A}^{(\kappa)} \times_{\mathsf{A}(\mathbf{1}_\mathcal{X})} \mathsf{A}(\mathbf{1}_\mathcal{X})_{/\alpha}.$$

Fix an object $\beta \in \mathcal{A}_{/\alpha}$, which we represent as a pair $\beta = (\beta, f)$, where $\beta \in \mathcal{A}$ and $f \colon \lambda_\mathsf{A}(\beta) \to \alpha$ is a morphism in $\mathsf{A}(\mathbf{1}_\mathcal{X})$. Write $\mathcal{A}_{\beta/\!/\alpha} \coloneqq (\mathcal{A}_{/\alpha})_{\beta/}$. Then, we have to prove that ∞-category

$$\mathcal{A}_{\beta/\!/\alpha}^{(\kappa)} \coloneqq \mathcal{A}_{/\alpha}^{(\kappa)} \times_{\mathcal{A}_{/\alpha}} \mathcal{A}_{\beta/\!/\alpha}$$

is weakly contractible. We claim that it is cofiltered. To see this, start by writing $x \coloneqq p(\beta)$, so that β can be seen as an element of $\mathsf{A}(x)$, and β as an element of $\mathsf{A}(x)_{/\alpha} \coloneqq \mathsf{A}(x) \times_{\mathsf{A}(\mathbf{1}_\mathcal{X})} \mathsf{A}(\mathbf{1}_\mathcal{X})_{/\alpha}$. We can thus set $\mathsf{A}(x)_{\beta/\!/\alpha} \coloneqq (\mathsf{A}(x)_{/\alpha})_{\beta/}$. Observe now that since π_A is a cartesian fibration, there is an induced commutative diagram

$$\begin{array}{ccc}
\mathcal{A}_{\beta/\!/\alpha}^{(\kappa)} & \longrightarrow & \mathcal{A}_{\beta/}^{(\kappa)} \\
\downarrow & & \downarrow \\
\mathsf{A}(x)_{\beta/\!/\alpha} & \longrightarrow & \mathsf{A}(x)_{\beta/} \ .
\end{array}$$

Inspection immediately reveals that it is a pullback square. Moreover, $\mathsf{A}(x)_{\beta/\!/\alpha}$ has an initial object, and it is in particular cofiltered. Thus, Lemma 3.5 reduces us to check that the map $\mathcal{A}_{\beta/}^{(\kappa)} \to \mathsf{A}(x)_{\beta/}$ is cofiltered. Unraveling the definitions, we see that it is enough to check that for every finite category I the canonical map

$$\operatorname*{colim}_{y \in \mathcal{X}_{/x}^\kappa} \operatorname{Fun}(I, \mathsf{A}(y)_{\beta/}) \longrightarrow \operatorname{Fun}(I, \mathsf{A}(x)_{\beta/})$$

is an equivalence. This, however, is guaranteed from the assumption that $\mathcal{X}$ is κ-filtered and that the functor A is κ-accessible. □

Fix a presentable ∞-category $\mathcal{E}$. Lemma 3.3 provides a functor

$$\Psi_{\mathcal{X},\mathsf{A}}^{\mathcal{E}} \coloneqq \pi_{\mathsf{A},*} \circ \lambda_\mathsf{A}^* \colon \operatorname{Fun}(\mathsf{A}(\mathbf{1}_\mathcal{X}), \mathcal{E}) \longrightarrow \operatorname{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{E}).$$

From Lemma 3.3, this functor is concretely described by the formula

$$(3.7) \qquad\qquad \Psi_{\mathcal{X},\mathsf{A}}^{\mathcal{E}}(F)(x) \simeq \lim_{\mathsf{A}(x)} F|_{\mathsf{A}(x)} \ ,$$

where the restriction is performed along the map $\mathsf{A}(x) \to \mathsf{A}(\mathbf{1}_\mathcal{X})$ induced by the canonical morphism $x \to \mathbf{1}_\mathcal{X}$. On the other hand, Lemma 3.6 shows the existence of a second functor

$$\Phi_{\mathcal{X},\mathsf{A}}^{\mathcal{E}} \coloneqq \lambda_{A,!} \circ \pi_\mathcal{A}^* \colon \operatorname{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{E}) \longrightarrow \operatorname{Fun}(\mathsf{A}(\mathbf{1}_\mathcal{X}), \mathcal{E}).$$

By construction, $\Phi_{\mathcal{X},\mathsf{A}}^{\mathcal{E}}$ is left adjoint to $\Psi_{\mathcal{X},\mathsf{A}}^{\mathcal{E}}$.

3.2. **Change of coefficients.** Let $\mathcal{E}$ be a presentable ∞-category. For every presentable ∞-category $\mathcal{E}'$ and for every $\mathcal{C} \in \operatorname{Cat}_\infty$, there is a canonical equivalence

$$\operatorname{Fun}(\mathcal{C}, \mathcal{E}) \otimes \mathcal{E}' \simeq \operatorname{Fun}(\mathcal{C}, \mathcal{E} \otimes \mathcal{E}') \ .$$

This gives an evident functoriality in $\mathcal{E}'$ with respect to morphisms in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. To see that the exodromy adjunction is compatible with this functoriality, consider first the following lemma:

Lemma 3.8. *Let $\mathcal{X}$ be an ∞-category with a terminal object $1_\mathcal{X}$. Let $\mathsf{A} \colon \mathcal{X} \to \operatorname{Cat}_\infty$ be a functor. For every presentable ∞-categories $\mathcal{E}$ and $\mathcal{E}'$, there is a canonical equivalence*

$$\Phi_{\mathcal{X},\mathsf{A}}^{\mathcal{E} \otimes \mathcal{E}'} \simeq \Phi_{\mathcal{X},\mathsf{A}}^{\mathcal{E}} \otimes \operatorname{id}_{\mathcal{E}'} \ .$$

Proof. Since $\Phi_{\mathcal{X},\mathsf{A}}^{\mathcal{E}\otimes\mathcal{E}'}$ commutes with colimits, it is enough to observe that for every $F \in \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{E})$ and $E' \in \mathcal{E}'$, one has

$$\Phi_{\mathcal{X},\mathsf{A}}^{\mathcal{E}}(F \otimes E') := \lambda_{\mathsf{A},!}(\pi_{\mathsf{A}}^*(F \otimes E')) \simeq \lambda_{\mathsf{A},!}(\pi_{\mathsf{A}}^*(F) \otimes E') \simeq \lambda_{\mathsf{A},!}(\pi_{\mathsf{A}}^*(F)) \otimes E' \ ,$$

where the last equivalence follows from the fact that the external tensor product $\otimes \colon \mathcal{E} \times \mathcal{E}' \to \mathcal{E} \otimes \mathcal{E}'$ commutes with the colimits computing the left Kan extension in the first variable. $\qquad\square$

Corollary 3.9. *Let $\mathcal{X}$ be an ∞-category with a terminal object $1_{\mathcal{X}}$. Let $\mathsf{A} \colon \mathcal{X} \to \mathrm{Cat}_\infty$ be a functor. Let $f \colon \mathcal{E} \to \mathcal{E}'$ be a morphism in Pr^{L}. Then the diagram*

$$
\begin{array}{ccc}
\mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{E}) & \xrightarrow{\Phi_{\mathcal{X},\mathsf{A}}^{\mathcal{E}}} & \mathrm{Fun}(\mathsf{A}(1_{\mathcal{X}}), \mathcal{E}) \\
{\scriptstyle f \circ -}\downarrow & & \downarrow{\scriptstyle f \circ -} \\
\mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{E}') & \xrightarrow{\Phi_{\mathcal{X},\mathsf{A}}^{\mathcal{E}'}} & \mathrm{Fun}(\mathsf{A}(1_{\mathcal{X}}), \mathcal{E}')
\end{array}
$$

is canonically commutative.

Proof. This follows from Lemma 3.8 and the observation that under the identification $\mathrm{Fun}(\mathcal{C}, \mathcal{E}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{S}) \otimes \mathcal{E}$ for $\mathcal{C} \in \mathrm{Cat}_\infty$, the functor $f \circ -$ corresponds to $\mathrm{id}_{\mathrm{Fun}(\mathcal{C}, \mathcal{S})} \otimes f$. $\qquad\square$

3.3. **The sheaf condition.** Assume now that $\mathcal{X}$ is presentable and that the functor A commutes with colimits. Then the adjunction $\Phi_{\mathcal{X},\mathsf{A}}^{\mathcal{E}} \dashv \Psi_{\mathcal{X},\mathsf{A}}^{\mathcal{E}}$ can be refined thanks to the following observation:

Lemma 3.10. *Assume that $\mathsf{A} \colon \mathcal{X} \to \mathrm{Cat}_\infty$ commutes with colimits. Then $\Psi_{\mathcal{X},\mathsf{A}}^{\mathcal{E}}$ factors through the full subcategory $\mathcal{X} \otimes \mathcal{E} \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{X}^{\mathrm{op}}, \mathcal{E})$ of $\mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{E})$.*

Proof. Fix a functor $F \colon \mathsf{A}(1_{\mathcal{X}}) \to \mathcal{E}$ and let $I \to \mathcal{X}^{\mathrm{op}}$ be a diagram, noted $i \mapsto x_i$, and let x denote the limit of this diagram. By assumption, the canonical map

$$\operatorname*{colim}_{i \in I^{\mathrm{op}}} \mathsf{A}(x_i) \longrightarrow \mathsf{A}(x)$$

is an equivalence, the colimit being computed in Cat_∞. It follows that the canonical map

$$\mathrm{Fun}(\mathsf{A}(x), \mathcal{E}) \longrightarrow \lim_{i \in I} \mathrm{Fun}(\mathsf{A}(x_i), \mathcal{E})$$

is an equivalence as well. Applying [PY, §8.2], we deduce a canonical equivalence

$$\lim_{\mathsf{A}(x)} F|_{\mathsf{A}(x)} \simeq \lim_{i \in I} \lim_{\mathsf{A}(x_i)} F|_{\mathsf{A}(x_i)} \ .$$

Thus, $\Psi_{\mathcal{X},\mathsf{A}}^{\mathcal{E}}(F)$ commutes with limits in $\mathcal{X}^{\mathrm{op}}$. $\qquad\square$

Corollary 3.11. *Assume that $\mathsf{A} \colon \mathcal{X} \to \mathrm{Cat}_\infty$ commutes with colimits. Then for any presentable ∞-category $\mathcal{E}$, the functors $\Phi_{\mathcal{X},\mathsf{A}}^{\mathcal{E}}$ and $\Psi_{\mathcal{X},\mathsf{A}}^{\mathcal{E}}$ induce an adjunction*

$$\Phi_{\mathcal{X},\mathsf{A}}^{\mathcal{E}} \colon \mathcal{X} \otimes \mathcal{E} \leftrightarrows \mathrm{Fun}(\mathsf{A}(1_{\mathcal{X}}), \mathcal{E}) \colon \Psi_{\mathcal{X},\mathsf{A}}^{\mathcal{E}} \ .$$

Example 3.12. Let us place ourselves again in the context of Example 3.1. Let (X, P) be a conically stratified space. To lighten the notation, we write

$$\Phi_{X,P}^{\mathrm{hyp},\mathcal{E}} := \Phi_{\mathrm{Sh}^{\mathrm{hyp}}(X),\Pi_\infty^{\Sigma}}^{\mathcal{E}} \quad \text{and} \quad \Psi_{X,P}^{\mathrm{hyp},\mathcal{E}} := \Psi_{\mathrm{Sh}^{\mathrm{hyp}}(X),\Pi_\infty^{\Sigma}}^{\mathcal{E}} \ .$$

When the stratification is trivial we further simplify these notations by removing the subscript P, so that in this case $\Pi_\infty = \Pi_\infty^{\Sigma}$. When X and $\mathcal{E}$ are clear from the context we also remove the corresponding decoration. Given $F \in \mathrm{Fun}(\Pi_\infty^{\Sigma}(X, P), \mathcal{E})$, Lemma 3.10 shows that the functor $\Psi(F)$ belongs to $\mathrm{Fun}^{\mathrm{R}}(\mathrm{Sh}^{\mathrm{hyp}}(X)^{\mathrm{op}}; \mathcal{E}) \simeq \mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E})$. The previous discussion shows that for an open subset U of X, $\Psi(F)(U)$ is canonically given by the formula

$$(3.13) \qquad\qquad \Psi(F)(U) \simeq \lim_{\Pi_\infty^{\Sigma}(U, P|_U)} F|_{\Pi_\infty^{\Sigma}(U, P|_U)}.$$

Finally, we have obvious variants

$$\Phi_{X,P}^{\mathrm{psh},\mathcal{E}} := \Phi_{\mathrm{PSh}(X),\Pi_\infty^\Sigma}^{\mathcal{E}} , \qquad \Psi_{X,P}^{\mathrm{psh},\mathcal{E}} := \Psi_{\mathrm{PSh}(X),\Pi_\infty^\Sigma}^{\mathcal{E}} ,$$
$$\Phi_{X,P}^{\mathcal{E}} := \Phi_{\mathrm{Open}(X),\Pi_\infty^\Sigma}^{\mathcal{E}} , \qquad \Psi_{X,P}^{\mathcal{E}} := \Psi_{\mathrm{Open}(X),\Pi_\infty^\Sigma}^{\mathcal{E}} ,$$

obtained replacing $\mathrm{Sh}^{\mathrm{hyp}}(X)$ by $\mathrm{PSh}(X)$ and $\mathrm{Open}(X)$ respectively.
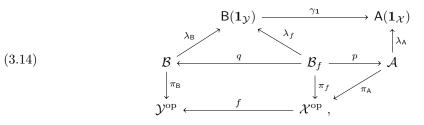
3.4. **Functoriality.** Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a functor. We assume that $\mathcal{X}$ and $\mathcal{Y}$ are either small with final objects, or presentable (importantly, we allow $\mathcal{X}$ to be of the first kind and $\mathcal{Y}$ to be of the second). Furthermore, we assume that $f(\mathbf{1}_{\mathcal{X}}) \simeq \mathbf{1}_{\mathcal{Y}}$, and if both $\mathcal{X}$ and $\mathcal{Y}$ are presentable then we additionally assume that f is a left adjoint. Let

$$\mathsf{B} \colon \mathcal{Y} \longrightarrow \mathrm{Cat}_\infty \quad \text{and} \quad \mathsf{A} \colon \mathcal{X} \longrightarrow \mathrm{Cat}_\infty$$

be two functors and let

$$\gamma \colon \mathsf{B} \circ f \longrightarrow \mathsf{A}$$

be a natural transformation. We can summarize this information in the following diagram:

(3.14)

where $\mathcal{B}_f$ and the maps π_f and q are defined by declaring that the bottom left square is a pullback, p is induced by the natural transformation γ and $\gamma_{\mathbf{1}} := \gamma_{\mathbf{1}_{\mathcal{X}}}$ is the value of γ on the final object of $\mathcal{X}$.

Lemma 3.15. *For every presentable ∞-category $\mathcal{E}$, the diagram*

$$\begin{array}{ccc} \mathrm{Fun}(\mathcal{Y}^{\mathrm{op}}, \mathcal{E}) & \xrightarrow{f^*} & \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{E}) \\ \downarrow{\scriptstyle \pi_{\mathsf{B}}^*} & & \downarrow{\scriptstyle \pi_f^*} \\ \mathrm{Fun}(\mathcal{B}, \mathcal{E}) & \xrightarrow{q^*} & \mathrm{Fun}(\mathcal{B}_f, \mathcal{E}) \end{array}$$

is vertically right adjointable.

Proof. The existence of the right adjoints to π_{A}^* and π_{B}^* follows from Lemma 3.3. In order to check that the Beck-Chevalley transformation

$$f^* \circ \pi_{\mathsf{B},*} \longrightarrow \pi_{f,*} \circ q^*$$

is an equivalence, it is enough to fix $F \in \mathrm{Fun}(\mathcal{B}, \mathcal{E})$ and $x \in \mathcal{X}^{\mathrm{op}}$. Then Lemma 3.3 identifies $f^*(\pi_{\mathsf{B},*}(F))(x)$ and $\pi_{f,*}(q^*(F))(x)$ with

$$\lim_{\mathsf{B}(f(x))} F|_{\mathsf{B}(f(x))}.$$

The conclusion follows. $\qquad\square$

Notation 3.16. Assume that $\mathcal{X}$ and $\mathcal{Y}$ are presentable and that $f \colon \mathcal{X} \to \mathcal{Y}$ commutes with colimits. For any presentable ∞-category $\mathcal{E}$ set $f_{\mathcal{E}} := f \otimes \mathrm{id}_{\mathcal{E}} \colon \mathcal{X} \otimes \mathcal{E} \to \mathcal{Y} \otimes \mathcal{E}$. We let $g_{\mathcal{E}}$ denote the right adjoint to $f_{\mathcal{E}}$.

Corollary 3.17. *Fix a presentable ∞-category $\mathcal{E}$.*

(1) Assume that $\mathcal{X}$ and $\mathcal{Y}$ are small. Then the diagrams

$$
\begin{array}{c}
\mathrm{Fun}(\mathsf{B}(\mathbf{1}_{\mathcal{Y}}),\mathcal{E}) \\
{\scriptstyle\Psi^{\mathcal{E}}_{\mathcal{Y},\mathsf{B}}}\swarrow \qquad \searrow {\scriptstyle\Psi^{\mathcal{E}}_{\mathcal{X},\mathsf{B}\circ f}} \\
\mathrm{Fun}(\mathcal{Y}^{\mathrm{op}},\mathcal{E}) \xrightarrow{\ \ f^*\ \ } \mathrm{Fun}(\mathcal{X}^{\mathrm{op}},\mathcal{E})
\end{array}
\qquad and \qquad
\begin{array}{c}
\mathrm{Fun}(\mathcal{X}^{\mathrm{op}},\mathcal{E}) \xrightarrow{\ \ f_!\ \ } \mathrm{Fun}(\mathcal{Y}^{\mathrm{op}},\mathcal{E}) \\
{\scriptstyle\Phi^{\mathcal{E}}_{\mathcal{X},\mathsf{B}\circ f}}\searrow \qquad \swarrow {\scriptstyle\Phi^{\mathcal{E}}_{\mathcal{Y},\mathsf{B}}} \\
\mathrm{Fun}(\mathsf{B}(\mathbf{1}_{\mathcal{Y}}),\mathcal{E})
\end{array}
$$

are canonically commutative.

(2) Assume that $\mathcal{X}$ and $\mathcal{Y}$ are presentable and that B and f commute with colimits. Then the diagrams

$$
\begin{array}{c}
\mathrm{Fun}(\mathsf{B}(\mathbf{1}_{\mathcal{Y}}),\mathcal{E}) \\
{\scriptstyle\Psi^{\mathcal{E}}_{\mathcal{Y},\mathsf{B}}}\swarrow \qquad \searrow {\scriptstyle\Psi^{\mathcal{E}}_{\mathcal{X},\mathsf{B}\circ f}} \\
\mathcal{Y}\otimes\mathcal{E} \xrightarrow{\ \ g_{\mathcal{E}}\ \ } \mathcal{X}\otimes\mathcal{E}
\end{array}
\qquad and \qquad
\begin{array}{c}
\mathcal{X}\otimes\mathcal{E} \xrightarrow{\ \ f_{\mathcal{E}}\ \ } \mathcal{Y}\otimes\mathcal{E} \\
{\scriptstyle\Phi^{\mathcal{E}}_{\mathcal{X},\mathsf{B}\circ f}}\searrow \qquad \swarrow {\scriptstyle\Phi^{\mathcal{E}}_{\mathcal{Y},\mathsf{B}}} \\
\mathrm{Fun}(\mathsf{B}(\mathbf{1}_{\mathcal{Y}}),\mathcal{E})
\end{array}
$$

are canonically commutative.

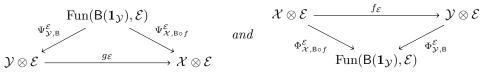*Proof.* It is enough to prove the commutativity of the left triangle of (1). Breaking $\Psi^{\mathcal{E}}_{\mathcal{Y},\mathsf{B}}$ into its components, we have to show that the squares of the following diagram commute:

$$
\begin{array}{ccc}
\mathrm{Fun}(\mathsf{B}(\mathbf{1}_{\mathcal{Y}}),\mathcal{E}) \xrightarrow{\ \lambda_{\mathsf{B}}^*\ } & \mathrm{Fun}(\mathcal{B},\mathcal{E}) \xrightarrow{\ \pi_{\mathsf{B},*}\ } & \mathrm{Fun}(\mathcal{Y}^{\mathrm{op}},\mathcal{E}) \\
\Big\| & \Big\downarrow{\scriptstyle q^*} & \Big\downarrow{\scriptstyle f^*} \\
\mathrm{Fun}(\mathsf{B}(\mathbf{1}_{\mathcal{Y}}),\mathcal{E}) \xrightarrow{\ \lambda_f^*\ } & \mathrm{Fun}(\mathcal{B}_f,\mathcal{E}) \xrightarrow{\ \pi_{f,*}\ } & \mathrm{Fun}(\mathcal{X}^{\mathrm{op}},\mathcal{E}) \ .
\end{array}
$$

For the left square, this just follows obviously from the definition of the functors. For the right square, this is a consequence of Lemma 3.15. $\qquad\square$

Example 3.18. Let (X,P) be a conically stratified topological space.

(1) Take $\mathcal{X} \coloneqq \mathrm{Open}(X)$, $\mathcal{Y} \coloneqq \mathrm{PSh}(X)$ and take f to be the Yoneda embedding. We take $\mathsf{A} \coloneqq \Pi_\infty^\Sigma$, the exit paths ∞-functor and we take B to be the left Kan extension of A along f. Then

$$
f^* \colon \mathrm{Fun}^{\mathrm{R}}(\mathrm{PSh}(X)^{\mathrm{op}},\mathcal{E}) \longrightarrow \mathrm{Fun}(\mathrm{Open}(X)^{\mathrm{op}},\mathcal{E})
$$

is an equivalence as a consequence of [HTT, Theorem 5.1.5.6]. Thus Lemma 3.10 (1) and Corollary 3.17 yield a natural equivalence

$$
\Psi^{\mathcal{E}}_{X,P} \simeq \Psi^{\mathrm{psh},\mathcal{E}}_{X,P} \ .
$$

Corollary 3.17 also identifies $\Phi^{\mathrm{psh},\mathcal{E}}_{X,P}$ with the left Kan extension of $\Phi^{\mathcal{E}}_{X,P}$ along the Yoneda embedding.

(2) Take $\mathcal{X} \coloneqq \mathrm{PSh}(X)$, $\mathcal{Y} \coloneqq \mathrm{Sh}^{\mathrm{hyp}}(X)$ and take $f \coloneqq (-)^{\mathrm{hyp}}$ to be the hypersheafification functor. In this case, Corollary 3.17 yields natural equivalences

$$
\Psi^{\mathrm{psh},\mathcal{E}}_{X,P} \simeq \Psi^{\mathrm{hyp},\mathcal{E}}_{X,P} \qquad and \qquad \Phi^{\mathrm{psh},\mathcal{E}}_{X,P} \simeq \Phi^{\mathrm{hyp},\mathcal{E}}_{X,P} \circ (-)^{\mathrm{hyp}} \ .
$$

In particular, we see that for every $F \in \mathrm{PSh}(X;\mathcal{E})$, the functor $\Phi^{\mathcal{E}}_{X,P}$ takes the natural transformation $F \to F^{\mathrm{hyp}}$ to an equivalence.

Construction 3.19. Associated to the top right square in the diagram (3.14) there is, for every presentable ∞-category $\mathcal{E}$, a Beck-Chevalley transformation

$$
\mathsf{BC} \colon \lambda_{f,!} \circ p^* \longrightarrow \gamma_{\mathbf{1}}^* \circ \lambda_{\mathsf{A},!} \ ,
$$

which induces the following transformation

$$\phi_{\mathsf{A},\mathsf{B},f,\gamma}\colon \Phi^{\mathcal{E}}_{\mathcal{Y},\mathsf{B}}\circ f_{\mathcal{E}}\simeq \Phi^{\mathcal{E}}_{\mathcal{X},\mathsf{B}\circ f}\simeq \lambda_{f,!}\circ \pi_f^*\simeq \lambda_{f,!}p^*\pi_{\mathsf{A}}^*\xrightarrow{\mathsf{BC}}\gamma_1^*\circ \lambda_{\mathsf{A},!}\circ \pi_{\mathsf{A}}^*\simeq \gamma_1^*\circ \Phi^{\mathcal{E}}_{\mathcal{X},\mathsf{A}}\ .$$

In turn, associated to $\phi_{\mathsf{A},\mathsf{B},f,\gamma}$ there is a natural exchange transformation

$$\psi_{\mathsf{A},\mathsf{B},f,\gamma}\colon f_{\mathcal{E}}\circ \Psi^{\mathcal{E}}_{\mathcal{X},\mathsf{A}}\xrightarrow{\eta_{\mathcal{Y}}}\Psi^{\mathcal{E}}_{\mathcal{Y},\mathsf{B}}\Phi^{\mathcal{E}}_{\mathcal{Y},\mathsf{B}}f_{\mathcal{E}}\Psi^{\mathcal{E}}_{\mathcal{X},\mathsf{A}}\xrightarrow{\phi_{\mathsf{A},\mathsf{B},f,\gamma}}\Psi^{\mathcal{E}}_{\mathcal{Y},\mathsf{B}}\gamma_1^*\Phi^{\mathcal{E}}_{\mathcal{X},\mathsf{A}}\Psi^{\mathcal{E}}_{\mathcal{X},\mathsf{A}}\xrightarrow{\varepsilon_{\mathcal{X}}}\Psi^{\mathcal{E}}_{\mathcal{Y},\mathsf{B}}\circ \gamma_1^*\ ,$$

where $\eta_{\mathcal{Y}}$ denotes the unit of the adjunction $\Phi^{\mathcal{E}}_{\mathcal{Y},\mathsf{B}}\dashv \Psi^{\mathcal{E}}_{\mathcal{Y},\mathsf{B}}$ and $\varepsilon_{\mathcal{X}}$ denotes the counit of the adjunction $\Phi^{\mathcal{E}}_{\mathcal{X},\mathsf{A}}\dashv \Psi^{\mathcal{E}}_{\mathcal{X},\mathsf{A}}$. In the rest of the paper we will often write ϕ_f and ψ_f instead of $\phi_{\mathsf{A},\mathsf{B},f,\gamma}$ and $\psi_{\mathsf{A},\mathsf{B},f,\gamma}$, respectively.

Remark 3.20.

(1) The triangular identities imply that the exchange transformation associated to $\psi_{\mathsf{A},\mathsf{B},f,\gamma}$ is once again $\phi_{\mathsf{A},\mathsf{B},f,\gamma}$. In particular, if one knows that both $\Phi^{\mathcal{E}}_{\mathcal{Y},\mathsf{B}}\dashv \Psi^{\mathcal{E}}_{\mathcal{Y},\mathsf{B}}$ and $\Phi^{\mathcal{E}}_{\mathcal{X},\mathsf{A}}\dashv \Psi^{\mathcal{E}}_{\mathcal{X},\mathsf{A}}$ are equivalences, then $\phi_{\mathsf{A},\mathsf{B},f,\gamma}$ is an equivalence if and only if $\psi_{\mathsf{A},\mathsf{B},f,\gamma}$ is one. Furthermore, triangular identities also imply that the square

$$
\begin{array}{ccc}
f & \xrightarrow{\ \eta_{\mathcal{Y}}f\ } & \Psi^{\mathcal{E}}_{\mathcal{Y},\mathsf{B}}\Phi^{\mathcal{E}}_{\mathcal{Y},\mathsf{B}}f \\
{\scriptstyle f\eta_{\mathcal{X}}}\downarrow & & \downarrow{\scriptstyle \Psi_{\mathcal{Y}}\phi_f} \\
f\Psi^{\mathcal{E}}_{\mathcal{X},\mathsf{A}}\Phi^{\mathcal{E}}_{\mathcal{X},\mathsf{A}} & \xrightarrow{\ \psi_f\Phi^{\mathcal{E}}_{\mathcal{X},\mathsf{A}}\ } & \Psi^{\mathcal{E}}_{\mathcal{Y},\mathsf{B}}\gamma_1^*\Phi^{\mathcal{E}}_{\mathcal{X},\mathsf{B}}
\end{array}
$$

is canonically commutative.

(2) Suppose that the natural transformation ψ_f is an equivalence, so that it renders the square

$$
\begin{array}{ccc}
\mathrm{Fun}(\mathsf{A}(\mathbf{1}_{\mathcal{X}}),\mathcal{E}) & \xrightarrow{\ \Psi^{\mathcal{E}}_{\mathcal{X},\mathsf{A}}\ } & \mathcal{X}\otimes \mathcal{E} \\
{\scriptstyle \gamma_1^*}\downarrow & & \downarrow{\scriptstyle f_{\mathcal{E}}} \\
\mathrm{Fun}(\mathsf{B}(\mathbf{1}_{\mathcal{Y}}),\mathcal{E}) & \xrightarrow{\ \Psi^{\mathcal{E}}_{\mathcal{Y},\mathsf{B}}\ } & \mathcal{Y}\otimes \mathcal{E}
\end{array}
$$

commutative. Then this square is horizontally right adjointable *if and only if* the natural transformation ϕ_f is an equivalence as well. Thus, if both ϕ_f and ψ_f are equivalences, it follows formally that $f_{\mathcal{E}}$ takes the unit of the adjunction $\Phi^{\mathcal{E}}_{\mathcal{X},\mathsf{A}}\dashv \Psi^{\mathcal{E}}_{\mathcal{X},\mathsf{A}}$ to the unit of the adjunction $\Phi^{\mathcal{E}}_{\mathcal{Y},\mathsf{B}}\dashv \Psi^{\mathcal{E}}_{\mathcal{Y},\mathsf{B}}$. Similarly, γ_1^* takes the counit of the adjunction $\Phi^{\mathcal{E}}_{\mathcal{X},\mathsf{A}}\dashv \Psi^{\mathcal{E}}_{\mathcal{X},\mathsf{A}}$ to the counit of the adjunction $\Phi^{\mathcal{E}}_{\mathcal{Y},\mathsf{B}}\dashv \Psi^{\mathcal{E}}_{\mathcal{Y},\mathsf{B}}$.

Corollary 3.21. *Assume that $\mathcal{X}$ and $\mathcal{Y}$ are presentable, that B commutes with colimits, that $f\colon \mathcal{X}\to \mathcal{Y}$ has a left adjoint $h\colon \mathcal{Y}\to \mathcal{X}$ and that the induced transformation*

$$\mathsf{B}\xrightarrow{\eta}\mathsf{B}\circ f\circ h\xrightarrow{\gamma}\mathsf{A}\circ h$$

is an equivalence, where η is the unit of the adjunction $h\dashv f$. Then the natural transformation

$$\psi_{\mathsf{A},\mathsf{B},f,\gamma}\colon f_{\mathcal{E}}\circ \Psi^{\mathcal{E}}_{\mathcal{X},\mathsf{A}}\longrightarrow \Psi^{\mathcal{E}}_{\mathcal{Y},\mathsf{B}}\circ \gamma_1^*$$

is an equivalence.

Proof. Indeed, since f has a left adjoint h, the same goes for $f_{\mathcal{E}}$ and moreover under the equivalences $\mathcal{X}\otimes \mathcal{E}\simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{X}^{\mathrm{op}},\mathcal{E})$ and $\mathcal{Y}\otimes \mathcal{E}\simeq \mathrm{Fun}^{\mathrm{L}}(\mathcal{Y}^{\mathrm{op}},\mathcal{E})$, the functor $f_{\mathcal{E}}$ corresponds to h^*. Unwinding the definitions and using Lemma 3.3, we see that for every $F\in \mathrm{Fun}(\mathsf{A}(\mathbf{1}_{\mathcal{X}}),\mathcal{E})$ and every $y\in \mathcal{Y}^{\mathrm{op}}$, the map $\psi_{\mathsf{A},\mathsf{B},f,\gamma}(F)$ at y is given by

$$\lim_{\mathsf{A}(h(y))}F|_{\mathsf{A}(h(y))}\longrightarrow \lim_{\mathsf{B}(y)}(F\circ u)|_{\mathsf{B}(y)}\ .$$

Thus, the conclusion follows from the fact that the functor $\mathsf{B}(y)\to \mathsf{A}(h(y))$ is an equivalence. $\square$

Example 3.22.

(1) Let $f\colon (Y,Q) \to (X,P)$ be a morphism of conically stratified spaces. Take $\mathcal{X} \coloneqq \mathrm{Sh}^{\mathrm{hyp}}(X)$ and $\mathcal{Y} \coloneqq \mathrm{Sh}^{\mathrm{hyp}}(Y)$ and $f^{*,\mathrm{hyp}}\colon \mathrm{Sh}^{\mathrm{hyp}}(X) \to \mathrm{Sh}^{\mathrm{hyp}}(Y)$ the induced geometric morphism. We let A and B be the exit paths functors

$$\Pi_\infty^\Sigma \colon \mathrm{Sh}^{\mathrm{hyp}}(X) \longrightarrow \mathrm{Cat}_\infty \qquad \text{and} \qquad \Pi_\infty^\Sigma \colon \mathrm{Sh}^{\mathrm{hyp}}(Y) \longrightarrow \mathrm{Cat}_\infty,$$

respectively. The functoriality of the exit path construction yields a natural transformation $\gamma\colon \mathsf{B} \circ f^{*,\mathrm{hyp}} \to \mathsf{A}$. In this setup, we denote by

$$\phi_f^{\mathrm{hyp}}\colon \Phi_Y^{\mathcal{E}} \circ f^{*,\mathrm{hyp}} \longrightarrow \Pi_\infty^\Sigma(f)^* \circ \Phi_X^{\mathcal{E}} \quad \text{and} \quad \psi_f^{\mathrm{hyp}}\colon f^{*,\mathrm{hyp}} \circ \Psi_X^{\mathcal{E}} \longrightarrow \Psi_Y^{\mathcal{E}} \circ \Pi_\infty^\Sigma(f)^*$$

the natural transformations of Construction 3.19. Taking $\mathcal{X} = \mathrm{PSh}(X)$ and $\mathcal{Y} = \mathrm{PSh}(Y)$ or $\mathcal{X} = \mathrm{Open}(X)$ and $\mathcal{Y} = \mathrm{Open}(Y)$ we obtain obvious variants of this construction, that we denote respectively by ϕ_f^{psh}, ψ_f^{psh}, ϕ_f and ψ_f.

(2) In the setting of the previous point, assume furthermore that the underlying morphism of topological spaces $f\colon Y \to X$ is an open immersion. Then $f^{*,\mathrm{hyp}}\colon \mathrm{Sh}^{\mathrm{hyp}}(X) \to \mathrm{Sh}^{\mathrm{hyp}}(Y)$ admits a fully faithful *left* adjoint, denoted $f_!^{\mathrm{hyp}}$. Furthermore, if U is an open subset of Y, seen as an object in $\mathrm{Sh}^{\mathrm{hyp}}(Y)$, then $f_!^{\mathrm{hyp}}(U) \in \mathrm{Sh}^{\mathrm{hyp}}(X)$ represents again U, seen as an open in X. It follows that the assumptions of Corollary 3.21 are satisfied and therefore that ψ_f^{hyp} is an equivalence in this case.

3.5. A glimpse of higher-functoriality. The natural transformations ϕ_f and ψ_f of Construction 3.19 enjoy themselves a form of higher functoriality. We fix two composable morphisms $f\colon \mathcal{X} \to \mathcal{Y}$ and $h\colon \mathcal{Y} \to \mathcal{Z}$ in Cat_∞. We further fix three functors

$$\mathsf{A}\colon \mathcal{X} \longrightarrow \mathrm{Cat}_\infty, \qquad \mathsf{B}\colon \mathcal{Y} \longrightarrow \mathrm{Cat}_\infty, \qquad \mathsf{C}\colon \mathcal{Z} \longrightarrow \mathrm{Cat}_\infty\ ,$$

together with natural transformations

$$\gamma\colon \mathsf{B} \circ f \longrightarrow \mathsf{A} \qquad \text{and} \qquad \delta\colon \mathsf{C} \circ h \longrightarrow \mathsf{B}\ .$$

As in the previous section, we assume that $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ are either small with a final objects or presentable. As for f and h, we assume that they preserve final objects and, if both their source and target are presentable, that they commute with colimits.

Fix a presentable ∞-category $\mathcal{E}$. In this situation, Construction 3.19 provides three natural transformations

$$\phi_h\colon \Phi_{\mathcal{Z},\mathsf{C}}^{\mathcal{E}} \circ h_{\mathcal{E}} \longrightarrow \delta_1^* \circ \Phi_{\mathcal{Y},\mathsf{B}}^{\mathcal{E}}\ , \qquad \phi_f\colon \Phi_{\mathcal{Y},\mathsf{B}}^{\mathcal{E}} \circ f_{\mathcal{E}} \longrightarrow \gamma_1^* \circ \Phi_{\mathcal{X},\mathsf{A}}^{\mathcal{E}}$$

as well as

$$\phi_{hf}\colon \Phi_{\mathcal{Z},\mathsf{C}}^{\mathcal{E}} \circ h_{\mathcal{E}} \circ f_{\mathcal{E}} \longrightarrow \delta_1^* \circ \gamma_1^* \circ \Phi_{\mathcal{X},\mathsf{A}}^{\mathcal{E}}\ .$$

We have:

Proposition 3.23. *The triangle*

(3.24)

$$\begin{array}{ccc}
\Phi_{\mathcal{Z},\mathsf{C}}^{\mathcal{E}} \circ h_{\mathcal{E}} \circ f_{\mathcal{E}} & \xrightarrow{\ \phi_h \circ f_{\mathcal{E}}\ } & \delta_1^* \circ \Phi_{\mathcal{Y},\mathsf{B}}^{\mathcal{E}} \circ f_{\mathcal{E}} \\
 & {\scriptstyle \phi_{hf}}\searrow & \downarrow {\scriptstyle \phi_f} \\
 & & \delta_1^* \circ \gamma_1^* \circ \Phi_{\mathcal{X},\mathsf{A}}^{\mathcal{E}}
\end{array}$$

is canonically commutative.

Proof. Consider the following diagram

Using Corollary 3.17 twice, we find canonical equivalences

$$\alpha \colon \Phi^{\mathcal{E}}_{\mathcal{Z},\mathsf{C}} \circ h_{\mathcal{E}} \circ f_{\mathcal{E}} \simeq \lambda_{hf,!} \circ \pi^*_{hf}, \qquad \beta \colon \Phi^{\mathcal{E}}_{\mathcal{Y},\mathsf{B}} \circ f_{\mathcal{E}} \simeq \lambda_{f,!} \circ \pi^*_f \ .$$

Write σ_1 for the square whose edges are $(\lambda_f, \delta_{\mathbf{1}}, \lambda_{hf}, s)$ and σ_2 for the square with edges $(\lambda_\mathsf{A}, \gamma_{\mathbf{1}}, \lambda_f, u)$. We also let σ_{12} denote the rectangle obtained juxtaposing σ_1 and σ_2. Write

$$\mathsf{BC}(\sigma_1) \colon \lambda_{hf,!} \circ s^* \longrightarrow \delta^*_{\mathbf{1}} \circ \lambda_{f,!}$$

for the Beck-Chevalley transformation associated to the square σ_1, and define similarly $\mathsf{BC}(\sigma_2)$ and $\mathsf{BC}(\sigma_{12})$. Unwinding Construction 3.19 and under the equivalences α and β above, we are called to prove that the diagram

is canonically commutative. Since σ_{12} is the composite square of σ_1 and σ_2, this follows from the well-known componibility property of Beck-Chevalley transformations. $\square$

Remark 3.25. The above proposition is just a shadow of the actual functorial dependence of $\Phi^{\mathcal{E}}_{\mathcal{X},\mathsf{A}}$ on the pair $(\mathcal{X}, \mathsf{A})$. To make this construction into an actual functor requires, as clearly shown by the above proof, to use the $(\infty, 2)$-functoriality property of the Beck-Chevalley transformation. Since this is not needed for the current paper, we leave the details for the interested reader.

Corollary 3.26. *In the same setting of Proposition 3.23, the triangle*

is canonically commutative.

Proof. We consider the case where $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are presentable, as every other case is dealt with similarly. We start with the following diagram:

$$
\begin{array}{ccc}
\mathcal{X} \otimes \mathcal{E} & \xrightarrow{\ f_{\mathcal{E}}\ } & \mathcal{Y} \otimes \mathcal{E} & \xrightarrow{\ h_{\mathcal{E}}\ } & \mathcal{Z} \otimes \mathcal{E} \\
\downarrow{\Phi_{\mathcal{X},\mathsf{A}}^{\mathcal{E}}} & & \downarrow{\Phi_{\mathcal{Y},\mathsf{B}}^{\mathcal{E}}} & & \downarrow{\Phi_{\mathcal{Z},\mathsf{C}}^{\mathcal{E}}} \\
\mathrm{Fun}(\mathsf{A}(\mathbf{1}_{\mathcal{X}}),\mathcal{E}) & \xrightarrow{\ \gamma_1^*\ } & \mathrm{Fun}(\mathsf{B}(\mathbf{1}_{\mathcal{Y}}),\mathcal{E}) & \xrightarrow{\ \delta_1^*\ } & \mathrm{Fun}(\mathsf{C}(\mathbf{1}_{\mathcal{Z}}),\mathcal{E})
\end{array}
$$

It is made 2-commutative by the transformations ϕ_f, ϕ_h. Proposition 3.23 identifies their composite $\phi_f \circ (\phi_h \circ f_{\mathcal{E}})$ with ϕ_{hf}. Since $\psi_f = \mathsf{BC}(\phi_f)$, $\psi_h = \mathsf{BC}(\phi_h)$ and $\psi_{hf} = \mathsf{BC}(\phi_{hf})$, the conclusion follows directly from the componibility property of the Beck-Chevalley transformations. $\square$

3.6. **Monodromy, revisited.** Let us explain how to recover the monodromy equivalence as a particular instance of Corollary 3.11. Let X be a topological space and let $\mathcal{E}$ be a presentable ∞-category. From [HPT, Construction 2.1], we have a functor

$$
\Pi_{\infty}^{\mathcal{E}} \colon \mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E}) \longrightarrow \mathcal{E} \ .
$$

When $\mathcal{E} = \mathcal{S}$ is the ∞-category of spaces, this functor – denoted simply by Π_{∞} – is the left Kan extension of the homotopy type functor, sending an open $U \in \mathrm{Open}(X)$ to its homotopy type $\Pi_{\infty}(U)$. For a general $\mathcal{E}$, we have a canonical identification $\Pi_{\infty}^{\mathcal{E}} \simeq \Pi_{\infty} \otimes \mathrm{id}_{\mathcal{E}}$. As observed in *loc. cit.*, $\Pi_{\infty}^{\mathcal{E}}$ has a right adjoint

$$
\Pi_{\mathcal{E}}^{\infty} \colon \mathcal{E} \longrightarrow \mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E}) \ .
$$

Assume now that X is locally weakly contractible. Then, [HPT, Proposition 2.5] implies that $\Pi_{\mathcal{E}}^{\infty}$ is canonically identified with the constant hypersheaf functor $\Gamma_X^{*,\mathrm{hyp}}$.

If $\mathcal{E} = \mathcal{S}$, the functor Π_{∞} can be canonically lifted to a functor

$$
\widetilde{\Pi}_{\infty} \colon \mathrm{Sh}^{\mathrm{hyp}}(X) \longrightarrow \mathcal{S}_{/\Pi_{\infty}(X)} \ ,
$$

which in turn admits a right adjoint $\widetilde{\Pi}^{\infty}$ sending $f \colon K \to \Pi_{\infty}(X)$ to the hypersheaf defined by

$$
(3.27) \qquad\qquad \widetilde{\Pi}^{\infty}(K)(U) \coloneqq \mathrm{Map}_{/\Pi_{\infty}(X)}(\Pi_{\infty}(U), K) \ .
$$

Put $\widetilde{\Pi}_{\infty}^{\mathcal{E}} \coloneqq \widetilde{\Pi}_{\infty} \otimes \mathrm{id}_{\mathcal{E}}$ and denote by $\widetilde{\Pi}_{\mathcal{E}}^{\infty}$ the right adjoint of $\widetilde{\Pi}_{\infty}^{\mathcal{E}}$.

Lemma 3.28. *Let X be a weakly contractible and locally weakly contractible topological space. Let $\mathcal{E}$ be a presentable ∞-category. Then, $\widetilde{\Pi}_{\mathcal{E}}^{\infty}$ factors through the full subcategory $\mathrm{LC}^{\mathrm{hyp}}(X;\mathcal{E})$.*

Proof. Since X is weakly contractible, $\Pi_{\infty}(X) \simeq *$, and therefore $\widetilde{\Pi}_{\infty}$ identifies with Π_{∞}. By tensoring with $\mathcal{E}$, we deduce that $\widetilde{\Pi}_{\infty}^{\mathcal{E}}$ identifies with $\Pi_{\infty}^{\mathcal{E}}$. Hence, $\widetilde{\Pi}_{\mathcal{E}}^{\infty}$ identifies with $\Pi_{\mathcal{E}}^{\infty}$. Since X is locally weakly contractible, [HPT, Proposition 2.5] supplies a canonical identification of $\Pi_{\mathcal{E}}^{\infty}$ with $\Gamma_X^{*,\mathrm{hyp}} \colon \mathcal{E} \to \mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E})$. The conclusion follows. $\square$

Lemma 3.29. *Let X be a locally weakly contractible topological space and let $\mathcal{E}$ be a presentable ∞-category. Then $\Phi_X^{\mathrm{hyp},\mathcal{E}}$ and $\widetilde{\Pi}_{\infty}^{\mathcal{E}}$ are canonically equivalent as functors from $\mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E})$ to $\mathrm{Fun}(\Pi_{\infty}(X),\mathcal{E})$. In particular, $\Psi_X^{\mathrm{hyp},\mathcal{E}}$ and $\widetilde{\Pi}_{\mathcal{E}}^{\infty}$ are canonically equivalent.*

Proof. It is enough to show that $\Phi_X^{\mathrm{hyp},\mathcal{E}}$ and $\widetilde{\Pi}_{\infty}^{\mathcal{E}}$ are canonically equivalent. From Lemma 3.8, we have $\Phi_X^{\mathrm{hyp},\mathcal{E}} \simeq \Phi_X^{\mathrm{hyp}} \otimes \mathrm{id}_{\mathcal{E}}$. Since $\widetilde{\Pi}_{\infty}^{\mathcal{E}} = \widetilde{\Pi}_{\infty} \otimes \mathrm{id}_{\mathcal{E}}$, we are left to show that Φ_X^{hyp} and $\widetilde{\Pi}_{\infty}$ are canonically equivalent. To do this, it is enough to show that Ψ_X^{hyp} and $\widetilde{\Pi}^{\infty}$ are canonically equivalent. Under the straightening equivalence $\mathcal{S}_{/\Pi_{\infty}(X)} \simeq \mathrm{Fun}(\Pi_{\infty}(X),\mathcal{S})$, the formula Eq. (3.27) for $\widetilde{\Pi}^{\infty}$ identifies canonically with the formula Eq. (3.7) for Ψ_X^{hyp}. Thus Ψ_X^{hyp} and $\widetilde{\Pi}^{\infty}$ are canonically equivalent. $\square$

Corollary 3.30. *Let X be a locally weakly contractible topological space. Then for every presentable ∞-category $\mathcal{E}$ the functor $\Psi_X^{\mathrm{hyp},\mathcal{E}} \colon \mathrm{Fun}(\Pi_\infty(X), \mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E})$ factors through the full subcategory $\mathrm{LC}^{\mathrm{hyp}}(X; \mathcal{E})$.*

Proof. Let $F \colon \Pi_\infty(X) \to \mathcal{E}$ be a functor. Since X is locally weakly contractible, it is enough to prove that for every weakly contractible open subset U of X the restriction $\Psi_X^{\mathrm{hyp},\mathcal{E}}(F)|_U$ is locally hyperconstant. Using Corollary 3.21 in the form of Example 3.22-(2), we find a canonical equivalence

$$\Psi_X^{\mathrm{hyp},\mathcal{E}}(F)|_U \simeq \Psi_U^{\mathrm{hyp},\mathcal{E}}(F|_{\Pi_\infty(U)}).$$

Hence, we are left to suppose that X is weakly contractible and locally weakly contractible. In that case, Corollary 3.30 follows from the combination of Lemma 3.29 and Lemma 3.28. $\qquad\square$

In particular, Corollary 3.30 implies that the adjunction $\Phi_X^{\mathrm{hyp},\mathcal{E}} \dashv \Psi_X^{\mathrm{hyp},\mathcal{E}}$ restricts to an adjunction

$$(3.31) \qquad\qquad \Phi_X^{\mathrm{hyp},\mathcal{E}} \colon \mathrm{LC}^{\mathrm{hyp}}(X; \mathcal{E}) \leftrightarrows \mathrm{Fun}(\Pi_\infty(X), \mathcal{E}) \colon \Psi_X^{\mathrm{hyp},\mathcal{E}} \ .$$

At this point, we have:

Corollary 3.32. *Let X be a locally weakly contractible topological space. Let $\mathcal{E}$ be a presentable ∞-category. Then the adjunction (3.31) is an equivalence.*

Proof. Combine Lemma 3.29 with the monodromy equivalence [HPT, Corollary 3.9]. $\qquad\square$

4. Exodromy adjunction for constructible hypersheaves

Fix a conically stratified space (X, P). We specialize the discussion of Section 3 to the context of Example 3.1-(2). In other words, we take $\mathcal{X} \coloneqq \mathrm{Sh}^{\mathrm{hyp}}(X)$ and consider the functor

$$\Pi_\infty^\Sigma \colon \mathrm{Sh}^{\mathrm{hyp}}(X) \longrightarrow \mathrm{Cat}_\infty$$

left Kan extended from the functor sending an open U of X to the ∞-category of exit paths $\Pi_\infty^\Sigma(U, P|_U)$. Van Kampen's theorem for exit paths (see [HA, Theorem A.7.1]) shows that Π_∞^Σ is a colimit preserving functor. Thus, Corollary 3.11 provides for every presentable ∞-category $\mathcal{E}$ an adjunction

$$\Phi_{X,P}^{\mathrm{hyp},\mathcal{E}} \colon \mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E}) \leftrightarrows \mathrm{Fun}(\Pi_\infty^\Sigma(X, P), \mathcal{E}) \colon \Psi_{X,P}^{\mathrm{hyp},\mathcal{E}} \ ,$$

which we refer to as the exodromy adjunction. When the stratification on X is trivial, Corollaries 3.30 and 3.32 show that this adjunction restricts to an equivalence between $\mathrm{Fun}(\Pi_\infty^\Sigma(X, P), \mathcal{E})$ and the ∞-category of locally hyperconstant hypersheaves on X. Our goal in this section is to show that the functor $\Psi_{X,P}^{\mathrm{hyp},\mathcal{E}}$ factors through the full subcategory $\mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E})$ of constructible hypersheaves on X, and therefore that the exodromy adjunction restricts to an adjunction

$$\Phi_{X,P}^{\mathrm{hyp},\mathcal{E}} \colon \mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E}) \leftrightarrows \mathrm{Fun}(\Pi_\infty^\Sigma(X, P), \mathcal{E}) \colon \Psi_{X,P}^{\mathrm{hyp},\mathcal{E}} \ .$$

Since the case of trivial stratification is already known, the main point is to show that the functor $\Psi_{X,P}^{\mathrm{hyp},\mathcal{E}}$ is compatible with the restriction to strata.

Throughout this section, we fix a presentable ∞-category $\mathcal{E}$. To lighten the notation, we will suppress the $\mathcal{E}$ superscript in the exodromy adjunction.

4.1. **Criteria for ψ_f^{hyp} to be an equivalence.** Let $f \colon (Y, Q) \to (X, P)$ be a morphism of conically stratified spaces. Construction 3.19 associated to f a natural transformation

$$\psi_f^{\mathrm{hyp}} \colon f^{*,\mathrm{hyp}} \circ \Psi_{X,P}^{\mathrm{hyp}} \longrightarrow \Psi_{Y,Q}^{\mathrm{hyp}} \circ \Pi_\infty^\Sigma(f)^* \ .$$

Our goal is to prove that ψ_f^{hyp} is an equivalence when f is the inclusion of a single stratum. For this, we start laying out general locality properties of ψ_f^{hyp}.

Notation 4.1. Let $f \colon (Y, Q) \to (X, P)$ be a morphism of conically stratified spaces. For every open $U \subseteq X$ we let $Y_U \coloneqq U \times_X Y$ and we denote by $f_U \colon (Y_U, Q) \to (U, P)$ the induced morphism.

The formation of the hypersheaf associated to a functor from the Exit Paths is local. This is justified by the following

Lemma 4.2. Let (X, P) be a conically stratified space. Let $j \colon U \to X$ be the inclusion of an open subset of X. Then, the natural transformation

$$\psi_j^{\mathrm{hyp}} \colon j^{*,\mathrm{hyp}} \circ \Psi_{X,P}^{\mathrm{hyp}} \longrightarrow \Psi_{U,P}^{\mathrm{hyp}} \circ \Pi_\infty^\Sigma(j)^*$$

is an equivalence.

Proof. Since $j \colon U \to X$ is the inclusion of an open subset of X, the functor $j^{*,\mathrm{hyp}} \colon \mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(U; \mathcal{E})$ is computed as the presheaf pullback j^{-1}. On the other hand, j^{-1} admits a fully-faithful left adjoint $j_!$. Hence, $j_\sharp^{\mathrm{hyp}} \coloneqq \mathrm{hyp} \circ j_!$ is left adjoint to $j^{*,\mathrm{hyp}}$. Since the hypersheafification commutes with j^{-1}, the unit transformation $\mathrm{id} \to j^{*,\mathrm{hyp}} \circ j_\sharp^{\mathrm{hyp}}$ is an equivalence. Then, Lemma 4.2 follows from Corollary 3.21. $\square$

Lemma 4.3. Let $f \colon (Y, Q) \to (X, P)$ be a morphism of conically stratified spaces. Let $i \colon U \to X$ be the inclusion of an open subset of X. Let $j \colon f^{-1}(U) \to Y$ be the induced inclusion. Then, for every $F \in \mathrm{Fun}(\Pi_\infty^\Sigma(X, P), \mathcal{E})$, the transformation $j^{*,\mathrm{hyp}}(\psi_f^{\mathrm{hyp}}(F))$ is canonically equivalent to $\psi_{f_U}^{\mathrm{hyp}}(\Pi_\infty^\Sigma(i)^*(F))$.

Proof. Apply Corollary 3.26 and Lemma 4.2 twice. $\square$

Corollary 4.4. Let $f \colon (Y, Q) \to (X, P)$ be a morphism of conically stratified spaces. Let $\mathcal{B}$ be a basis for the topology of X. Assume that for every $U \in \mathcal{B}$, the transformation

$$\psi_{f_U}^{\mathrm{hyp}} \colon f_U^{*,\mathrm{hyp}} \circ \Psi_{U,P}^{\mathrm{hyp}} \longrightarrow \Psi_{Y_U,Q}^{\mathrm{hyp}} \circ \Pi_\infty^\Sigma(f_U)^*$$

is an equivalence. Then, ψ_f^{hyp} is an equivalence.

Proof. Fix $F \in \mathrm{Fun}(\Pi_\infty^\Sigma(X, P), \mathcal{E})$. We have to show that $\psi_f^{\mathrm{hyp}}(F)$ is an equivalence of hypersheaves on Y. This statement is local on Y. In particular, for $U \in \mathcal{B}$, if $i \colon U \to X$ and $j \colon f^{-1}(U) \to Y$ are the inclusions, it is enough to show that $j^{*,\mathrm{hyp}}(\psi_f^{\mathrm{hyp}}(F))$ is an equivalence. Then, Corollary 4.4 follows from Lemma 4.3. $\square$

Lemma 4.5. Let $f \colon (Y, Q) \to (X, P)$ be a morphism of conically stratified spaces. Then, ψ_f^{hyp} is canonically equivalent to $\mathrm{hyp}(\psi_f^{\mathrm{psh}})$.

Proof. Apply Corollary 3.26 and Example 3.18-(2) twice. $\square$

Corollary 4.6. Let $f \colon (Y, Q) \to (X, P)$ be a morphism of conically stratified spaces. Fix $F \in \mathrm{Fun}(\Pi_\infty^\Sigma(X, P), \mathcal{E})$. Assume that the collection $\mathcal{B} = \mathcal{B}(f, F)$ of open subsets $V \subseteq Y$ for which the transformation

$$\psi_f^{\mathrm{psh}}(F)(V) \colon f^{-1}\big(\Psi_{X,P}^{\mathrm{psh}}(F)\big)(V) \to \Psi_{Y,Q}^{\mathrm{psh}}\big(\Pi_\infty^\Sigma(f)^*(F)\big)(V)$$

is an equivalence is a basis for the topology of Y. Then, $\psi_f^{\mathrm{hyp}}(F)$ is an equivalence.

Proof. Let $j\colon \mathcal{B} \hookrightarrow \mathrm{Open}(Y)$ be the canonical inclusion. Consider the commutative square

$$
(4.7)
$$

$$
\begin{array}{ccc}
f^{-1}\big(\Psi^{\mathrm{psh}}_{X,P}(F)\big) & \xrightarrow{\ \psi^{\mathrm{psh}}_f(F)\ } & \Psi^{\mathrm{psh}}_{Y,Q}\big(\Pi^{\Sigma}_{\infty}(f)^*(F)\big) \\
\downarrow & & \downarrow \\
j_*j^{-1}f^{-1}\big(\Psi^{\mathrm{psh}}_{X,P}(F)\big) & \xrightarrow{j_*j^{-1}\psi^{\mathrm{psh}}_f(F)} & j_*j^{-1}\Psi^{\mathrm{psh}}_{Y,Q}\big(\Pi^{\Sigma}_{\infty}(f)^*(F)\big)
\end{array}
$$

From Example 3.18, the presheaf $\Psi^{\mathrm{psh}}_{Y,Q}\big(\Pi^{\Sigma}_{\infty}(f)^*(F)\big)$ is a hypersheaf on Y. From [HPT, Proposition 1.13], the adjunction $j^{-1} \dashv j_*$ induces an equivalence between $\mathrm{Sh}^{\mathrm{hyp}}(Y;\mathcal{E})$ and the full subcategory $\mathrm{Sh}^{\mathrm{hyp}}(\mathcal{B};\mathcal{E})$ of $\mathrm{PSh}(\mathcal{B};\mathcal{E})$ spanned by the functors $F\colon \mathcal{B}^{\mathrm{op}} \to \mathcal{E}$ such that $j_*(F)$ is a hypersheaf on Y. Hence, the right vertical arrow of Eq. (4.7) is an equivalence. Since $j^{-1}\psi^{\mathrm{psh}}_f(F)$ is an equivalence by assumption, the bottom arrow is an equivalence as well. Thus, $j_*j^{-1}f^{-1}\big(\Psi^{\mathrm{psh}}_{X,P}(F)\big)$ is a hypersheaf on Y. Then, [HPT, Remark 1.14] implies that the left vertical arrow exhibits its target as the hypersheafification of its source. Hence,

$$
j_*j^{-1}\psi^{\mathrm{psh}}_f(F) \simeq \mathrm{hyp}(\psi^{\mathrm{psh}}_f(F)) \simeq \psi^{\mathrm{hyp}}_f(F)
$$

where the last equivalence follows from Lemma 4.5. Corollary 4.6 thus follows. $\qquad\square$

Corollary 4.8. *Let $f\colon Y \to X$ be a morphism between locally weakly contractible topological spaces. Then for every presentable ∞-category $\mathcal{E}$, both transformations*

$$
\psi^{\mathrm{hyp}}_f\colon f^{*,\mathrm{hyp}} \circ \Psi^{\mathrm{hyp}}_X \longrightarrow \Psi^{\mathrm{hyp}}_Y \circ \Pi_{\infty}(f)^* \quad and \quad \phi^{\mathrm{hyp}}_f\colon \Phi^{\mathrm{hyp}}_Y \circ f^{*,\mathrm{hyp}} \longrightarrow \Pi_{\infty}(f)^* \circ \Phi^{\mathrm{hyp}}_X
$$

are equivalences once restricted to locally hyperconstant hypersheaves.

Proof. Combining Remark 3.20-(1) and Corollary 3.32 we see that it is enough to show that ψ_f is an equivalence. Using Corollary 4.4, we can reduce ourselves to the case where X is weakly contractible, so that the conclusion follows from Lemma 3.29 and [HPT, Proposition 2.5]. $\qquad\square$

4.2. **Restriction to strata for Ψ.**

Proposition 4.9. *Let (X,P) be a conically stratified space. Let $a \in P$ and let $i_a\colon X_a \to X$ be the inclusion of the associated stratum. Then the natural transformation*

$$
\psi^{\mathrm{hyp}}_a\colon i_a^{*,\mathrm{hyp}} \circ \Psi^{\mathrm{hyp}}_{X,P} \longrightarrow \Psi^{\mathrm{hyp}}_{X_a} \circ \Pi^{\Sigma}_{\infty}(i_a)^*
$$

is an equivalence.

Proof. Let $\mathcal{B}$ be the poset of open subsets U of X_a such that the set $\mathcal{B}^{\mathrm{exc}}(U)$ of open neighbourhoods V of U in X where $\Pi^{\Sigma}_{\infty}(U,\{a\}) \to \Pi^{\Sigma}_{\infty}(V,P)$ is final is a fundamental system of open neighbourhoods of U in X. From Proposition 2.20, the set $\mathcal{B}$ is a basis of X_a. Fix $F \in \mathrm{Fun}(\Pi^{\Sigma}_{\infty}(X,P),\mathcal{E})$. From Corollary 4.6, we are left to show that for every $U \in \mathcal{B}$, the morphism

$$
\psi^{\mathrm{psh}}_{i_a}(F)(U)\colon i_a^{-1}\big(\Psi^{\mathrm{psh}}_{X,P}(F)\big)(U) \to \Psi^{\mathrm{psh}}_{X_a,\{a\}}\big(\Pi^{\Sigma}_{\infty}(i_a)^*(F)\big)(U)
$$

is an equivalence. On the one hand

$$
i_a^{-1}\big(\Psi^{\mathrm{psh}}_{X,P}(F)\big)(U) \simeq \operatorname*{colim}_{V \in \mathcal{B}^{\mathrm{exc}}(U)} \Psi^{\mathrm{psh}}_{X,P}(F)(V) \simeq \operatorname*{colim}_{V \in \mathcal{B}^{\mathrm{exc}}(U)} \lim_{\Pi^{\Sigma}_{\infty}(V,P)} F|_{\Pi^{\Sigma}_{\infty}(V,P)} \ .
$$

Since for every $V \in \mathcal{B}^{\mathrm{exc}}(U)$, the functor $\Pi^{\Sigma}_{\infty}(U,\{a\}) \to \Pi^{\Sigma}_{\infty}(V,P)$ is final, we deduce

$$
i_a^{-1}\big(\Psi^{\mathrm{psh}}_{X,P}(F)\big)(U) \simeq \operatorname*{colim}_{V \in \mathcal{B}^{\mathrm{exc}}(U)} \lim_{\Pi^{\Sigma}_{\infty}(U,\{a\})} F|_{\Pi^{\Sigma}_{\infty}(U,\{a\})} \simeq \lim_{\Pi^{\Sigma}_{\infty}(U,\{a\})} F|_{\Pi^{\Sigma}_{\infty}(U,\{a\})} \ .
$$

On the other hand,

$$
\Psi^{\mathrm{hyp}}_{X_a}\big(\Pi^{\Sigma}_{\infty}(i_a)^*(F)\big)(U) \simeq \lim_{\Pi^{\Sigma}_{\infty}(U,\{a\})} F|_{\Pi^{\Sigma}_{\infty}(U,\{a\})} \ .
$$

This concludes the proof of Proposition 4.9. □

Corollary 4.10. *Let (X, P) be a conically stratified space with locally weakly contractible strata. Then the functor $\Psi_{X,P}^{\mathrm{hyp}} \colon \mathrm{Fun}(\Pi_\infty^\Sigma(X, P), \mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E})$ factors through the full subcategory $\mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E})$.*

Proof. This is an immediate consequence of Proposition 4.9 and Corollary 3.30. □

5. The exodromy equivalence

5.1. **A criterion for ϕ_f to be an equivalence.** Let $f \colon (Y, Q) \to (X, P)$ be a morphism of conically stratified spaces and let $\mathcal{E}$ be a presentable ∞-category. As usual $\mathcal{E}$ will be fixed throughout this section, and we therefore suppress it from the notations. Our first goal is to provide a general method to establish when the natural transformation

$$\phi_f^{\mathrm{hyp}} \colon \Phi_{Y,Q}^{\mathrm{hyp}} \circ f^{*,\mathrm{hyp}} \longrightarrow \Pi_\infty^\Sigma(f)^* \circ \Phi_{X,P}^{\mathrm{hyp}}$$

is an equivalence.

Lemma 5.1. *Let $f \colon (Y, Q) \to (X, P)$ be a morphism of conically stratified spaces. Then the following statements hold;*

    (1) the transformations ϕ_f^{psh} and $\phi_f^{\mathrm{hyp}} \circ \mathrm{hyp}$ are canonically equivalent.

    (2) for every diagram $F \colon I \to \mathrm{PSh}(X; \mathcal{E})$ with I a small ∞-category, the canonical morphism

$$\phi_f^{\mathrm{psh}}(\operatorname*{colim}_{i \in I} j(F_i)) \longrightarrow \operatorname*{colim}_{i \in I} \phi_f(F_i)$$

    is an equivalence, where $j \colon \mathrm{PSh}(X; \mathcal{E}) \to \mathrm{Fun}(\mathrm{PSh}(X; \mathcal{E})^{\mathrm{op}}; \mathcal{E})$ is the Yoneda embedding.

Proof. For the first point, apply Example 3.18-(2) and Proposition 3.23 twice. For the second point, we have

$$\phi_f^{\mathrm{psh}}(\operatorname*{colim}_{i \in I} j(F_i)) \simeq \operatorname*{colim}_{i \in I} \phi_f^{\mathrm{psh}}(j(F_i)) \simeq \operatorname*{colim}_{i \in I} \phi_f(F_i)$$

where the first equivalence follows from the fact that ϕ_f^{psh} is built out of left adjoint functors and where the second equivalence follows from the observation that $\Phi_{X,P}^{\mathrm{psh}}$ identifies with the left Kan extension of $\Phi_{X,P}$ along the Yoneda embedding. □

Corollary 5.2. *If ϕ_f is an equivalence, the same goes for ϕ_f^{psh} and ϕ_f^{hyp}.*

Proof. The first statement comes from Lemma 5.1-(2) and the fact that every presheaf is a small colimit of representable objects. The second statements then follows from the observation from Lemma 5.1-(1). □

The advantage of ϕ_f is that it can be explicitly computed in terms of open subsets of Y, rather than having to deal with all hypersheaves on Y. To fully exploit this, let us specialize the main construction of Section 3.4 to the current setting.

Notation 5.3. For a conically stratified space (X, P), let

$$\pi_X \colon \mathrm{E}_X \longrightarrow \mathrm{Open}(X)^{\mathrm{op}}$$

be the cartesian fibration classifying the exit paths ∞-functor $\Pi_\infty^\Sigma \colon \mathrm{Open}(X) \to \mathrm{Cat}_\infty$. We let

$$\lambda_X \colon \mathrm{E}_X \longrightarrow \Pi_\infty^\Sigma(X, P)$$

be the canonical localization functor. Given a morphism $f \colon (Y, Q) \to (X, P)$ of conically stratified spaces, we consider the following diagram:

(5.4)

$$
\begin{array}{ccc}
& \Pi^{\Sigma}_{\infty}(Y,Q) \xrightarrow{\quad \Pi^{\Sigma}_{\infty}(f) \quad} \Pi^{\Sigma}_{\infty}(X,P) &
\end{array}
$$

$$
\begin{array}{ccccc}
& \lambda_Y \nearrow & \lambda_f \nwarrow & & \lambda_X \uparrow \\
\mathrm{E}_Y & \xleftarrow{\quad q \quad} & \mathrm{E}_f & \xrightarrow{\quad p \quad} & \mathrm{E}_X \\
\downarrow{\pi_Y} & & \downarrow{\pi_f} & \pi_X \swarrow & \\
\mathrm{Open}(Y)^{\mathrm{op}} & \xleftarrow{\quad f^{-1} \quad} & \mathrm{Open}(X)^{\mathrm{op}} & &
\end{array}
$$

With these notations, Example 3.18-(1) shows that $\Psi^{\mathrm{psh}}_{X,P} \simeq \pi_{X,*} \circ \lambda_X^*$. In turn, we obtain:

Corollary 5.5.

(1) *There is a canonical identification* $\Phi^{\mathrm{psh}}_{X,P} \simeq \lambda_{X,!} \circ \pi_X^*$.

(2) *The Beck-Chevalley transformation*

$$(f^{-1})^* \circ \pi_{Y,*} \longrightarrow \pi_{f,*} \circ q^*$$

is an equivalence.

(3) *If the Beck-Chevalley transformation*

$$\lambda_{f,!} \circ p^* \longrightarrow \Pi^{\Sigma}_{\infty}(f)^* \circ \lambda_{X,!}$$

is an equivalence, then so is ϕ^{hyp}_f.

Proof. Point (1) follows from the identification $\Psi^{\mathrm{psh}}_{X,P} \simeq \pi_{X,*} \circ \lambda_X^*$ passing to left adjoints. Point (2) is just a reformulation of Lemma 3.15 in this specific situation. As for point (3), observe that by construction ϕ_f is the composition

$$\lambda_{Y,!} \circ \pi_Y^* \circ (f^{-1})_! \simeq \lambda_{f,!} \circ \pi_f^* \simeq \lambda_{f,!} \circ p^* \circ \pi_X^* \longrightarrow \Pi^{\Sigma}_{\infty}(f)^* \circ \lambda_{X,!} \circ \pi_X^* \ .$$

Therefore, if the Beck-Chevalley transformation of the statement is an equivalence, the same goes for ϕ_f. Thus, the conclusion follows from Corollary 5.2. $\qquad \square$

5.2. **Restriction to a closed union of strata for Φ.** Assume that $f \colon (Y, Q) \to (X, P)$ is the inclusion of a closed union of strata. The following lemma is a straightforward consequence of Quillen's theorem A and its proof is left to the reader:

Lemma 5.6. *Let*

$$
\begin{array}{ccc}
\mathcal{C}_0 & \xrightarrow{\ i\ } & \mathcal{C} \\
\downarrow{g} & & \downarrow{f} \\
\mathcal{D}_0 & \xrightarrow{\ j\ } & \mathcal{D}
\end{array}
$$

be a pullback square in Cat_{∞}. Assume that:
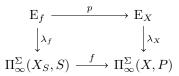
(1) *the functors i and j are fully faithful;*

(2) *for $d \in \mathcal{D}$, $d_0 \in \mathcal{D}_0$, if $\mathrm{Map}_{\mathcal{D}}(d, j(d_0)) \neq \emptyset$, then d belongs to the essential image of j.*

Then, for any $\mathcal{E}$ presentable ∞-category the Beck-Chevalley transformation $g_! \circ i^* \to j^* \circ f_!$ is an equivalence.

Corollary 5.7. *Let (X, P) be a conically stratified space. Let $S \subset P$ be a closed downwards subset and let $f: X_S \to X$ be the inclusion of the corresponding closed union of strata. The upper right square of the diagram (5.4)*

$$
\begin{array}{ccc}
\mathrm{E}_f & \xrightarrow{\ \ p\ \ } & \mathrm{E}_X \\
\downarrow{\scriptstyle\lambda_f} & & \downarrow{\scriptstyle\lambda_X} \\
\Pi^{\Sigma}_{\infty}(X_S, S) & \xrightarrow{\ \ f\ \ } & \Pi^{\Sigma}_{\infty}(X, P)
\end{array}
$$

induces the Beck-Chevalley transformation

$$
\beta\colon \lambda_{f!} \circ p^* \longrightarrow f^* \circ \lambda_{X!},
$$

which is an equivalence.

Proof. We check that the conditions of Lemma 5.6 are satisfied. First of all we observe that for every open subset U of X, one has $U \cap X_S \simeq U_S$, where we consider U equipped with the induced stratification $P|_U$. From Lemma 2.6, the functor

$$
f_U\colon \mathrm{Exit}(U_S, S|_U) \longrightarrow \mathrm{Exit}(U, P|_U)
$$

is fully faithful. It follows that both f and p are fully faithful functors. That the above square is a pullback is then obvious. Finally, the condition (3) from Lemma 5.6 holds since $S \subset P$ is closed downwards. $\qquad\square$

Combining Corollaries 5.5-(3) and 5.7 we immediately obtain the following:

Corollary 5.8. *Let (X, P) be a conically stratified space. Let $S \subset P$ be a closed downwards subset and let $f: X_S \to X$ be the inclusion of the corresponding closed union of strata. Then the natural transformation*

$$
\phi^{\mathrm{hyp}}_S\colon \Phi^{\mathrm{hyp}}_{X_S, S} \circ f^{*, \mathrm{hyp}} \longrightarrow \Pi^{\Sigma}_{\infty}(f)^* \circ \Phi^{\mathrm{hyp}}_{X, P}
$$

is an equivalence.

5.3. **Restriction to an open union of strata for Φ.** Dealing with the inclusion of an open union of strata is more complicated. We start collecting some general ∞-categorical facts:

Definition 5.9. We say that a functor $f: \mathcal{C} \to \mathcal{D}$ is weakly cofiltered if for every object $d \in \mathcal{D}$ the ∞-category $\mathcal{C}_{d/} \coloneqq \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{d/}$ is cofiltered.

Remark 5.10. If $f: \mathcal{C} \to \mathcal{D}$ is cofiltered in the sense of Definition 3.4 it is also weakly cofiltered: indeed, Lemma 3.5 shows that for every $d \in \mathcal{D}$ the induced map $\mathcal{C}_{d/} \to \mathcal{D}_{d/}$ is cofiltered and, on the other hand, $\mathcal{D}_{d/}$ is obviously cofiltered since it has an initial object. On the other hand, Quillen's theorem A implies that if $f: \mathcal{C} \to \mathcal{D}$ is weakly cofiltered, then it is also colimit-final.

Lemma 5.11. *Let $\mathcal{X}$ be an ∞-category. Let $\mathsf{A}, \mathsf{B}: \mathcal{X} \to \mathrm{Cat}_{\infty}$ be functors and let*

$$
\pi_{\mathsf{A}}\colon \mathcal{A} \longrightarrow \mathcal{X}^{\mathrm{op}}\ , \qquad \pi_{\mathsf{B}}\colon \mathcal{B} \longrightarrow \mathcal{X}^{\mathrm{op}}
$$

be the associated cartesian fibrations. Let $f: \mathsf{A} \to \mathsf{B}$ be a natural transformation. If for every $x \in \mathcal{X}$ the functor $f_x: \mathsf{A}(x) \to \mathsf{B}(x)$ is weakly cofiltered, then the induced functor $\mathcal{A} \to \mathcal{B}$ is colimit-final.

Proof. Let $b \in \mathcal{B}$ be an object. In virtue of Quillen's theorem A, we have to prove that the ∞-category

$$
\mathcal{A}_{b/} \coloneqq \mathcal{A} \times_{\mathcal{B}} \mathcal{B}_{b/}
$$

is weakly contractible. We claim that it is cofiltered. Set $x \coloneqq \pi_{\mathsf{B}}(b)$, so that we can review b as an element in $\mathcal{B}_x \simeq \mathsf{B}(x)$. Write $\mathsf{A}(x)_{b/} \coloneqq \mathsf{A}(x) \times_{\mathsf{B}(x)} \mathsf{B}(x)_{b/}$ and let

$$
j\colon \mathsf{A}(x)_{b/} \longrightarrow \mathcal{A}_{b/}
$$

be the natural functor. Let now $F \colon I \to \mathcal{A}_{b/}$ be a finite diagram. Composing F with the canonical projection $\mathcal{A}_{b/} \to \mathcal{X}_{x/}$ and using the fact that both π_{A} and π_{B} are cartesian fibrations, we deduce the existence of a functor

$$F' \colon I \to \mathsf{A}(x)_{b/} := \mathsf{A}(x) \times_{\mathsf{B}(x)} \mathsf{B}(x)_{b/}$$

together with a natural transformation $\gamma \colon j \circ F' \to F$. By assumption the functor $f_x \colon \mathsf{A}(x) \to \mathsf{B}(x)$ is weakly cofiltered, and therefore $\mathsf{A}(x)_{b/}$ is cofiltered. It follows that F' admits an extension $\widetilde{F}' \colon I^{\triangleleft} \to \mathsf{A}(x)_{b/}$. At this point, the natural transformation γ allows us to prolong $\widetilde{F}'$ into an extension $\widetilde{F} \colon I^{\triangleleft} \to \mathcal{A}_{b/}$ of F. The conclusion follows. $\qquad\square$

Let (X, P) be a conically stratified space. Fix an object $x \in \Pi_{\infty}^{\Sigma}(X, P)$. Define $\Pi_{\infty}^{\Sigma}(X, P)_{/x}^{\simeq}$ as the full subcategory of $\Pi_{\infty}^{\Sigma}(X, P)_{/x}$ spanned by its final objects. For every functor $\mathcal{C} \to \Pi_{\infty}^{\Sigma}(X, P)$ we set

$$\mathcal{C}_{/x} := \mathcal{C} \times_{\Pi_{\infty}^{\Sigma}(X, P)} \Pi_{\infty}^{\Sigma}(X, P)_{/x} \quad \text{and} \quad \mathcal{C}_{/x}^{\simeq} := \mathcal{C} \times_{\Pi_{\infty}^{\Sigma}(X, P)} \Pi_{\infty}^{\Sigma}(X, P)_{/x}^{\simeq} \ .$$

The following lemma contains the key geometrical argument of this section:

Lemma 5.12. *The functor* $(\mathrm{E}_X)_{/x}^{\simeq} \longrightarrow (\mathrm{E}_X)_{/x}$ *is cofinal.*

Proof. We apply Lemma 5.11 taking $\mathcal{X} = \mathrm{Open}(X)$, and A and B to be the functors given by

$$\mathsf{A}(U) := \Pi_{\infty}^{\Sigma}(U, P)_{/x}^{\simeq} \quad \text{and} \quad \mathsf{B}(U) := \Pi_{\infty}^{\Sigma}(U, P)_{/x} \ .$$

By construction, the cartesian fibration classified by B is given by the natural projection $(\mathrm{E}_X)_{/x} \to \mathrm{Open}(X)^{\mathrm{op}}$, and similarly the cartesian fibration classified by A is given by $(\mathrm{E}_X)_{/x}^{\simeq} \to \mathrm{Open}(X)^{\mathrm{op}}$. Thus, Lemma 5.11 shows that it is enough to show that for every open U of X, the functor

$$\Pi_{\infty}^{\Sigma}(U, P)_{/x}^{\simeq} \longrightarrow \Pi_{\infty}^{\Sigma}(U, P)_{/x}$$

is weakly cofiltered. Let $\gamma \in \Pi_{\infty}^{\Sigma}(U, P)_{/x}$ and write $\Pi_{\infty}^{\Sigma}(U, P)_{\gamma /\!\!/ x} := (\Pi_{\infty}^{\Sigma}(U, P)_{/x})_{\gamma /}$. We show that the ∞-category

$$\Pi_{\infty}^{\Sigma}(U, P)_{\gamma /\!\!/ x}^{\simeq} := \Pi_{\infty}^{\Sigma}(U, P)_{/x}^{\simeq} \times_{\Pi_{\infty}^{\Sigma}(U, P)_{/x}} \Pi_{\infty}^{\Sigma}(U, P)_{\gamma /\!\!/ x}$$

is cofiltered. Let therefore I be a finite category and consider the lifting problem

$$
\begin{array}{ccc}
I & \xrightarrow{\ g\ } & \Pi_{\infty}^{\Sigma}(U, P)_{\gamma /\!\!/ x}^{\simeq} \\
{\scriptstyle j}\downarrow & \nearrow & \\
I^{\triangleleft} & {\scriptstyle \overline{g}} &
\end{array}
$$

Since $\Pi_{\infty}^{\Sigma}(U, P)_{\gamma /\!\!/ x}^{\simeq}$ is fully faithful inside $\Pi_{\infty}^{\Sigma}(U, P)_{\gamma /\!\!/ x}$, we can rewrite the above lifting problem as

$$
\begin{array}{ccc}
I & \xrightarrow{\ g\ } & \Pi_{\infty}^{\Sigma}(U, P)_{\gamma /\!\!/ x} \\
{\scriptstyle j}\downarrow & \nearrow & \\
I^{\triangleleft} & {\scriptstyle \overline{g}} &
\end{array}
$$

where g sends an object $i \in I$ to a morphism $\gamma \to \delta_i$ in $\Pi_{\infty}^{\Sigma}(U, P)_{/x}$ with $\delta_i \in \Pi_{\infty}^{\Sigma}(U, P)_{/x}^{\simeq}$ and where $\overline{g}$ is required to send the vertex v_0 of $I^{\triangleleft}$ to a morphism $\gamma \to \delta$ in $\Pi_{\infty}^{\Sigma}(U, P)_{/x}$ with

$\delta \in \Pi_\infty^\Sigma(U, P)_{\widetilde{/x}}$. This is equivalent to a lifting problem

$$
\begin{array}{ccc}
I^\triangleleft & \xrightarrow{\ g'\ } & \Pi_\infty^\Sigma(U, P)_{/x} \\
{\scriptstyle j^\triangleleft}\downarrow & \nearrow & \\
(I^\triangleleft)^\triangleleft & \overset{\overline{g}'}{} &
\end{array}
$$

where g' sends the vertex v_1 of the exterior cone $(I^\triangleleft)^\triangleleft$ to γ and the other objects in $\Pi_\infty^\Sigma(U, P)_{\widetilde{/x}}$, and where $\overline{g}'$ is required to send v_0 in $\Pi_\infty^\Sigma(U, P)_{\widetilde{/x}}$. Unraveling the definitions and denoting by v_2 the final object of $((I^\triangleleft)^\triangleleft)^\triangleright$, we can further reduce this lifting problem to the following one

$$
\begin{array}{ccc}
(I^\triangleleft)^\triangleright & \xrightarrow{\ h\ } & \Pi_\infty^\Sigma(X, P) \\
{\scriptstyle j^\triangleleft}\downarrow & \nearrow & \\
((I^\triangleleft)^\triangleleft)^\triangleright & \overset{\overline{h}}{} &
\end{array} \quad ,
$$

where we ask for $\overline{h}$ to represent a continuous morphism

$$\overline{h}\colon |((I^\triangleleft)^\triangleleft)^\triangleright| \longrightarrow X,$$

with the following properties:

(1) the morphism h takes the segment $[v_1, v_2]$ to the exit path γ underlying γ.

(2) the morphism h takes the complement of v_1 to the same stratum of x;

(3) the morphism h takes the double cone $(I^\triangleleft)^\triangleleft$ inside the open U.

Observe that there is a canonical equivalence

$$((I^\triangleleft)^\triangleleft)^\triangleright \simeq (\Delta^1 \star I) \star \Delta^0 \simeq \Delta^1 \star (I \star \Delta^0) \simeq ((I^\triangleright)^\triangleleft)^\triangleleft \ ,$$

where $\star$ denotes the join operation. Set $J := I^\triangleright$. By definition of $\Pi_\infty^\Sigma(X, P)$, we can represent h by an explicit continuous morphism

$$h\colon |J^\triangleleft| \longrightarrow X,$$

with the property that it takes the segment $[v_1, v_2]$ to γ, that the complement of v_1 is sent in the same stratum as x, and that the cone over I with vertex v_1 is sent in U. Since h is continuous, the preimage of U is an open of $|J^\triangleleft|$ containing the cone over I with vertex v_1. Then, the inclusion of the exterior cone

$$i := |j^\triangleleft|\colon |J^\triangleleft| \hookrightarrow |(J^\triangleleft)^\triangleleft|$$

admits a retraction r sending v_0 inside $[v_1, v_2]$ close enough to v_1 and satisfying the following extra properties:

  (i) r sends $|(I^\triangleleft)^\triangleleft|$ inside $h^{-1}(U)$;

  (ii) the vertex v_1 is the unique point $s \in |(J^\triangleleft)^\triangleleft|$ such that $r(s) = v_1$.

Define

$$\overline{h} := h \circ r.$$

It is then straightforward to verify that $\overline{h}$ satisfies the conditions (1),(2) and (3) listed above. $\qquad \square$

Corollary 5.13. *Let (X, P) be a conically stratified space. Let $S \subset P$ be a closed upwards subset and let $f\colon X_S \to X$ be the inclusion of the corresponding open union of strata. The upper right*

square of the diagram (5.4)

(5.14)
$$
\begin{array}{ccc}
\mathrm{E}_f & \xrightarrow{\ \ p\ \ } & \mathrm{E}_X \\
\downarrow{\scriptstyle \lambda_f} & & \downarrow{\scriptstyle \lambda_X} \\
\Pi_\infty^\Sigma(X_S, S) & \xrightarrow{\ \ f\ \ } & \Pi_\infty^\Sigma(X, P)
\end{array}
$$

induces the Beck-Chevalley transformation

$$
\beta \colon \lambda_{f!} \circ p^* \longrightarrow f^* \circ \lambda_{X!}
$$

which is an equivalence.

Proof. As in the proof of Corollary 5.7, f and p are fully faithful and the above square is a pullback. Let now $x \in \Pi_\infty^\Sigma(X_S, S)$ be an object. Committing a slight abuse of notation, we still denote by x its image in $\Pi_\infty^\Sigma(X, P)$ via the functor f. Put

$$
(\mathrm{E}_f)_{/x} \coloneqq \mathrm{E}_f \times_{\Pi_\infty^\Sigma(X_S, S)} \Pi_\infty^\Sigma(X_S, S)_{/x}
$$

and consider the commutative diagram

$$
\begin{array}{ccc}
(\mathrm{E}_f)_{/x} & \longrightarrow & (\mathrm{E}_X)_{/x} \\
\downarrow & & \downarrow \\
\Pi_\infty^\Sigma(X_S, S)_{/x} & \longrightarrow & \Pi_\infty^\Sigma(X, P)_{/x}.
\end{array}
$$

Unraveling the definitions, we reduce ourselves to check that $(\mathrm{E}_f)_{/x} \longrightarrow (\mathrm{E}_X)_{/x}$ is cofinal. Observe that since p is fully faithful, the same goes for this map. By definition, $(\mathrm{E}_f)_{/x}$ is the full subcategory of triples (U, y, γ) in $(\mathrm{E}_X)_{/x}$ such that y lies in X_S. Using the same notations of Lemma 5.12, we find the following commutative diagram of ∞-categories over $\Pi_\infty^\Sigma(X, P)_{/x}$:

$$
\begin{array}{ccc}
(\mathrm{E}_f)_{\widetilde{/x}} & \longrightarrow & (\mathrm{E}_X)_{\widetilde{/x}} \\
\downarrow & & \downarrow \\
(\mathrm{E}_f)_{/x} & \longrightarrow & (\mathrm{E}_X)_{/x}.
\end{array}
$$

Since the vertical functors and the bottom horizontal one are fully faithful, the same goes for the top horizontal one. Furthermore, if $(U, y, \gamma) \in (\mathrm{E}_X)_{\widetilde{/x}}$ then $\gamma \colon y \to x$ is an equivalence in $\Pi_\infty^\Sigma(X, P)$. Since x belongs to X_S, the same goes for y. In other words, (U, y, γ) belongs to $(\mathrm{E}_f)_{\widetilde{/x}}$. This shows that the top horizontal arrow is also essentially surjective, hence an equivalence. As a consequence, to prove that the bottom horizontal functor is cofinal it is enough to prove that both vertical functors are cofinal.

We now show that it is enough to prove that the right vertical functor is cofinal. By Quillen's Theorem A and the above discussion, it is enough to show that for $\mathbf{y} = (U, y, \gamma) \in (\mathrm{E}_f)_{/x}$, the fully faithful functor $((\mathrm{E}_f)_{/x})_{\mathbf{y}/} \to ((\mathrm{E}_X)_{/x})_{\mathbf{y}/}$ is an equivalence. Let $\mathbf{z} \coloneqq (V, z, \delta) \in (\mathrm{E}_X)_{/x}$ and let $\mathbf{y} \to \mathbf{z}$ be a morphism in $(\mathrm{E}_X)_{/x}$. Observe that this morphism produces an exit path $y \to z$. Since S is closed upwards, this implies that z lies in X_S. Thus, we have $\mathbf{z} \in (\mathrm{E}_f)_{/x}$. Hence, the fully faithful functor $((\mathrm{E}_f)_{/x})_{\mathbf{y}/} \to ((\mathrm{E}_X)_{/x})_{\mathbf{y}/}$ is an equivalence and the sought-after reduction is complete. We then conclude the proof of Corollary 5.13 using Lemma 5.12. $\qquad\square$

Combining Corollaries 5.5-(3) and 5.13 we immediately obtain the following:

Corollary 5.15. *Let (X, P) be a conically stratified space. Let $S \subset P$ be a closed upwards subset and let $f \colon X_S \to X$ be the inclusion of the corresponding open union of strata. Then the natural transformation*

$$\phi_S^{\mathrm{hyp}} \colon \Phi_{X_S, S}^{\mathrm{hyp}} \circ f^{*,\mathrm{hyp}} \longrightarrow \Pi_\infty^\Sigma (f)^* \circ \Phi_{X, P}^{\mathrm{hyp}}$$

is an equivalence.

Corollary 5.16. *Let (X, P) be a conically stratified space. Let $a \in P$ and let $i_a \colon X_a \to X$ be the inclusion of the corresponding stratum. Then the natural transformation*

$$\phi_a^{\mathrm{hyp}} \colon \Phi_{X_a}^{\mathrm{hyp}} \circ i_a^{*,\mathrm{hyp}} \longrightarrow \Pi_\infty^\Sigma (i_a)^* \circ \Phi_{X, P}^{\mathrm{hyp}}$$

is an equivalence.

Proof. It follows by first applying first Corollary 5.15 to (X, P) and the subset $P_{\geqslant a} \subseteq P$ and subsequently Corollary 5.8 to $(X_{\geqslant a}, P_{\geqslant a})$ and the subset $\{a\} \subseteq P_{\geqslant a}$. $\qquad\square$

5.4. **The equivalence.** We are now ready for the main theorem:

Theorem 5.17. *Let (X, P) be a conically stratified space and let $\mathcal{E}$ be a presentable ∞-category. Assume that:*

(1) *for every $a \in P$, the associated stratum X_a is locally weakly contractible;*

(2) *the hyper-restrictions*

$$\left\{ i_a^{*,\mathrm{hyp}} \colon \mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E}) \longrightarrow \mathrm{Sh}^{\mathrm{hyp}}(X_a; \mathcal{E}) \right\}_{a \in P}$$

are jointly conservative.

Then the exodromy adjunction

$$\Phi_{X, P}^{\mathrm{hyp}} \colon \mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E}) \leftrightarrows \mathrm{Fun}(\Pi_\infty^\Sigma(X, P), \mathcal{E}) \colon \Psi_{X, P}^{\mathrm{hyp}}$$

is an equivalence.

Proof. We have to prove that for every functor $F \colon \Pi_\infty^\Sigma(X, P) \to \mathcal{E}$ and every hyperconstructible hypersheaf $G \in \mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E})$, the unit and counit

$$\eta_G \colon G \longrightarrow \Psi_{X, P}^{\mathrm{hyp}}(\Phi_{X, P}^{\mathrm{hyp}}(G)) \quad \text{and} \quad \varepsilon_F \colon \Phi_{X, P}^{\mathrm{hyp}}(\Psi_{X, P}^{\mathrm{hyp}}(F)) \longrightarrow F$$

are equivalences. By assumption, it is enough to check that for every $a \in P$, the (hyper-)restrictions $i_a^{*,\mathrm{hyp}}(\eta_G)$ and $\Pi_\infty^\Sigma(i_a)^*(\varepsilon_F)$ are equivalences. Consider the following square:

$$
\begin{array}{ccc}
\mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E}) & \xrightarrow{\;\Phi_{X, P}^{\mathrm{hyp}}\;} & \mathrm{Fun}\left(\Pi_\infty^\Sigma(X, P), \mathcal{E}\right) \\
\downarrow{\scriptstyle i_a^{*,\mathrm{hyp}}} & & \downarrow{\scriptstyle \Pi_\infty^\Sigma(i_a)^*} \\
\mathrm{Sh}^{\mathrm{hyp}}(X_a; \mathcal{E}) & \xrightarrow{\;\Phi_{X_a}^{\mathrm{hyp}}\;} & \mathrm{Fun}\left(\Pi_\infty(X_a), \mathcal{E}\right) .
\end{array}
$$

Corollary 5.16 shows that the natural transformation ϕ_a^{hyp} makes this into a commutative diagram. Moreover, combining Remark 3.20-(2) and Proposition 4.9 we deduce that it is horizontally right adjointable. Therefore we obtain canonical identifications

$$i_a^{*,\mathrm{hyp}}(\eta_G) \simeq \eta_{i_a^{*,\mathrm{hyp}}(G)} \qquad \text{and} \qquad \Pi_\infty^\Sigma(i_a)^*(\varepsilon_F) \simeq \varepsilon_{\Pi_\infty^\Sigma(i_a)^*(F)} ,$$

where the latter are the unit and the counit of the adjunction $\Phi_{X_a}^{\mathrm{hyp}} \dashv \Psi_{X_a}^{\mathrm{hyp}}$. Observe now that since G is hyperconstructible, $i_a^{*,\mathrm{hyp}}(G)$ is locally constant. Thus the conclusion follows directly from Corollary 3.32. $\qquad\square$

Remark 5.18. The joint conservativity of the hyper-restrictions to strata is satisfied in the following two cases of interest:

(1) the category $\mathcal{E}$ is compactly generated. See [HPT, Corollary 5.16].

(2) the poset P is noetherian and the category $\mathcal{E}$ is stable and presentable. See [HPT, Corollary 5.21].

Let us emphasize some immediate consequences of the above theorem.

Corollary 5.19. *Let (X, P) be a conically stratified space and let $\mathcal{E}$ be a presentable ∞-category. If the assumptions of Theorem 5.17 are satisfied, then:*

(1) the ∞-category $\mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E})$ is presentable;

(2) if $\mathcal{E}$ is an ∞-topos, then so is $\mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E})$.

The following corollary generalizes at the same time [HA, Lemma A.9.14] and [HPT, Corollary 3.2]

Corollary 5.20. *Let (X, P) be a conically stratified space and let $\mathcal{E}$ be a presentable ∞-category. If the assumptions of Theorem 5.17 are satisfied, then the full subcategory $\mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E}) \hookrightarrow \mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E})$ is closed under small limits and small colimits.*

Proof. For colimits it is enough to observe that if $a \in P$ and if $i_a \colon X_a \to X$ is the inclusion of the corresponding stratum, then $i_a^{*,\mathrm{hyp}} \colon \mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(X_a; \mathcal{E})$ commutes with colimits. Thus, we are reduced to show that $\mathrm{LC}^{\mathrm{hyp}}(X; \mathcal{E})$ is closed under colimits in $\mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E})$. This was shown in [HPT, Corollary 3.2]. Concerning limits, it is enough to observe that $\Psi_{X,P}^{\mathrm{hyp}} \colon \mathrm{Fun}(\Pi_\infty^\Sigma(X, P), \mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E})$ is fully faithful and right adjoint to $\Phi_{X,P}^{\mathrm{hyp}}$. Thus $\Psi_{X,P}^{\mathrm{hyp}}$ commutes with small limits. On the other hand Theorem 5.17 shows that the essential image of $\Psi_{X,P}^{\mathrm{hyp}}$ is $\mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E})$, whence the conclusion. $\qquad\square$

Remark 5.21.

(1) Combining Corollaries 5.19 and 5.20 it follows that if the assumptions of Theorem 5.17 are satisfied then the inclusion $\iota \colon \mathrm{Cons}_P(X; \mathcal{E}) \hookrightarrow \mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E})$ admits both a left adjoint $\mathrm{L}_P^{\mathrm{hyp}}$ and a right adjoint $\mathrm{R}_P^{\mathrm{hyp}}$.

(2) The functor $\Psi_{X,P}^{\mathrm{hyp}} \colon \mathrm{Fun}(\Pi_\infty^\Sigma(X, P), \mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E})$ factors as $\iota \circ \Psi_{X,P}^{\mathrm{hyp}}$. Hence, if the assumptions of Theorem 5.17 are satisfied, it admits both $\Phi_{X,P}^{\mathrm{hyp}}$ and $\Phi_{X,P}^{\mathrm{hyp}}|_{\mathrm{Cons}_P(X;\mathcal{E})} \circ \mathrm{L}_P^{\mathrm{hyp}}$ as left adjoints. Thus, for every $F \in \mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E})$, the unit adjunction $F \to \iota \circ L_P^{\mathrm{hyp}}(F)$ induces an equivalence $\Phi_{X,P}^{\mathrm{hyp}}(F) \simeq \Phi_{X,P}^{\mathrm{hyp}} L_P^{\mathrm{hyp}}(F)$.

Corollary 5.22. *Let (X, P) be a conically stratified space and let $\mathcal{E}$ be a presentable ∞-category. Under the assumptions of Theorem 5.17, for a hypersheaf $F \in \mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E})$ the following statements are equivalent:*

(1) F is hyperconstructible on (X, P);

(2) for every open inclusion $U \subset V$ for which the induced map $\Pi_\infty^\Sigma(U, P) \to \Pi_\infty^\Sigma(V, P)$ is a categorical equivalence, the restriction map $F(V) \to F(U)$ is an equivalence;

(3) for every conical chart $Z \times C(Y)$ of (X, P), F satisfies the following two conditions:
  (i) for every open subset $W \subset Z$ and every $0 < \varepsilon < \varepsilon' \leqslant 1$, the restriction map

$$F(W \times C_{\varepsilon'}(Y)) \longrightarrow F(W \times C_\varepsilon(Y))$$

  is an equivalence;

(ii) *for every inclusion $U \subseteq V$ of weakly contractible open subsets of Z, the restriction map*

$$F(V \times C(Y)) \longrightarrow F(U \times C(Y))$$

is an equivalence.

Proof. The implication (1) $\Rightarrow$ (2) is a direct consequence of Theorem 5.17. For the implication (2) $\Rightarrow$ (3), it is enough to observe that, on the one hand, the inclusion $W \times C_\varepsilon(Y) \hookrightarrow W \times C_\varepsilon(Y)$ is a stratified homotopy equivalence (and hence it satisfies the assumption of point (2)); and on the other hand, for an inclusion $U \subseteq V$ of weakly contractible open subsets of Z, the induced map $\Pi_\infty^\Sigma(U \times C(Y)) \to \Pi_\infty^\Sigma(V \times C(Y))$ is also a categorical equivalence, thanks to Lemma 2.14. We are therefore left to prove that (3) implies (1). The question is local, so we can replace X by a conical chart of the form $Z \times C(Y)$. Letting $i \colon Z \to Z \times C(Y)$ be the natural inclusion, we only have to check that $i^{*,\mathrm{hyp}}(F)$ is locally hyperconstant. Consider first the presheaf-theoretic pullback $i^{-1}(F)$. For every open subset W of Z, one has

$$i^{-1}(F)(W) \simeq \operatorname*{colim}_{0 < \varepsilon \leqslant 1} F(W \times C_\varepsilon(Y)) \ .$$

Our assumption (i) guarantees that the transition maps in the above colimit diagram are equivalences, and hence that the canonical restriction map

$$F(W \times C(Y)) \longrightarrow i^{-1}(F)(W)$$

is an equivalence. In particular, it follows that $i^{-1}(F)$ is a hypersheaf, and therefore that it coincides with $i^{*,\mathrm{hyp}}(F)$.

At this point, in order to prove that $i^{-1}(F) \simeq i^{*,\mathrm{hyp}}(F)$ is locally hyperconstant it is enough, in virtue of [HPT, Proposition 3.1], to prove that for every inclusion $U \subset V$ of weakly contractible open subsets of Z, the restriction map

$$i^{-1}(F)(V) \longrightarrow i^{-1}(F)(U)$$

is an equivalence. As we showed above, this amounts to check that the restriction map

$$F(V \times C(Y)) \longrightarrow F(U \times C(Y))$$

is an equivalence. Since this holds by assumption (ii), the conclusion follows. $\square$

For later use, let us remark that in the previous proof we established the following fact:

Corollary 5.23. *Let Z be a locally weakly contractible topological space and let (Y, Q) be a stratified space so that $(X, P) \coloneqq (Z \times C(Y), Q^\lhd)$ is conically stratified. Let $\mathcal{E}$ be a presentable ∞-category and assume that the assumptions of Theorem 5.17 are satisfied by (X, P) and $\mathcal{E}$. Let $F \in \mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E})$ and let U be an open subset of Z. Then the canonical map*

$$F(U \times C(Y)) \longrightarrow i^{*,\mathrm{hyp}}(F)(U)$$

is an equivalence, where i denotes the canonical inclusion $i \colon Z \hookrightarrow X$.

6. Consequences

Having proven Theorem 5.17, we now explore some of its consequences. In this section, we mainly focus on improved functoriality results for the exodromy equivalence and some structural results for the ∞-category of hyperconstructible hypersheaves.

6.1. **Structural results for hyperconstructible hypersheaves.** Let (X, P) be a conically stratified with locally weakly contractible strata. Applying Corollary 5.19, we see that $\mathrm{Cons}_P^{\mathrm{hyp}}(X)$ is a presentable ∞-category. In particular, for every $\mathcal{E} \in \mathcal{P}\mathrm{r}^{\mathrm{L}}$, the tensor product $\mathrm{Cons}_P^{\mathrm{hyp}}(X) \otimes \mathcal{E}$ is well defined. Our next task is to compare it with $\mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E})$.

Since $\mathrm{Cons}_P^{\mathrm{hyp}}(X)$ is presentable, it follows that the inclusion $\mathrm{Cons}_P^{\mathrm{hyp}}(X) \hookrightarrow \mathrm{Sh}^{\mathrm{hyp}}(X)$ is a morphism in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. Therefore, tensoring with $\mathcal{E}$ yields a fully faithful functor

$$(6.1) \qquad \mathrm{Cons}_P^{\mathrm{hyp}}(X) \otimes \mathcal{E} \longrightarrow \mathrm{Sh}^{\mathrm{hyp}}(X) \otimes \mathcal{E} \ .$$

Let $a \in P$ and let $i_a \colon X_a \hookrightarrow X$ be the inclusion of the corresponding stratum. Functoriality of the tensor product of presentable ∞-categories immediately implies the commutativity of the following diagram:

$$\begin{array}{ccc} \mathrm{Cons}_P^{\mathrm{hyp}}(X) \otimes \mathcal{E} & \longrightarrow & \mathrm{Sh}^{\mathrm{hyp}}(X) \otimes \mathcal{E} \\ {\scriptstyle i_a^{*,\mathrm{hyp}} \otimes \mathrm{id}_{\mathcal{E}}} \downarrow & & \downarrow {\scriptstyle i_a^{*,\mathrm{hyp}} \otimes \mathrm{id}_{\mathcal{E}}} \\ \mathrm{LC}^{\mathrm{hyp}}(X_a) \otimes \mathcal{E} & \longrightarrow & \mathrm{Sh}^{\mathrm{hyp}}(X_a) \otimes \mathcal{E} \ . \end{array}$$

Recall from [HPT, Observation 3.11] that the equivalence $\mathrm{Sh}^{\mathrm{hyp}}(X_a) \otimes \mathcal{E} \simeq \mathrm{Sh}^{\mathrm{hyp}}(X_a; \mathcal{E})$ restricts to an equivalence $\mathrm{LC}^{\mathrm{hyp}}(X_a) \otimes \mathcal{E} \simeq \mathrm{LC}^{\mathrm{hyp}}(X_a; \mathcal{E})$. Thus it follows that under the equivalence $\mathrm{Sh}^{\mathrm{hyp}}(X) \otimes \mathcal{E} \simeq \mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E})$, the functor (6.1) factors through $\mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E})$, yielding the following commutative diagram:

$$(6.2) \qquad \begin{array}{ccc} \mathrm{Cons}_P^{\mathrm{hyp}}(X) \otimes \mathcal{E} & \longrightarrow & \mathrm{Sh}^{\mathrm{hyp}}(X) \otimes \mathcal{E} \\ \downarrow & & \downarrow \\ \mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E}) & \longrightarrow & \mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E}) \ . \end{array}$$

Corollary 6.3. *Let (X, P) be a conically stratified space and let $\mathcal{E}$ be a presentable ∞-category. If the assumptions of Theorem 5.17 are satisfied, then the diagram*

$$\begin{array}{ccc} \mathrm{Cons}_P^{\mathrm{hyp}}(X) \otimes \mathcal{E} & \xrightarrow{\ \Phi_{X,P}^{\mathrm{hyp}} \otimes \mathrm{id}_{\mathcal{E}}\ } & \mathrm{Fun}\left(\Pi_\infty^\Sigma(X, P), \mathcal{S}\right) \otimes \mathcal{E} \\ {\scriptstyle \boxtimes} \downarrow & & \downarrow {\scriptstyle \boxtimes} \\ \mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E}) & \xrightarrow{\ \Phi_{X,P}^{\mathrm{hyp},\mathcal{E}}\ } & \mathrm{Fun}\left(\Pi_\infty^\Sigma(X, P), \mathcal{E}\right) \end{array}$$

is canonically commutative. In particular, the left vertical functor is an equivalence.

Proof. The first half is a direct consequence of Lemma 3.8 and the commutativity of the diagram (6.2). The second half follows from the first one and the 2-out-of-3 property of the equivalences. $\square$

Corollary 6.4. *Let (X, P) be a conically stratified space and let $\mathcal{E}$ be a presentable ∞-category. If the assumptions of Theorem 5.17 are satisfied, then the ∞-category $\mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E})$ is κ-presentable.*

Proof. Indeed, $\mathrm{Fun}(\Pi_\infty^\Sigma(X, P), \mathcal{S})$ is always compactly generated, and therefore $\mathrm{Fun}(\Pi_\infty^\Sigma(X, P), \mathcal{S}) \otimes \mathcal{E}$ is κ-presentable by [HA, Lemma 5.3.2.11]. $\square$

Corollary 6.5 (Categorical Künneth formula)**.** *Let (X, P) and (Y, Q) be conically stratified spaces with locally weakly contractible strata. Then there is a canonical equivalence*

$$\mathrm{Cons}_{P \times Q}^{\mathrm{hyp}}(X \times Y) \simeq \mathrm{Cons}_P^{\mathrm{hyp}}(X) \otimes \mathrm{Cons}_Q^{\mathrm{hyp}}(Y) \ .$$

Proof. This follows from the fact that $\mathrm{Fun}(-, \mathcal{S})$ takes products of ∞-categories to tensor products of presentable ∞-categories, and the fact that Π_∞^Σ commutes with finite products of stratified spaces. $\qquad\square$

6.2. **Exodromic morphisms.** Let $f \colon (Y, Q) \to (X, P)$ be a morphism of conically stratified spaces. Let $\mathcal{E}$ be a presentable ∞-category. Although the natural transformations

$$\psi_f^{\mathrm{hyp}} \colon f^{*,\mathrm{hyp}} \circ \Psi_{X,P}^{\mathrm{hyp}} \longrightarrow \Psi_{Y,Q}^{\mathrm{hyp}} \circ \Pi_\infty^\Sigma(f)^* \quad \text{and} \quad \phi_f^{\mathrm{hyp}} \colon \Phi_{Y,Q}^{\mathrm{hyp}} \circ f^{*,\mathrm{hyp}} \longrightarrow \Pi_\infty^\Sigma(f)^* \circ \Phi_{X,P}^{\mathrm{hyp}}$$

are always defined, they are not always equivalences.

Definition 6.6. We say that the morphism $f \colon (Y, Q) \to (X, P)$ is:

- *right exodromic* if the natural transformation ψ_f^{hyp} is an equivalence;

- *left exodromic* if the natural transformation ϕ_f^{hyp} is an equivalence when evaluated on hyperconstructible hypersheaves on (X, P);

- *strongly left exodromic* if the natural transformation ϕ_f^{hyp} is an equivalence;

- *(strongly) exodromic* if it is right and (strongly) left exodromic.

With this terminology and with the help of Theorem 5.17, Remark 3.20-(2) implies:

Lemma 6.7. *Let $f \colon (Y, Q) \to (X, P)$ be a morphism of conically stratified spaces with locally weakly contractible strata. Let $\mathcal{E}$ be a presentable ∞-category satisfying the assumptions of Theorem 5.17. Then f is left exodromic if and only if it is right exodromic.*

Proposition 6.8. *Let $f \colon (Y, Q) \to (X, P)$ be a morphism of conically stratified spaces with locally weakly contractible strata. Let $\mathcal{E}$ be a presentable ∞-category satisfying the assumptions of Theorem 5.17. Then f is left exodromic.*

Proof. From Lemma 6.7, it is enough to prove that ψ_f^{hyp} is an equivalence. For this, it is enough to show that for every $a \in Q$, the hyper-restriction $i_a^{*,\mathrm{hyp}}(\psi_f^{\mathrm{hyp}})$ is an equivalence, where $i_a \colon Y_a \to Y$ is the inclusion. From Proposition 4.9 and Corollary 3.26, we find for every $F \colon \Pi_\infty^\Sigma(X, P) \to \mathcal{E}$ a canonical identification

$$i_a^{*,\mathrm{hyp}}(\psi_f^{\mathrm{hyp}}(F)) \simeq \psi_{f \circ i_a}^{\mathrm{hyp}}(F) \ .$$

Let $b \in P$ be the image of a, let $i_b \colon X_b \to X$ be the inclusion and let $g \colon Y_a \to X_b$ be the morphism morphism induced by f. Then, $f \circ i_a = i_b \circ g$ so that a second application of Proposition 4.9 and Corollary 3.26 gives a canonical identification

$$i_a^{*,\mathrm{hyp}}(\psi_f^{\mathrm{hyp}}(F)) \simeq \psi_g^{\mathrm{hyp}}(\Pi_\infty^\Sigma(i_b)^*(F)) \ .$$

Thus, the conclusion follows from Corollary 4.8. $\qquad\square$

Remark 6.9. In particular, if $f \colon (Y, Q) \to (X, P)$ is an open immersion, Proposition 6.8 implies that ϕ_f^{hyp} is an equivalence on hyperconstructible hypersheaves. When f is the inclusion of an open union of strata of (X, P), Corollaries 3.21 and 5.15 show that more is true: indeed, in this case f is strongly exodromic.

Corollary 6.10. *Let (X, P) be a conically stratified space with locally weakly contractible strata. Let $\mathcal{E}$ be a presentable ∞-category satisfying the assumptions of Theorem 5.17. Let $x \in X$ be a point and let $F \in \mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E})$. We have:*

$$\Phi_{X,P}^{\mathrm{hyp}}(F)(x) \simeq \operatorname*{colim}_{x \in U} F(U) \ ,$$

where the colimit ranges over the open subsets of X containing x.

Proof. Review x as a morphism $x \colon * \to X$. The induced stratification on $*$ is the trivial one, and in particular it is conical. Thus, Proposition 6.8 implies that x is exodromic, and therefore that the natural transformation ϕ_x^{hyp} is an equivalence. The conclusion follows since the colimit in the statement is canonically identified with $x^{*,\mathrm{hyp}}(F)$. $\qquad\square$

The next lemma asserts that the failure for the formula from Corollary 6.10 to hold exactly measures the defect for a hypersheaf to be hyperconstructible.

Corollary 6.11. *Let (X, P) be a conically stratified space with locally weakly contractible strata. Let $\mathcal{E}$ be a presentable ∞-category satisfying the assumptions of Theorem 5.17. Let $F \in \mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E})$. Then F is hyperconstructible on (X, P) if and only if for every $x \in X$, the canonical morphism*

$$\phi_x^{\mathrm{hyp}}(F) \colon \Phi_{X,P}^{\mathrm{hyp}}(F)(x) \longrightarrow \operatorname*{colim}_{x \in U} F(U)$$

is an equivalence.

Proof. The direct implication follows from Corollary 6.10. Assume now that $\phi_x^{\mathrm{hyp}}(F)$ is an equivalence for every $x \in X$. Let L_P^{hyp} be the left adjoint to the inclusion $\mathrm{Cons}_P(X; \mathcal{E}) \hookrightarrow \mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E})$ as obtained in Remark 5.21-(2). To prove Corollary 6.11, it is enough to prove that the unit transformation $F \to L_P^{\mathrm{hyp}}(F)$ is an equivalence. Since both source and target are hypersheaves it is enough to show that for every $x \in X$, the induced morphism $F_x \to L_P^{\mathrm{hyp}}(F)_x$ is an equivalence. By assumption, F_x identifies with $\Phi_{X,P}^{\mathrm{hyp}}(F)(x)$. On the other hand, Corollary 6.10 combined with Remark 5.21-(3) gives the following chain of equivalences

$$L_P^{\mathrm{hyp}}(F)_x \simeq \Phi_{X,P}^{\mathrm{hyp}}(L_P^{\mathrm{hyp}}(F))(x) \simeq \Phi_{X,P}^{\mathrm{hyp}}(F)(x)$$

Corollary 6.11 is thus proved. $\qquad\square$

Corollary 6.12. *Let (X, P) be a conically stratified space and let (X, Q) be a conical refinement of P. Assume that the strata of (X, P) and (X, Q) are locally weakly contractible. Then the natural map*

$$f \colon \Pi_\infty^\Sigma(X, Q) \longrightarrow \Pi_\infty^\Sigma(X, P)$$

is a localization.

Proof. We first reduce to the case where the stratification on X is trivial. Let W be the collection of f-local equivalences in $\Pi_\infty^\Sigma(X, Q)$. Since f is essentially surjective, [C, 7.1.7, 7.1.11] shows that it is enough to prove that the functor

$$f^* \colon \mathrm{Fun}(\Pi_\infty^\Sigma(X, P), \mathcal{S}) \longrightarrow \mathrm{Fun}(\Pi_\infty^\Sigma(X, Q), \mathcal{S})$$

is fully faithful and its essential image consists of W-local objects. Since by definition $f = \Pi_\infty^\Sigma(\mathrm{id}_X)$, Proposition 6.8 shows that ψ_f^{hyp} makes the diagram

$$
\begin{array}{ccc}
\mathrm{Fun}(\Pi_\infty^\Sigma(X, P), \mathcal{S}) & \xrightarrow{\ f^*\ } & \mathrm{Fun}(\Pi_\infty^\Sigma(X, Q), \mathcal{S}) \\
\downarrow{\scriptstyle \Psi_{X,P}^{\mathrm{hyp}}} & & \downarrow{\scriptstyle \Psi_{X,Q}^{\mathrm{hyp}}} \\
\mathrm{Cons}_P^{\mathrm{hyp}}(X) & \xrightarrow{\ \mathrm{id}_X^{*,\mathrm{hyp}}\ } & \mathrm{Cons}_Q^{\mathrm{hyp}}(X)
\end{array}
$$

commutative, while Theorem 5.17 shows that the vertical functors are equivalences. Considering $\mathrm{Cons}_Q^{\mathrm{hyp}}(X)$ and $\mathrm{Cons}_P^{\mathrm{hyp}}(X)$ as full subcategories of $\mathrm{Sh}^{\mathrm{hyp}}(X)$, we see that $f^{*,\mathrm{hyp}}$ acts as the identity and it is therefore fully faithful. Thus, the commutativity of the above diagram implies that the functor f^* is fully faithful as well. Since f takes (by definition) arrows in W to equivalences, we see that f^* factors through the full subcategory of W-local objects. Thus, we are left to check the essential surjectivity. Let $F \colon \Pi_\infty^\Sigma(X, Q) \to \mathcal{S}$ be a functor and assume that it is

W-local. In virtue of Theorem 5.17, it is enough to prove that $\Psi_{X,Q}^{\mathrm{hyp}}(F)$ belongs to $\mathrm{Cons}_P^{\mathrm{hyp}}(X)$. Denote by $\varphi\colon Q \to P$ be the given morphism at the level of posets. Let $a \in P$ be a point and let $Q_a \coloneqq \varphi^{-1}(a)$. Denote by $j_a\colon X_{Q_a} \to X$ be the inclusion of the corresponding union of strata. Applying Proposition 4.9 we see that the transformation

$$\psi_{j_a}^{\mathrm{hyp}}\colon j_a^{*,\mathrm{hyp}}\big(\Psi_{X,Q}^{\mathrm{hyp}}(F)\big) \longrightarrow \Psi_{X_{Q_a},Q_a}^{\mathrm{hyp}}\big(\Pi_\infty^\Sigma(j_a)^*(F)\big)$$

is an equivalence. Thus, we can replace X by X_a. Equivalently, we can assume from the very beginning that P is the trivial stratification (and therefore that X is locally weakly contractible).

Equipping $*$ with the trivial stratification S, we can see the canonical map $\Gamma_X\colon X \to *$ as a map of stratified spaces $(X,Q) \to (*,S)$. Then Proposition 6.8 implies that $\psi_{\Gamma_X}^{\mathrm{hyp}}$ yields a canonical identification

$$\Gamma_X^{*,\mathrm{hyp}} \simeq \Psi_{X,Q}^{\mathrm{hyp}} \circ \Gamma_{\Pi_\infty^\Sigma(X,Q)}^* \ .$$

Since X is locally weakly contractible, [HPT, § 2.2] shows that $\Gamma_X^{*,\mathrm{hyp}}$ has a left adjoint $\Gamma_{X,\sharp}^{\mathrm{hyp}}$. The above identification implies

$$\Gamma_{X,\sharp}^{\mathrm{hyp}} \simeq \Gamma_{\Pi_\infty^\Sigma(X,Q),!} \circ \Phi_{X,Q}^{\mathrm{hyp}} \ .$$

Using [HPT, Corollary 3.5] we find a canonical identification $\Pi_\infty(X) \simeq \Gamma_{X,\sharp}^{\mathrm{hyp}}(\Gamma_X^{*,\mathrm{hyp}}(*))$. Thus, the above identification yields:

$$\Pi_\infty(X) \simeq \Gamma_{X,\sharp}^{\mathrm{hyp}}\Gamma_X^{*,\mathrm{hyp}}(*) \simeq \operatorname*{colim}_{\Pi_\infty^\Sigma(X,Q)} \Gamma_{\Pi_\infty^\Sigma(X,Q)}^*(*) \simeq \mathrm{Env}(\Pi_\infty^\Sigma(X,Q)) \ ,$$

whence the conclusion. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We conclude this subsection with Observation 6.13 showing that exodromy implies enhanced functorialities at the level of hyperconstructible hypersheaves:

Observation 6.13. Let $f\colon (X,P) \to (Y,Q)$ be an exodromic morphism between conically stratified spaces with locally weakly contractible strata, and let $\mathcal{E}$ be a presentable ∞-category satisfying the assumption of Theorem 5.17. The functor $\Pi_\infty^\Sigma(f)^*$ has both a left adjoint given by the left Kan extension $\Pi_\infty^\Sigma(f)_!$ and a right adjoint given by the right Kan extension $\Pi_\infty^\Sigma(f)_*$. Thus, it follows that the functor

$$f^{*,\mathrm{hyp}}\colon \mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathcal{E}) \longrightarrow \mathrm{Cons}_Q^{\mathrm{hyp}}(Y;\mathcal{E})$$

also has a left and a right adjoint. We set

$$f_\sharp^{\mathrm{cons}} \coloneqq \Psi_{X,P}^{\mathrm{hyp}} \circ \Pi_\infty^\Sigma(f)_! \circ \Phi_{X,P}^{\mathrm{hyp}} \quad \text{and} \quad f_*^{\mathrm{cons}} \coloneqq \Psi_{X,P}^{\mathrm{hyp}} \circ \Pi_\infty^\Sigma(f)_* \circ \Phi_{X,P}^{\mathrm{hyp}} \ .$$

Since the functor $f^{*,\mathrm{hyp}}\colon \mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(Y;\mathcal{E})$ has a right adjoint given by f_*, and the inclusions of $\mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathcal{E})$ and $\mathrm{Cons}_Q^{\mathrm{hyp}}(Y;\mathcal{E})$ have right adjoints $\mathrm{R}_P^{\mathrm{hyp}}$ and $\mathrm{R}_Q^{\mathrm{hyp}}$ (see Remark 5.21), it follows from formal reasons that

$$(6.14) \qquad\qquad\qquad\qquad\qquad f_*^{\mathrm{cons}} \simeq \mathrm{R}_P^{\mathrm{hyp}} \circ f_* \ .$$

When $f^{*,\mathrm{hyp}}$ has itself a left adjoint[3] $f_\sharp^{\mathrm{hyp}}$, it follows similarly that

$$f_\sharp^{\mathrm{cons}} \simeq \mathrm{L}_P^{\mathrm{hyp}} \circ f_\sharp^{\mathrm{hyp}} \ .$$

---

[3]This is e.g. the case if f is the projection $S \times X \to S$ for locally weakly contractible topological spaces, see [HPT, § 2.2].

6.3. Change of coefficients revisited.

Lemma 6.15. *Let (X, P) be a conically stratified space. Let $f : \mathcal{E} \to \mathcal{D}$ be a cocontinuous functor between presentable ∞-categories with right adjoint $g : \mathcal{D} \to \mathcal{E}$. Assume that the conditions of Theorem 5.17 are satisfied. Let $p \in P$ and let $i_p : X_p \to X$ be the inclusion. Then the following commutative diagram*

$$
\begin{array}{ccc}
\mathrm{Sh}^{\mathrm{hyp}}(X_p; \mathcal{D}) & \xrightarrow{g \circ -} & \mathrm{Sh}^{\mathrm{hyp}}(X_p; \mathcal{E}) \\
\downarrow{\scriptstyle i_{p,*}} & & \downarrow{\scriptstyle i_{p,*}} \\
\mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{D}) & \xrightarrow{g \circ -} & \mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E}) \ .
\end{array}
$$

is verticaly left adjointable on $\mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{D})$. That is, for every $F \in \mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{D})$, the Beck-Chevalley transformation

$$
i_p^{*,\mathrm{hyp}} \circ g(F) \longrightarrow g \circ i_p^{*,\mathrm{hyp}}(F)
$$

is an equivalence.

Proof. The question is local on X. Hence, we can suppose that (X, P) is of the form $Z \times C(Y)$ where Z is a locally weakly contractible topological space and where $(Y, P_{>p})$ is a stratified space. Let U be an open subset of Z. Then, Corollary 5.23 gives

$$
(g \circ i_p^{*,\mathrm{hyp}}(F))(U) = g(i_p^{*,\mathrm{hyp}}(F)(U)) \simeq g(F(U \times C(Y)))
$$

On the other hand, we have

$$
(i_p^{*,\mathrm{hyp}} \circ g(F))(U) \simeq \operatorname*{colim}_{\varepsilon \in (0,1)} g(F(U \times C_\varepsilon(Y)))
$$

We know from Corollary 5.22 that for every $\varepsilon \in (0,1)$, the restriction morphism

$$
F(U \times C(Y)) \longrightarrow F(U \times C_\varepsilon(Y))
$$

is an equivalence. Hence, the above colimit is constant and we get

$$
(i_p^{*,\mathrm{hyp}} \circ g(F))(U) \simeq g(F(U \times C(Y)))
$$

Lemma 6.15 is thus proved. $\qquad\square$

The following Corollary 6.16 contrasts with Remark 2.29.

Corollary 6.16. *Let (X, P) be a conically stratified space. Let $f : \mathcal{E} \to \mathcal{D}$ be a cocontinuous functor between presentable ∞-categories with right adjoint $g : \mathcal{D} \to \mathcal{E}$. Assume that the conditions of Theorem 5.17 are satisfied. Then, the following statements hold :*

(1) *For every $F \in \mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E})$, the functor $f \circ F : \mathrm{Open}(X)^{\mathrm{op}} \to \mathcal{D}$ lies in $\mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{D})$.*

(2) *For every $F \in \mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{D})$, the functor $g \circ F : \mathrm{Open}(X)^{\mathrm{op}} \to \mathcal{E}$ lies in $\mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E})$. In particular, the adjunction Eq. (2.28) induces an adjunction*

$$
f \circ - : \ \mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E}) \leftrightarrows \mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{D}) : g \circ -
$$

(3) *The Exodromy equivalence induces an equivalence of adjunctions*

$$
\begin{array}{ccc}
f \circ - : \mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E}) & \rightleftarrows & \mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{D}) : g \circ - \\
\downarrow{\scriptstyle \Phi_{X,P}^{\mathrm{hyp},\mathcal{E}}} & & \downarrow{\scriptstyle \Phi_{X,P}^{\mathrm{hyp},\mathcal{D}}} \\
f \circ - : \mathrm{Fun}(\Pi_\infty^\Sigma(X, P), \mathcal{E}) & \rightleftarrows & \mathrm{Fun}(\Pi_\infty^\Sigma(X, P), \mathcal{D}) : g \circ - \ .
\end{array}
$$

Proof. The first statement follows from Theorem 5.17 and Corollary 3.9. The statement (2) follows from Lemma 6.15 and Lemma 2.30. The statement (3) follows from Corollary 3.9. $\qquad\square$

6.4. **Monoidal structures.** Assume now that our ∞-category of coefficients $\mathcal{E}$ carries a presentably symmetric monoidal structure. In other words, we fix $\mathcal{E} \in \mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L},\otimes})$.

Recollection 6.17.

(1) For every topological space X, the ∞-category $\mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E})$ inherits a canonical symmetric monoidal structure, and every morphism $f\colon X \to Y$ induces a symmetric monoidal functor

$$f^{*,\mathrm{hyp}}\colon \mathrm{Sh}^{\mathrm{hyp}}(Y;\mathcal{E}) \longrightarrow \mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E}) \ .$$

In particular, it follows that $\mathrm{LC}^{\mathrm{hyp}}(X;\mathcal{E})$ is stable under tensor product, and consequently that for every stratified space (X,P), $\mathrm{Cons}^{\mathrm{hyp}}_P(X,P)$ acquires a symmetric monoidal structure.

(2) Similarly, $\mathrm{Fun}\big(\Pi^{\Sigma}_{\infty}(X,P),\mathcal{E}\big)$ inherits a symmetric monoidal structure, where the tensor product is computed objectwise.

Proposition 6.18. *Let (X,P) be a conically stratified space and let $\mathcal{E}^{\otimes}$ be a symmetric monoidal ∞-category. Then the functor*

$$\Phi^{\mathrm{hyp},\mathcal{E}}_{X,P}\colon \mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E}) \longrightarrow \mathrm{Fun}\big(\Pi^{\Sigma}_{\infty}(X,P),\mathcal{E}\big) \ ,$$

has a natural lax symmetric monoidal structure. Under the assumptions of Theorem 5.17, $\Phi^{\mathrm{hyp},\mathcal{E}}_{X,P}$ restricts to a symmetric monoidal functor on the full subcategory $\mathrm{Cons}^{\mathrm{hyp}}_P(X;\mathcal{E})$.

Proof. Thanks to Lemma 3.8, we immediately reduce to the case where $\mathcal{E} = \mathcal{S}$. In this case, the monoidal structure induced on both $\mathrm{Cons}^{\mathrm{hyp}}_P(X)$ and $\mathrm{Fun}\big(\Pi^{\Sigma}_{\infty}(X,P),\mathcal{S}\big)$ is the cartesian one. Then $\Phi^{\mathrm{hyp}}_{X,P}$ has a canonical lax monoidal structure given by [HA, Proposition 2.4.1.7]. Since $\Phi^{\mathrm{hyp}}_{X,P}$ is an equivalence, it commutes with products and hence Corollary 2.4.1.8 in *loc. cit.* guarantees that $\Phi^{\mathrm{hyp}}_{X,P}$ is strong monoidal. The conclusion follows. $\qquad\square$

As a consequence of our criterion for constructibility (see Corollary 5.22), we obtain:

Corollary 6.19. *Let (X,P) be a conically stratified space and let $\mathcal{F},\mathcal{G} \in \mathrm{Cons}^{\mathrm{hyp}}_P(X;\mathcal{E})$. If the assumptions of Theorem 5.17 are satisfied, then the internal hom $\mathcal{H}\mathrm{om}_X(\mathcal{F},\mathcal{G})$, computed inside $\mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E})$, belongs to $\mathrm{Cons}^{\mathrm{hyp}}_P(X;\mathcal{E})$.*

Proof. We apply Corollary 5.22. It is enough to prove that for every inclusion $i\colon U \subset V$ for which the induced map $\Pi^{\Sigma}_{\infty}(i)\colon \Pi^{\Sigma}_{\infty}(U,P) \to \Pi^{\Sigma}_{\infty}(V,P)$ is an equivalence, the restriction map

$$\mathcal{H}\mathrm{om}_X(\mathcal{F},\mathcal{G})(V) \longrightarrow \mathcal{H}\mathrm{om}_X(\mathcal{F},\mathcal{G})(U)$$

is an equivalence as well. By definition, this map can be rewritten as

(6.20) $$\mathrm{Hom}_{\mathrm{Cons}^{\mathrm{hyp}}_P(V;\mathcal{E})}(\mathcal{F}|_V,\mathcal{G}|_V) \longrightarrow \mathrm{Hom}_{\mathrm{Cons}^{\mathrm{hyp}}_P(U;\mathcal{E})}(\mathcal{F}|_U,\mathcal{G}|_U) \ .$$

Theorem 5.17 allows to write the target of this morphism

$$\mathrm{Hom}_{\mathrm{Fun}(\Pi^{\Sigma}_{\infty}(U,P),\mathcal{E})}(\Phi^{\mathrm{hyp}}_{U,P}(\mathcal{F}|_U),\Phi^{\mathrm{hyp}}_{U,P}(\mathcal{G}|_U)) \ ,$$

and similarly for the source. Moreover, since the natural transformation

$$\phi^{\mathrm{hyp}}_i\colon \Phi^{\mathrm{hyp}}_{U,P} \circ i^{*,\mathrm{hyp}} \longrightarrow \Pi^{\Sigma}_{\infty}(i)^* \circ \Phi^{\mathrm{hyp}}_{V,P}$$

is an equivalence thanks to Proposition 6.8, the morphism (6.20) becomes canonically identified with the morphism induced by $\Pi^{\Sigma}_{\infty}(i)^*$

$$\mathrm{Hom}_V(\Phi^{\mathrm{hyp}}_{V,P}(\mathcal{F}|_V),\Phi^{\mathrm{hyp}}_{V,P}(\mathcal{G}|_V)) \longrightarrow \mathrm{Hom}_U(\Pi^{\Sigma}_{\infty}(i)^*\Phi^{\mathrm{hyp}}_{V,P}(\mathcal{F}|_V),\Pi^{\Sigma}_{\infty}(i)^*\Phi^{\mathrm{hyp}}_{V,P}(\mathcal{G}|_V)) \ ,$$

where we wrote Hom_V to denote the $\mathcal{E}$-enriched hom object computed in $\mathrm{Fun}(\Pi^{\Sigma}_{\infty}(V,P),\mathcal{E})$, and similarly for Hom_U. Since $\Pi^{\Sigma}_{\infty}(i)^*$ is an equivalence, so is the above morphism, whence the conclusion. $\qquad\square$

6.5. **Compact generation.** Using the extra functorialities obtained at the end of the previous subsection, we are now able to prove the following result, strengthening Corollary 6.4:

Theorem 6.21. *Let (X, P) be a conically stratified space with locally weakly contractible strata. Let $\mathcal{E}$ be a compactly generated presentable ∞-category with a single compact generator. Assume that the poset P is finite and that for every $p \in P$, the set $\pi_0(X_p)$ is finite as well. Then the ∞-category $\mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E})$ has a single compact generator.*

The proof relies on the following two simple observations:

Observation 6.22. Let (X, P) be a conically stratified space with locally weakly contractible strata. Let $x \in X$ be a point and let $j_x \colon \{x\} \to X$ be the natural inclusion. The stratification on $\{x\}$ induced by P is the trivial one. In particular, Observation 6.13 guarantees that the functor

$$j_x^{*,\mathrm{hyp}} \colon \mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E}) \longrightarrow \mathcal{E}$$

admits a left adjoint $j_{x,\sharp}^{\mathrm{cons}}$ which can be identified with the left Kan extension along the induced morphism $\Pi_\infty^\Sigma(j_x) \colon \Pi_\infty^\Sigma(\{x\}) \to \Pi_\infty^\Sigma(X, P)$. Since $j_x^{*,\mathrm{hyp}}$ has also a right adjoint, it commutes with filtered colimits, and hence $j_{x,\sharp}^{\mathrm{cons}}$ preserves κ-compact objects for every regular cardinal κ.

Observation 6.23. Let X be a locally weakly contractible topological space. Assume that X is connected and let $x \in X$ be a point. Then the induced morphism $\Pi_\infty(\{x\}) \to \Pi_\infty(X)$ is an epimorphism in $\mathcal{S}$. It follows that for every presentable ∞-category $\mathcal{E}$ the restriction functor

$$\Pi_\infty(j_x)^* \colon \mathrm{Fun}\left(\Pi_\infty(X), \mathcal{E}\right) \longrightarrow \mathrm{Fun}\left(\Pi_\infty(\{x\}), \mathcal{E}\right) \simeq \mathcal{E}$$

is conservative. If furthermore $\mathcal{E}$ satisfies the assumptions of Theorem 5.17, Proposition 6.8 implies that the functor

$$j_x^{*,\mathrm{hyp}} \colon \mathrm{LC}^{\mathrm{hyp}}(X; \mathcal{E}) \longrightarrow \mathcal{E}$$

is conservative.

Proof of Theorem 6.21. Since $\mathcal{E}$ is compactly generated, Remark 5.18 ensures that the hyper-restriction functors

$$\left\{ j_p^{*,\mathrm{hyp}} \colon \mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E}) \to \mathrm{LC}^{\mathrm{hyp}}(X_p; \mathcal{E}) \right\}_{p \in P}$$

are jointly conservative. Furthermore, Observation 6.13 guarantees that they have left adjoints $j_{p,\sharp}^{\mathrm{cons}}$. Therefore, their product

$$\mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E}) \longrightarrow \prod_{p \in P} \mathrm{LC}^{\mathrm{hyp}}(X_p; \mathcal{E}) \ ,$$

is conservative and admits a left adjoint. Furthermore, it commutes with (filtered) colimits. Thus, [AGV, Lemma 2.8.3] reduces us to check that the target has a single compact generator. Since this properties are closed under finite products, it is enough to consider the case of a single stratum.

When the stratification on X is trivial, our assumption guarantees that the number of connected components of X is finite. Hence, we are left to treat the case where X is connected. Let then $x \in X$ be a point and consider the functor

$$j_x^{*,\mathrm{hyp}} \colon \mathrm{LC}^{\mathrm{hyp}}(X; \mathcal{E}) \longrightarrow \mathcal{E} \ .$$

It follows from Observation 6.23 that this functor is conservative, while Observation 6.22 shows that it has a left adjoint. Since it also commutes with filtered colimits, using once more [AGV, Lemma 2.8.3] we see that if $\{E_i\}_{i \in I}$ is a family of compact generators for $\mathcal{E}$, then $\left\{ j_{x,\sharp}^{\mathrm{cons}}(E_i) \right\}_{i \in I}$ is a family of compact generators for $\mathrm{LC}^{\mathrm{hyp}}(X; \mathcal{E})$. In particular, since $\mathcal{E}$ has a single compact generator, then so does $\mathrm{LC}^{\mathrm{hyp}}(X; \mathcal{E})$. $\qquad\square$

For future reference, let us record the following more specific statement that can be deduced from the previous proof:

Corollary 6.24. *Let (X, P) be a conically stratified space with locally weakly contractible strata. Let $\mathcal{E}$ be a compactly generated presentable ∞-category with compact generators $\{E_\alpha\}_{\alpha \in J}$. Assume that the poset P is finite and that for every $p \in P$, the set $\pi_0(X_p)$ is finite as well. Then*

$$\left\{ j^{\mathrm{cons}}_{x_i,\sharp}(E_\alpha) \right\}_{i \in I, \alpha \in J}$$

is a system of compact generators for $\mathrm{Cons}^{\mathrm{hyp}}_P(X; \mathcal{E})$.

This result is already interesting in the case of the trivial stratification:

Example 6.25. Let X be a connected, locally weakly contractible topological space and let $x \in X$ be a point. Let $\mathcal{E}$ be a presentable ∞-category with a single compact generator $E \in \mathcal{E}$ (e.g. $\mathcal{E} = \mathrm{Mod}_A$ for some $\mathbb{E}_\infty$-algebra A, and $E = A$). Then $j_{x,\sharp}(A) \in \mathrm{LC}^{\mathrm{hyp}}(X; \mathcal{E})$ is a single compact generator.

As a consequence, we can express $\mathrm{LC}^{\mathrm{hyp}}(X; \mathcal{E})$ as the ∞-category of modules over the endomorphism ring of $j_{x,\sharp}(A)$. It turns out that it is possible to compute quite explicitly such endomorphism ring:[4]

Theorem 6.26. *Let X be a connected and locally weakly contractible topological space. Fix a point $x \in X$ and consider the x-based loop space $\Omega_x(X)$. Then for every $\mathbb{E}_\infty$-algebra A, the endomorphism ring of $j_{x,\sharp}(A)$ canonically coincides with the chain algebra $\mathrm{C}_*(\Omega_x(X); A)$. In particular, there is an equivalence*
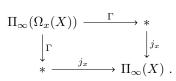
$$\mathrm{LC}^{\mathrm{hyp}}(X; \mathrm{Mod}_A) \simeq \mathrm{Mod}_{\mathrm{C}_*(\Omega_x(X); A)} \ .$$

Remark 6.27. This theorem recovers [Ha, Theorem 26] and [Hb, Theorem 12]. Besides it generalizes it in several directions, as A is now allowed to be an $\mathbb{E}_\infty$-algebra, and X is now only required to be locally weakly contractible rather than admitting a bounded locally finite good hypercover. In particular, the above theorem holds even without any paracompactness assumption on X.

Proof of Theorem 6.26. The second half is a direct consequence of the first half and Lurie-Barr-Beck's theorem [HA, Theorem 4.7.3.5]. To compute the endomorphism ring of $j_{x,\sharp}(A)$, we first observe that there is the following natural equivalence:

$$\mathrm{Hom}_{\mathrm{LC}^{\mathrm{hyp}}(X; \mathrm{Mod}_A)} \left(j^{\mathrm{hyp}}_{x,\sharp}(A), j^{\mathrm{hyp}}_{x,\sharp}(A) \right) \simeq j^{*,\mathrm{hyp}}_x j^{\mathrm{hyp}}_{x,\sharp}(A) \ .$$

Observe now that the square

$$\begin{array}{ccc}
\Pi_\infty(\Omega_x(X)) & \xrightarrow{\ \Gamma\ } & * \\
\Big\downarrow{\scriptstyle \Gamma} & & \Big\downarrow{\scriptstyle j_x} \\
* & \xrightarrow{\ j_x\ } & \Pi_\infty(X) \ .
\end{array}$$

is a pullback in $\mathcal{S}$, where we wrote Γ instead of $\Gamma_{\Omega_x(X)}$ for brevity. Unraveling the definitions, this implies that $\Pi_\infty(\Omega_x(X))$ coincides with the comma category

$$\{x\} \times_{\Pi_\infty(X)} \Pi_\infty(X)_{j_x/} \ ,$$

so that combining Corollary 3.32 with the formula for the left Kan extension, we deduce that the Beck-Chevalley transformation

$$\Gamma^{\mathrm{hyp}}_\sharp \circ \Gamma^{*,\mathrm{hyp}} \longrightarrow j^{*,\mathrm{hyp}}_x \circ j^{\mathrm{hyp}}_{x,\sharp}$$

---

[4]We warmly thank Julian V. Holstein and Alexandru Oancea for discussions on this point.

is an equivalence. Thus, the endomorphism ring of $j_{x,\sharp}(A)$ is canonically identified with

$$\Gamma_\sharp^{\mathrm{hyp}}\Gamma^{*,\mathrm{hyp}}(A) \simeq \Gamma_\sharp^{\mathrm{hyp}}\Gamma^{*,\mathrm{hyp}}(*) \otimes A \ .$$

By definition, $\Gamma_\sharp^{\mathrm{hyp}}\Gamma^{*,\mathrm{hyp}}(*) \in \mathcal{S}$ is the shape of the ∞-topos $\mathrm{Sh}^{\mathrm{hyp}}(\Omega_x(X))$. Applying [HPT, Corollary 3.5], we see that this shape is simply identified with the homotopy type $\Pi_\infty(\Omega_x(X))$. Thus, the endomorphism ring of $j_{x,\sharp}(A)$ is identified with

$$\Gamma_\sharp^{\mathrm{hyp}}\Gamma^{*,\mathrm{hyp}}(A) \simeq \Pi_\infty(\Omega_x(X)) \otimes A \simeq \mathrm{C}_*(\Omega_x(X); A) \ ,$$

where the last equivalence holds by definition of singular chains. $\qquad\square$

Remark 6.28. It is possible to obtain a similar description in the stratified case, in line with [HL, §6]. It should be possible to obtain an explicit description of the endomorphism ring of the single compact generator provided by Theorem 6.21 akin to the one of Theorem 6.26, at least in the setting of conically smooth stratified spaces. Indeed, any such description should see the chain algebras on single strata, but at the same time, it should also have a contribution from the links of the stratification. We will come back to this subject in a later work.

6.6. **Exodromy and stalkwise compactness.** Let (X, P) be a conically stratified space with locally weakly contractible strata and let $\mathcal{E}$ be a presentable ∞-category. The construction of $\Phi_{X,P}^{\mathrm{hyp}}$ and $\Psi_{X,P}^{\mathrm{hyp}}$ relies a priori on the existence of limits and colimits of diagrams that are typically not finite. However, the "regularity" of conical charts paired with the homotopy-invariance property of hyperconstructible hypersheaves [HPT, Theorem 0.4] actually implies that all infinite colimits that are involved in the construction of the exodromy adjunction can be ignored. To give a proper formulation of this idea, let us first introduce the following notation:

Notation 6.29. Let (X, P) be a stratified space and let $\mathcal{E}$ be a presentable ∞-category. Let κ be a regular cardinal. We denote by $\mathrm{Cons}_{P,\kappa}^{\mathrm{hyp}}(X; \mathcal{E})$ the full subcategory of $\mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E})$ spanned by $\mathcal{E}$-valued hyperconstructible hypersheaves F whose stalks are κ-compact objects of $\mathcal{E}$. When the stratification P is trivial, we denote this ∞-category by $\mathrm{LC}_\kappa^{\mathrm{hyp}}(X; \mathcal{E})$.

Warning 6.30. The ∞-category $\mathrm{Cons}_{P,\kappa}^{\mathrm{hyp}}(X; \mathcal{E})$ does not coincide neither with $\mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E}^\kappa)$ nor with $\mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E})^\kappa$. We offer two counterexamples:

(1) Take $X := \coprod_\mathbb{N} *$ to be an infinite disjoint union of points equipped with the trivial stratification. Take also $\mathcal{E} = \mathcal{S}$ and $\kappa = \omega$. Fix $K \in \mathcal{S}^\kappa$ and let $F := \Gamma_X^*(K)$ be the hyperconstant hypersheaf associated to K. Then $F \in \mathrm{Cons}_{P,\kappa}^{\mathrm{hyp}}(X; \mathcal{E})$, but $F(X) \simeq \prod_\mathbb{N} K$, which does not belong to $\mathcal{S}^\omega$ unless K is contractible. In particular, F does not belong to $\mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{S}^\omega)$.

(2) Take $X = S^1$, $\mathcal{E} = \mathrm{Mod}_\mathbb{C}$ and $\kappa = \omega$. Once again, equip X with the trivial stratification. Then, we have canonical identifications

$$\mathrm{LC}^{\mathrm{hyp}}(S^1; \mathrm{Mod}_\mathbb{C}) \simeq \mathrm{Fun}(\Pi_\infty(S^1), \mathrm{Mod}_\mathbb{C}) \simeq \mathrm{QCoh}(\mathbb{G}_{m,\mathbb{C}}) \ .$$

Therefore,

$$\mathrm{LC}^{\mathrm{hyp}}(S^1; \mathrm{Mod}_\mathbb{C})^\omega \simeq \mathrm{Perf}(\mathbb{G}_{m,\mathbb{C}}) \ .$$

In particular, the structure sheaf of $\mathbb{G}_{m,\mathbb{C}}$ corresponds to the functor $F \colon S^1 \to \mathrm{Mod}_\mathbb{C}$ selecting $\mathbb{C}[T, T^{-1}]$ with endomorphism given by multiplication by T. Then Corollary 6.10 implies that the stalk of $\Psi_{S^1}^{\mathrm{hyp}}(F)$ at any point of S^1 coincides with $\mathbb{C}[T, T^{-1}]$, which is not compact as an object in $\mathrm{Mod}_\mathbb{C}$.

Remark 6.31. In the second example above, it follows from [BZNP] that $\mathrm{LC}^{\mathrm{hyp}}_\kappa(S^1; \mathrm{Mod}_{\mathbb{C}})$ corresponds to the full subcategory of $\mathrm{QCoh}(\mathbb{G}_{m,\mathbb{C}})$ spanned by perfect complexes *with proper support.*

Proposition 6.32. *Let (X, P) be a conically stratified space and let $\mathcal{E}$ be a presentable ∞-category. Let κ be a regular cardinal. If the assumptions of Theorem 5.17 are satisfied, then the exodromy equivalence $\Phi^{\mathrm{hyp}}_{X,P} \dashv \Psi^{\mathrm{hyp}}_{X,P}$ restricts to an equivalence*

$$\Phi^{\mathrm{hyp}}_{X,P} \colon \mathrm{Fun}(\Pi^\Sigma_\infty(X, P), \mathcal{E}^\kappa) \leftrightarrows \mathrm{Cons}^{\mathrm{hyp}}_{P,\kappa}(X; \mathcal{E}) \colon \Psi^{\mathrm{hyp}}_{X,P} \ .$$

Proof. It is enough to prove that both $\Phi^{\mathrm{hyp}}_{X,P}$ and $\Psi^{\mathrm{hyp}}_{X,P}$ respect these two full subcategories. Let first $F \in \mathrm{Cons}^{\mathrm{hyp}}_{P,\kappa}(X; \mathcal{E})$. Then we have to check that the functor $\Phi^{\mathrm{hyp}}_{X,P}(F)$ takes values in $\mathcal{E}^\kappa$. To see this, let $x \in \Pi^\Sigma_\infty(X, P)$. Then Corollary 6.10 provides a canonical identification

$$\Phi^{\mathrm{hyp}}_{X,P}(F)(x) \simeq F_x$$

so that $\Phi^{\mathrm{hyp}}_{X,P}(F)$ belongs to $\mathrm{Fun}(\Pi^\Sigma_\infty(X, P), \mathcal{E}^\kappa)$.

Let now $G \colon \Pi^\Sigma_\infty(X, P) \to \mathcal{E}^\kappa$ be a functor. Let $x \in X$. Then Corollary 6.10 and Theorem 5.17 provide the following canonical identifications:

$$\Psi^{\mathrm{hyp}}_{X,P}(G)_x \simeq \left(\Phi^{\mathrm{hyp}}_{X,P}\Psi^{\mathrm{hyp}}_{X,P}(G)\right)(x) \simeq G(x) \ .$$

Thus, the stalks of $\Psi^{\mathrm{hyp}}_{X,P}(G)$ belong to $\mathcal{E}^\kappa$. In other words, $\Psi^{\mathrm{hyp}}_{X,P}(G)$ belongs to $\mathrm{Cons}^{\mathrm{hyp}}_{P,\kappa}(X; \mathcal{E})$. $\square$

6.7. **Constructibility and pushforward. A general criterion.** Let $f \colon (Y, Q) \to (X, P)$ be an exodromic morphism of conically stratified spaces. Then the natural transformation ψ^{hyp}_f makes the square

$$\begin{array}{ccc}
\mathrm{Fun}(\Pi^\Sigma_\infty(X, P), \mathcal{E}) & \xrightarrow{\Pi^\Sigma_\infty(f)^*} & \mathrm{Fun}(\Pi^\Sigma_\infty(Y, Q), \mathcal{E}) \\
\downarrow{\scriptstyle \Psi^{\mathrm{hyp}}_{X,P}} & & \downarrow{\scriptstyle \Psi^{\mathrm{hyp}}_{Y,Q}} \\
\mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E}) & \xrightarrow{f^{*,\mathrm{hyp}}} & \mathrm{Sh}^{\mathrm{hyp}}(Y; \mathcal{E})
\end{array}$$

commutative. In particular, there is an associated Beck-Chevalley transformation

$$\chi^{\mathrm{hyp}}_f \colon \Psi^{\mathrm{hyp}}_{X,P} \circ \Pi^\Sigma_\infty(f)_* \longrightarrow f_* \circ f^{*,\mathrm{hyp}} \circ \Psi^{\mathrm{hyp}}_{X,P} \circ \Pi^\Sigma_\infty(f)_* \xrightarrow{\psi^{\mathrm{hyp}}_f} f_* \circ \Psi^{\mathrm{hyp}}_{Y,Q} \circ \Pi^\Sigma_\infty(f)^* \circ \Pi^\Sigma_\infty(f)_* \longrightarrow f_* \circ \Psi^{\mathrm{hyp}}_{Y,Q} \ .$$

The identification (6.14) has the following immediate consequence:

Lemma 6.33. *Let $f \colon (Y, Q) \to (X, P)$ be an exodromic morphism of conically stratified spaces with locally weakly contractible strata. Let $\mathcal{E}$ be a presentable ∞-category and assume that the conditions of Theorem 5.17 are satisfied. Then for every $F \colon \Pi^\Sigma_\infty(Y, Q) \to \mathcal{E}$, the following statements are equivalent:*

(1) the hypersheaf $f_*\left(\Psi^{\mathrm{hyp}}_{Y,Q}(F)\right) \in \mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E})$ is hyperconstructible on (X, P);

(2) the transformation $\chi^{\mathrm{hyp}}_f(F) \colon \Psi^{\mathrm{hyp}}_{X,P}\left(\Pi^\Sigma_\infty(f)_*(F)\right) \to f_*\left(\Psi^{\mathrm{hyp}}_{Y,Q}(F)\right)$ is an equivalence.

The following observation gives a convenient sufficient condition to check whether pushforward along a morphism $f \colon (Y, Q) \to (X, P)$ preserves hyperconstructible hypersheaves:

Lemma 6.34. *Let $f \colon (Y, Q) \to (X, P)$ be a strongly exodromic morphism of conically stratified spaces with locally weakly contractible strata. Let $\mathcal{E}$ be a presentable ∞-category and assume that the conditions of Theorem 5.17 are satisfied. Then f_* preserves hyperconstructible hypersheaves.*

Proof. By assumption, the natural transformation ϕ_f^{hyp} makes the diagram

$$
\begin{array}{ccc}
\mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E}) & \xrightarrow{\ f^{*,\mathrm{hyp}}\ } & \mathrm{Sh}^{\mathrm{hyp}}(Y;\mathcal{E}) \\
\downarrow{\scriptstyle \Phi_{X,P}^{\mathrm{hyp}}} & & \downarrow{\scriptstyle \Phi_{Y,Q}^{\mathrm{hyp}}} \\
\mathrm{Fun}(\Pi_\infty^\Sigma(X,P),\mathcal{E}) & \xrightarrow{\ \Pi_\infty^\Sigma(f)^*\ } & \mathrm{Fun}(\Pi_\infty^\Sigma(Y,Q),\mathcal{E})
\end{array}
$$

commutative. Passing to right adjoints, we deduce that the natural transformation χ_f^{hyp} is an equivalence. Then Lemma 6.34 follows from Corollary 4.10. $\qquad\square$

6.8. **Constructibility and pushforward along immersions.**

Proposition 6.35. *Let (X,P) be a conically stratified space and let $\mathcal{E}$ be a presentable ∞-category. Let $S \subseteq P$ be a locally closed subset of P and let $i_S\colon X_S \to X$ be the inclusion of the corresponding union of strata. If the assumption of Theorem 5.17 are satisfied, then*

$$ i_{S,*}\colon \mathrm{Sh}^{\mathrm{hyp}}(X_S;\mathcal{E}) \longrightarrow \mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E}) $$

takes hyperconstructible hypersheaves on (X_S,S) to hyperconstructible hypersheaves on (X,P).

Proof. Combine Corollaries 5.8 and 5.15 with Lemma 6.34. $\qquad\square$

Proposition 6.35 has the following two immediate corollaries below which are true for hyperconstructible hypersheaves since they are already true for hypersheaves.

Corollary 6.36. *Let (X,P) be a conically stratified space and let $S \subseteq P$ be a locally closed subset. Let $\mathcal{E}$ and $\mathcal{D}$ be presentable ∞-categories and let $f\colon \mathcal{E} \to \mathcal{D}$ be a cocontinuous functor. If the assumption of Theorem 5.17 are satisfied, then the diagram*

$$
\begin{array}{ccc}
\mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathcal{E}) & \xrightarrow{\ i_S^{*,\mathrm{hyp}}\ } & \mathrm{Cons}_S^{\mathrm{hyp}}(X_S;\mathcal{E}) \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\
\mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathcal{D}) & \xrightarrow{\ i_S^{*,\mathrm{hyp}}\ } & \mathrm{Cons}_S^{\mathrm{hyp}}(X_S;\mathcal{D})
\end{array}
$$

is horizontally right adjointable. Than is, the Beck-Chevalley transformation

$$ f \circ i_{S,*}^{\mathrm{hyp}} \longrightarrow i_{S,*}^{\mathrm{hyp}} \circ f $$

is an equivalence.

For the notion of recollement of ∞-categories, let us refer to [HA, A.8.1].

Corollary 6.37. *Let (X,P) be a conically stratified space and let $\mathcal{E}$ be a presentable ∞-category. Let $S \subseteq P$ be a closed subset and let $i_S\colon X_S \to X$ be the inclusion. Put $U := P \setminus S$ and let $i_U\colon X_U \to X$ be the inclusion. If the assumptions of Theorem 5.17 are satisfied, then the fully-faithful functors*

$$ i_{S,*}\colon \mathrm{Cons}_S^{\mathrm{hyp}}(X_S;\mathcal{E}) \longrightarrow \mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathcal{E}) \longleftarrow \mathrm{Cons}_U^{\mathrm{hyp}}(X_U;\mathcal{E})\colon i_{U,*} $$

exhibits $\mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathcal{E})$ as a recollement of $\mathrm{Cons}_S^{\mathrm{hyp}}(X_S;\mathcal{E})$ and $\mathrm{Cons}_U^{\mathrm{hyp}}(X_U;\mathcal{E})$.

Proof. Corollary 6.37 follows immediately from the fact that the fully-faithful functors

$$ i_{S,*}\colon \mathrm{Sh}^{\mathrm{hyp}}(X_S;\mathcal{E}) \longrightarrow \mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E}) \longleftarrow \mathrm{Sh}^{\mathrm{hyp}}(X_U;\mathcal{E})\colon i_{U,*} $$

exhibits $\mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E})$ as a recollement of $\mathrm{Sh}^{\mathrm{hyp}}(X_S;\mathcal{E})$ and $\mathrm{Sh}^{\mathrm{hyp}}(X_U;\mathcal{E})$. $\qquad\square$

As for the hyperrestriction of a hyperconstructible hypersheaf to a stratum computed in Corollary 5.23, one can compute the fibre of $i_{S,*}$ at a point x explicitly in terms of a conical chart containing x. This is the following

Lemma 6.38. *Let Z be a weakly contractible locally weakly contractible topological space and let (Y, Q) be a stratified space such that $\mathrm{Exit}(Y, Q)$ is an ∞-category and $(X, P) \coloneqq (Z \times C(Y), Q^{\triangleleft})$ is conically stratified. Let $\mathcal{E}$ be a presentable ∞-category. Let $F \in \mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E})$ and let $x \in Z$. If the assumptions of Theorem 5.17 are satisfied, there is a canonical identification*

$$(i_{Q,*}F)_x \simeq \lim_{\mathrm{Exit}(Y,Q)} \Phi_{X,P}^{\mathrm{hyp}}(F)|_{\mathrm{Exit}(Y,Q)}$$

where in the right-hand side, the restriction is performed along the equivalence $\mathrm{Exit}(Z \times \mathbb{R}_{>0} \times Y, Q) \to \mathrm{Exit}(Y, Q)$ induced by the canonical projection.

Proof. Write $\mathcal{B}_x$ for the collection of weakly contractible open neighborhoods of x inside Z. Then

$$i_{Q,*}(F)_x \simeq \operatorname*{colim}_{U \in \mathcal{B}_x} \operatorname*{colim}_{\varepsilon \in (0,1)} F(U \times (0, \varepsilon) \times Y) \ .$$

From Theorem 5.17 and Eq. (3.13), we deduce

$$i_{Q,*}(F)_x \simeq \operatorname*{colim}_{U \in \mathcal{B}_x} \operatorname*{colim}_{\varepsilon \in (0,1)} \lim_{\mathrm{Exit}(U \times (0,\epsilon) \times Y, Q)} \Phi_{X,P}^{\mathrm{hyp}}(F) \ .$$

Since the $U \in \mathcal{B}_x$ are weakly contractible, we have further

$$\mathrm{Exit}(U \times (0, \epsilon) \times Y, Q) \simeq \mathrm{Exit}(U, *) \times \mathrm{Exit}((0, \epsilon), *) \times \mathrm{Exit}(Y, Q) \simeq \mathrm{Exit}(Y, Q) \ .$$

Hence, the above colimits are constant and Lemma 6.38 thus follows. $\qquad\square$

In view of the construction of moduli carried out in Section 7, we need to understand the interaction of push-forward with proper hyperconstructible hypersheaves. To this end, additional finiteness assumptions on (X, P) will be needed. They will be justified by the following general

Lemma 6.39. *Let $\mathcal{C} \in \mathrm{Cat}_\infty^\omega$ be a compact object in Cat_∞ and let $\mathcal{E}$ be a presentable ∞-category. Assume that filtered colimits are left exact in $\mathcal{E}$. Then the functor*

$$\Gamma_{\mathcal{C},*} \colon \mathrm{Fun}(\mathcal{C}, \mathcal{E}) \longrightarrow \mathcal{E}$$

commutes with filtered colimits. When $\mathcal{E}$ is stable, both the left adjoint $\Gamma_{\mathcal{C},!}$ and the right adjoint $\Gamma_{\mathcal{C},*}$ to $\Gamma_{\mathcal{C}}^*$ restrict to functors

$$\Gamma_{\mathcal{C},!}, \ \Gamma_{\mathcal{C},*} \colon \mathrm{Fun}(\mathcal{C}, \mathcal{E}^\omega) \longrightarrow \mathcal{E}^\omega \ .$$

Proof. The functor $\Gamma_{\mathcal{C}}^*$ takes $\mathcal{E}^\omega$ in $\mathrm{Fun}(\mathcal{C}, \mathcal{E}^\omega)$ by definition. It is therefore enough to prove that $\Gamma_{\mathcal{C},*}$ commutes with filtered colimits and that in the stable case it restricts it takes $\mathrm{Fun}(\mathcal{C}, \mathcal{E}^\omega)$ to $\mathcal{E}^\omega$. Let $\mathcal{T}_0(\mathcal{E})$ be the full subcategory of Cat_∞ spanned by those ∞-categories $\mathcal{C}$ for which the functor $\Gamma_{\mathcal{C},*}$ commutes with filtered colimits. Let $\mathcal{T}(\mathcal{E})$ be the full subcategory of Cat_∞ spanned by those ∞-categories $\mathcal{C}$ for which the functors $\Gamma_{\mathcal{C},!}$ and $\Gamma_{\mathcal{C},*}$ take $\mathrm{Fun}(\mathcal{C}, \mathcal{E}^\omega)$ to $\mathcal{E}^\omega$. Inspection reveals that $\mathcal{T}(\mathcal{E})$ is closed under finite colimits, retractions and contain all finite 1-categories (that is, 1-categories having a finite number of objects and of morphisms). Since Cat_∞ is compactly generated by the 1-categories Δ^n, the conclusion follows. $\qquad\square$

Definition 6.40. We say that a stratified space (X, P) is *categorically compact* if $\mathrm{Exit}(X, P)$ is a compact object in Cat_∞.

Definition 6.41. Let (X, P) be a conically stratified space. We say that (X, P) is *locally categorically compact* if for every $p \in P$ and every $x \in X_p$, there exists a conical chart of the form $Z \times C(Y)$ containing x such that $(Y, P_{>p})$ is categorically compact and Z is weakly contractible and locally weakly contractible.

Proposition 6.42. *Let (X, P) be a conically stratified space with locally weakly contractible strata. Assume that (X, P) is locally categorically compact and that P is finite. Let $S \subseteq P$ be a locally closed subset and let $\mathcal{E}$ be a stable presentable ∞-category. Then the functor*

$$i_{S,*} \colon \mathrm{Cons}_S^{\mathrm{hyp}}(X_S; \mathcal{E}) \longrightarrow \mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E})$$

commutes with filtered colimits and it restricts to a functor

$$i_{S,*} \colon \mathrm{Cons}_{S,\omega}^{\mathrm{hyp}}(X_S; \mathcal{E}) \longrightarrow \mathrm{Cons}_{P,\omega}^{\mathrm{hyp}}(X; \mathcal{E}) \ .$$

Proof. When S is closed inside P, the functor $i_{S,*}$ coincides with the extension by zero and Proposition 6.42 is trivial in that case. It is then enough to consider the case where S is open inside P. We proceed by induction on $\mathrm{depth}(P)$. When $\mathrm{depth}(P) = 0$, the stratification is trivial and therefore we have either $S = \emptyset$ or $S = P$. In both cases, the statement is obvious.

Assume now that $\mathrm{depth}(X) > 0$. Since P is finite, the subset $M \subseteq P$ of its minimal elements is finite as well. Writing $M = \{p_1, \dots, p_n\}$, we see that $\{X_{\geqslant p_i}\}_{i=1,\dots,n}$ is an open cover of X. Since both the compacity of stalks and the formation of filtered colimits are local statements on X, we can assume that P has a minimum p. Using the inductive hypothesis, we further reduce to the case where $S = P \smallsetminus M$.

Let now $F \in \mathrm{Cons}_{S,\omega}^{\mathrm{hyp}}(X_S; \mathcal{E})$. We have to prove that the stalks of $i_{S,*}(F)$ belong to $\mathcal{E}^\omega$. Since $i_{S,*}$ is fully faithful, we only have to prove this statement for the stalks at a point $x \in X_p$. Choose a conical chart of the form $Z \times C(Y)$, where (Y, S) is a categorically compact stratified space and where Z is weakly contractible and locally weakly contractible. From Lemma 6.38, we have a canonical identification

$$(i_{S,*}F)_x \simeq \lim_{\mathrm{Exit}(Y,S)} \Phi_{X,P}^{\mathrm{hyp}}(F)|_{\mathrm{Exit}(Y,S)}$$

Since $\mathrm{Exit}(Y, S)$ is a compact object of Cat_∞ and since the functor $\Phi_{X,P}^{\mathrm{hyp}}(F)$ takes values in $\mathcal{E}^\omega$, Proposition 6.42 thus follows from Lemma 6.39. The same method also guarantees that $i_{S,*}$ commutes with filtered colimits. $\qquad\square$

Corollary 6.37 immediately gives the following

Corollary 6.43. *Let (X, P) be a conically stratified space and let $\mathcal{E}$ be a presentable ∞-category. Let $S \subseteq P$ be a closed subset and let $i_S \colon X_S \to X$ be the inclusion. Put $U := P \setminus S$ and let $i_U \colon X_U \to X$ be the inclusion. If the assumptions of Theorem 5.17 are satisfied, then the fully-faithful functors*

$$i_{S,*} \colon \mathrm{Cons}_{S,\omega}^{\mathrm{hyp}}(X_S; \mathcal{E}) \longrightarrow \mathrm{Cons}_{P,\omega}^{\mathrm{hyp}}(X; \mathcal{E}) \longleftarrow \mathrm{Cons}_{U,\omega}^{\mathrm{hyp}}(X_U; \mathcal{E}) \colon i_{U,*}$$

exhibits $\mathrm{Cons}_{P,\omega}^{\mathrm{hyp}}(X; \mathcal{E})$ as a recollement of $\mathrm{Cons}_{S,\omega}^{\mathrm{hyp}}(X_S; \mathcal{E})$ and $\mathrm{Cons}_{U,\omega}^{\mathrm{hyp}}(X_U; \mathcal{E})$.

Corollary 6.36 and Proposition 6.42 immediately gives the following

Corollary 6.44. *Let (X, P) be a conically stratified space with locally weakly contractible strata. Assume that (X, P) is locally categorically compact and that P is finite. Let $S \subseteq P$ be a locally closed subset and let $\mathcal{E}$ and $\mathcal{D}$ stable presentable ∞-categories. Let $f \colon \mathcal{E} \to \mathcal{D}$ be a cocontinuous functor preserving compact objects. Then, the diagram*

$$
\begin{array}{ccc}
\mathrm{Cons}_{P,\omega}^{\mathrm{hyp}}(X; \mathcal{E}) & \xrightarrow{\ i_S^{*,\mathrm{hyp}}\ } & \mathrm{Cons}_{S,\omega}^{\mathrm{hyp}}(X_S; \mathcal{E}) \\
\Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle f} \\
\mathrm{Cons}_{P,\omega}^{\mathrm{hyp}}(X; \mathcal{D}) & \xrightarrow{\ i_S^{*,\mathrm{hyp}}\ } & \mathrm{Cons}_{S,\omega}^{\mathrm{hyp}}(X_S; \mathcal{D})
\end{array}
$$

is horizontally right adjointable. Than is, the Beck-Chevalley transformation

$$f \circ i_{S,*}^{\mathrm{hyp}} \longrightarrow i_{S,*}^{\mathrm{hyp}} \circ f$$

is an equivalence.

6.9. **Constructibility, pushforward and weakly stratified bundles.** We now present a second result concerning proper pushforward. Lurie proved proper non abelian base change for sheaves on locally compact Hausdorff spaces. See [HTT, Corollary 7.3.1.18]. Note however that it is not clear that proper base change holds for hypersheaves. We are going to see that under some additional assumptions on the stratifications involved, proper base change holds for *hyperconstructible* hypersheaves.

Observation 6.45. Let $f \colon Y \to X$ be a proper morphism between topological spaces. Following [HTT, Definition 7.3.1.14], this means for us that f is universally closed. Let $C \subseteq X$ be a locally closed subset of X. Write $\mathcal{B}_C$ for the collection of open neighborhoods of C inside X. Then $\{f^{-1}(U)\}_{U \in \mathcal{B}_C}$ is a fundamental system of open neighborhoods for the inverse image $f^{-1}(C)$. Indeed, since f is universally closed we can localize on X and therefore assume that C is closed. Let now V be an open neighborhood of $f^{-1}(C)$ inside Y. Since f is closed, $f(Y \smallsetminus V)$ is a closed subset of X. Furthermore $f(Y \smallsetminus V) \cap C = \emptyset$, so $U \coloneqq X \smallsetminus f(Y \smallsetminus V)$ is an open neighborhood of C inside X. We now observe that if $y \in f^{-1}(U)$, then $f(y) \notin f(Y \smallsetminus V)$, which in turn implies that $y \in V$. Therefore, $f^{-1}(U) \subseteq V$.

Before proving the sought after proper base change for hyperconstructible hypersheaves, we need a strengthening of the notion of excellency introduced in Definition 2.18.

Definition 6.46. Let (X, P) be a conically stratified space and let $S \subseteq P$ be a subset. We say that (X, P) is *hereditary excellent at S* if for every open subset $U \subseteq X$, the stratified space (U, P) is excellent at S.

Proposition 6.47. *Let $f \colon (Y, Q) \to (X, P)$ be a morphism of conically stratified spaces with locally weakly contractible strata and let $S \subseteq P$ be a locally closed subset. Let $\varphi \colon Q \to P$ be the underlying morphism of posets and set $R \coloneqq \varphi^{-1}(S)$. Consider the induced commutative square*

$$
\begin{array}{ccc}
(Y_R, R) & \xrightarrow{\ j\ } & (Y, Q) \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\
(X_S, S) & \xrightarrow{\ i\ } & (X, P) \ .
\end{array}
$$

Assume that:

   (1) the underlying morphism $f \colon Y \to X$ is proper;

   (2) (Y, Q) is hereditary excellent at R.

Then for every presentable ∞-category $\mathcal{E}$ satisfying the assumptions of Theorem 5.17 and every $F \in \mathrm{Cons}_Q^{\mathrm{hyp}}(Y; \mathcal{E})$ the canonical map

$$i^{*,\mathrm{hyp}}(f_*(F)) \longrightarrow g_*(j^{*,\mathrm{hyp}}(F))$$

is an equivalence.

Proof. The statement is local on X. We can therefore suppose that S is a closed downwards subset of P. Since source and target are hypersheaves, it is enough to prove that for every open subset U of X_S the induced morphism

$$i^{-1}(f_*(F))(U) \longrightarrow g_*(j^{*,\mathrm{hyp}}(F))(U)$$

is an equivalence. Let $\mathcal{B}_U$ for the collection of open neighborhoods of U inside X. Since f is proper and U is locally closed inside X, Observation 6.45 shows that $\{f^{-1}(V)\}_{V \in \mathcal{B}_U}$ is a fundamental system of open neighborhoods of $g^{-1}(U) = f^{-1}(U)$ inside Y. Since (Y, Q) is hereditary excellent at R, the collection $\mathcal{B}^{\mathrm{exc}}_{g^{-1}(U)}$ of excellent open neighborhoods of $g^{-1}(U)$ inside Y is a fundamental system of open neighborhoods for $g^{-1}(U)$. Thus, Theorem 5.17 and Lemma 3.3 provide the following chain of natural equivalences:

$$i^{-1}(f_*(F))(U) \simeq \operatorname*{colim}_{V \in \mathcal{B}_U} F(f^{-1}(V)) \simeq \operatorname*{colim}_{W \in \mathcal{B}^{\mathrm{exc}}_{g^{-1}(U)}} F(W) \simeq \operatorname*{colim}_{W \in \mathcal{B}^{\mathrm{exc}}_{g^{-1}(U)}} \lim_{\Pi^{\Sigma}_{\infty}(W,Q)} \Phi^{\mathrm{hyp}}_{Y,Q}(F) \ .$$

Since each $W \in \mathcal{B}^{\mathrm{exc}}_{g^{-1}(U)}$ is excellent at R, the functor $\Pi^{\Sigma}_{\infty}(g^{-1}(U), R) \to \Pi^{\Sigma}_{\infty}(W, Q)$ is final. Therefore the colimit on the right is constant, and we deduce

$$i^{-1}(f_*(F))(U) \simeq \lim_{\Pi^{\Sigma}_{\infty}(g^{-1}(U),R)} \Phi^{\mathrm{hyp}}_{Y,Q}(F)|_{\Pi^{\Sigma}_{\infty}(Y_R,R)}$$

Since $j : Y_R \to Y$ is exodromic, we deduce

$$i^{-1}(f_*(F))(U) \simeq \lim_{\Pi^{\Sigma}_{\infty}(g^{-1}(U),R)} \Phi^{\mathrm{hyp}}_{Y_R,R}(j^{*,\mathrm{hyp}}(F)) \simeq j^{*,\mathrm{hyp}}(F)(g^{-1}(U)) \simeq g_*(j^{*,\mathrm{hyp}}(F))(U) \ .$$

The conclusion follows. $\square$

Remark 6.48. In the above proof, properness is only used in the form of Observation 6.45. It would therefore be enough to ask that for every $x \in X$ the collection $\{f^{-1}(V)\}_{V \in \mathcal{B}_x}$ form a fundamental system of open neighborhood for the fiber $f^{-1}(x)$ inside Y (where $\mathcal{B}_x$ denotes the collection of open neighborhoods of x inside X).

Definition 6.49. We say that a morphism between conically stratified spaces $f : (Y, Q) \to (X, P)$ is a *weak stratified bundle* if for every $p \in P$ every point $x \in X_p$ admits an open neighborhood U in X_p such that there exists a conically stratified space (W, R) and an isomorphism of stratified spaces

$$(f^{-1}(U), Q) \xrightarrow{\sim} (U \times W, R)$$

over U.

Proposition 6.50. *Let $f : (Y, Q) \to (X, P)$ be a morphism of conically stratified spaces with locally weakly contractible strata. Let $\mathcal{E}$ be a presentable ∞-category. Assume that:*

(1) the underlying morphism $f : Y \to X$ is proper;

(2) f is a weak stratified bundle whose fibres have locally weakly contractible strata;

(3) (Y, Q) is hereditary excellent at every locally closed subset of Q;

(4) the assumptions of Theorem 5.17 are satisfied.

Then $f_* : \mathrm{Sh}^{\mathrm{hyp}}(Y; \mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E})$ preserves hyperconstructible hypersheaves and for every $F \in \mathrm{Cons}^{\mathrm{hyp}}_P(Y; \mathcal{E})$ and every $x \in X$ there is a canonical equivalence

$$f_*(F)_x \simeq \Gamma(Y_x, j^{*,\mathrm{hyp}}_x(F)) \ ,$$

where $j_x : Y_x \hookrightarrow Y$ denotes the inclusion of the fiber.

Proof. Using Proposition 6.47, we immediately reduce to the case where P is trivial. Since both statements are local on the target and since f is a weak stratified bundle, we can assume that $(Y, Q) \simeq (X \times W, R)$ for some conically stratified space (W, R) and that f coincides with the canonical projection to X. Let $F \in \mathrm{Cons}^{\mathrm{hyp}}_P(Y; \mathcal{E})$. We first show that $f_*(F)$ is locally hyperconstant on X. To do this, [HPT, Proposition 3.1] ensures that it is enough to prove that for every inclusion $U \subseteq V$ of weakly contractible open subsets in X, the restriction map

$$f_*(F)(V) \longrightarrow f_*(F)(U)$$

is an equivalence. Since F is hyperconstructible, Corollary 3.21 and Theorem 5.17 allow to rewrite this map as

$$\lim_{\Pi_\infty^\Sigma(V\times W,R)} \Phi^{\mathrm{hyp}}_{X\times W,R}(F) \longrightarrow \lim_{\Pi_\infty^\Sigma(U\times W,R)} \Phi^{\mathrm{hyp}}_{X\times W,R}(F) \ .$$

Since Π_∞^Σ commutes with finite products, we have

$$\Pi_\infty^\Sigma(V\times W,R) \simeq \Pi_\infty(V) \times \Pi_\infty^\Sigma(W,R), \qquad \Pi_\infty^\Sigma(U\times W,R) \simeq \Pi_\infty(U) \times \Pi_\infty^\Sigma(W,R) \ .$$

Since the map $\Pi_\infty(U) \to \Pi_\infty(V)$ is an equivalence, the conclusion follows.

Let now $x \in X$. To compute $f_*(F)_x$, we can further suppose that X is weakly contractible. In that case, the local hyperconstancy of $f_*(F)$ combined with [HPT, Proposition 3.1] ensures that the canonical map

$$f_*(F)(X) \longrightarrow f_*(F)_x$$

is an equivalence. On the other hand, Corollary 3.21 combined with the fact that the morphism $Y_x = \{x\} \times W \to Y$ is exodromic gives a chain of equivalences

$$f_*(F)(X) \simeq \lim_{\Pi_\infty^\Sigma(X\times W,R)} \Phi^{\mathrm{hyp}}_{X\times W,R}(F) \simeq \lim_{\Pi_\infty^\Sigma(\{x\}\times W,R)} \Phi^{\mathrm{hyp}}_{X\times W,R}(F)$$

$$\simeq \lim_{\Pi_\infty^\Sigma(Y_x,R)} \Phi^{\mathrm{hyp}}_{Y_x,R}(j_x^{*,\mathrm{hyp}}(F)) \simeq \Gamma(Y_x, j_x^{*,\mathrm{hyp}}(F))$$

The proof of Proposition 6.50 is thus complete. $\qquad\square$

7. Moduli spaces

Let (X,P) be a conically stratified space with locally weakly contractible strata and let R be an excellent commutative ring. Consider the derived prestack

$$\mathbf{Cons}_P(X)\colon \mathrm{dAff}_R^{\mathrm{op}} \longrightarrow \mathcal{S}$$

defined by sending a derived affine scheme $\mathrm{Spec}(A)$ over R to the maximal ∞-groupoid contained in

$$\mathrm{Cons}^{\mathrm{hyp}}_{P,\omega}(X; \mathrm{Mod}_A) \simeq \mathrm{Fun}\left(\Pi_\infty^\Sigma(X,P), \mathrm{Perf}(A)\right) \ ,$$

where the equivalence is the one provided by Proposition 6.32. Observe that the right hand side commutes with limits in $\mathrm{Perf}(A)$. In particular we obtain:

Lemma 7.1. *The derived prestack $\mathbf{Cons}_P(X)$ satisfies faithfully flat hyperdescent. In particular, it is a derived stack.*

Proof. Combine [SAG, Corollary D.6.3.3 and Proposition 2.8.4.2-(10)]. $\qquad\square$

Our goal is to prove that when (X,P) satisfies certain mild finiteness conditions, $\mathbf{Cons}_P(X)$ is locally geometric.

7.1. Finiteness conditions on stratified homotopy types.

Definition 7.2. For stratified space (X,P), we say that :

    (1) (X,P) is *categorically compact* if $\mathrm{Exit}(X,P)$ is a compact object in Cat_∞ ;

If furthermore (X,P) is conically stratified, we say that:

    (2) (X,P) is *locally categorically compact* if for every $p \in P$ and every $x \in X_p$, there exists a conical chart of the form $Z \times C(Y)$ containing x such that $(Y,P_{>p})$ is categorically compact and Z is weakly contractible and locally weakly contractible ;

    (3) (X,P) is *of finite stratified type* if the poset P is finite and for every $p \in P$, the homotopy type $\Pi_\infty(X_p)$ of the associated stratum is a compact object in $\mathcal{S}$.

When the stratification is trivial, there is an abundance of examples:

Example 7.3.

(1) Assume that X is Stein complex (smooth) manifold, equipped with a trivial stratification. Then classical results of Remmert [R], Bishop [B] and Narasimhan [N] show that X can be realized as a closed subvariety of $\mathbb{C}^n$ for $n \gg 0$. In particular, Andreotti-Frankel's theorem [AF] shows that X has the homotopy type of a finite CW complex, and in particular $\Pi_\infty(X)$ is compact.

(2) Assume now that X is a smooth algebraic variety. If X is affine, then its analytification X^{an} is smooth and Stein and therefore $\Pi_\infty(X^{\mathrm{an}})$ is compact by the previous point. In general, X admits a finite cover by affine smooth open subvarieties, whose intersections are again smooth and affine. This immediately implies that $\Pi_\infty(X^{\mathrm{an}})$ can be realized as a finite colimit of compact objects in $\mathcal{S}$, and henceforth that $\Pi_\infty(X^{\mathrm{an}})$ is compact itself.

(3) Assume that X is a *compact* topological manifold. Then the work of Kirby and Siebenmann [KS, Theorem III] implies that X has the homotopy type of a *finite* CW complex (although X might not be triangulable itself). In particular, $\Pi_\infty(X)$ is a compact object in $\mathcal{S}$.

Another big class of examples comes from the theory of conically smooth stratified spaces.

Recollection 7.4. We refer the reader to [AFTb, § 3.2] for the notion of conically smooth stratified space. Although their definition is long and involved, it is an extremely convenient setting to work in, as it enjoys the following properties:

(1) every stratum X_p is a smooth manifold;

(2) for every $p \in P$, every point $x \in X_p$ has a fundamental system of open neighborhoods of the form $U \times C(Z)$, where U is a weakly contractible (and in fact homeomorphic to $\mathbb{R}^i$ for some $i \geqslant 0$) open neighborhood of x in X_p and Z is a *compact* conically smooth stratified space over $P_{>p}$;

(3) if the stratification on (X, P) is Whitney, then it is conically smooth, as recently shown by Nocera and Volpe [NV, Theorem 2.7].

In other words, properties (1) and (2) enable inductive arguments on the depth of the stratification, while property (3) guarantees that every reasonable example considered in practice falls in this class.

Also, in [AFTb, Definition 8.3.6] the authors introduced the notion of *finitary* conically smooth stratified space, that intuitively corresponds to those conically smooth stratified spaces that admit a finite handlebody decomposition, such as the one provided by Morse theory. Observe that [AFTb, Theorem 8.3.10-(2)] implies that every conically smooth stratified space (X, P) whose underlying topological space is compact is automatically finitary.

Having recalled these facts concerning conically smooth stratified spaces, we can state the following recent theorem of Volpe:

Theorem 7.5 (Volpe, [V, Proposition 2.12 & Corollary 2.13])**.** *Let (X, P) be a finitary conically smooth stratified space. Then $\Pi_\infty^\Sigma(X, P)$ is categorically equivalent to a simplicial set having only a finite number of non-degenerate simplexes. Hence, (X, P) is categorically compact.*

Corollary 7.6. *Let (X, P) be a conically smooth stratified space. Then (X, P) is locally categorically compact.*

Proof. Since (X, P) is a finitary conically smooth stratified space, for every $p \in P$ and every $x \in X_p$ there exists an open neighborhood of x of the form $U \times C(Z)$, where U is weakly contractible locally weakly contractible and $(Z, P_{>p})$ is a *compact* conically smooth stratified space. In

particular, [AFTb, Theorem 8.3.10-(2)] implies that $(Z, P_{>p})$ is finitary. Then Theorem 7.5 implies that $(Z, P_{>p})$ is categorically compact. $\qquad\square$

When (X, P) is categorically compact, it is particularly easy to prove that $\mathbf{Cons}_P(X)$ is a locally geometric derived stack:

Proposition 7.7. *Let (X, P) be a categorically compact conically stratified space. Then the derived stack $\mathbf{Cons}_P(X)$ is locally geometric, and the tangent complex at a point $x \colon \mathrm{Spec}(A) \to \mathbf{Cons}_P(X)$ classifying a constructible sheaf $F \in \mathrm{Cons}^{\mathrm{hyp}}_{P,\omega}(X; \mathrm{Mod}_A)$ is given by*

$$x^* \mathbb{T}_{\mathbf{Cons}_P(X)} \simeq \mathrm{Hom}_{\mathrm{Cons}^{\mathrm{hyp}}_P(X;\mathrm{Mod}_A)}(F, F)[1] \ ,$$

where the right hand side denotes the Mod_A-enriched Hom of $\mathbf{Cons}_P(X; \mathrm{Mod}_A)$.

Proof. By assumption, $\Pi^{\Sigma}_{\infty}(X, P)$ is a compact object in Cat_{∞}. Since $(-)^{\mathrm{op}} \colon \mathrm{Cat}_{\infty} \to \mathrm{Cat}_{\infty}$ is a self-equivalence, we deduce that $\Pi^{\Sigma}_{\infty}(X, P)^{\mathrm{op}}$ is compact as well. Moreover, every self map in $\Pi^{\Sigma}_{\infty}(X, P)^{\mathrm{op}}$ is automatically an equivalence, which in turn implies that $\Pi^{\Sigma}_{\infty}(X, P)$ is idempotent complete. In particular,

$$\Pi^{\Sigma}_{\infty}(X, P)^{\mathrm{op}} \simeq \mathrm{PSh}\big(\Pi^{\Sigma}_{\infty}(X, P)^{\mathrm{op}}\big)^{\omega} \simeq \mathrm{Fun}\big(\Pi^{\Sigma}_{\infty}(X, P), \mathcal{S}\big)^{\omega} \ .$$

Now, recall from [HTT, Proposition 5.5.7.11 & Corollary 4.4.5.21] that $\mathrm{Cat}^{\mathrm{rex,idem}}_{\infty}$ is closed under filtered colimits in Cat_{∞}. It follows that $\Pi^{\Sigma}_{\infty}(X, P)^{\mathrm{op}}$ is compact in $\mathrm{Cat}^{\mathrm{rex,idem}}_{\infty} \simeq \mathcal{P}\mathrm{r}^{\mathrm{L},\omega}$. Thus,

$$\mathcal{C} \coloneqq \mathrm{PSh}\big(\Pi^{\Sigma}_{\infty}(X, P)^{\mathrm{op}}\big) \otimes \mathrm{Mod}_R \simeq \mathrm{Fun}\big(\Pi^{\Sigma}_{\infty}(X, P)^{\mathrm{op}}, \mathrm{Mod}_R\big) \ .$$

is automatically a compact object in $\mathcal{P}\mathrm{r}^{\mathrm{L},\omega}_R$, and hence it is smooth and proper in the sense of [SAG, §11]. In particular, its moduli of objects $\mathcal{M}_{\mathcal{C}}$ in the sense of [TV] is well defined. Unraveling the definitions, we see that the functor of points of $\mathcal{M}_{\mathcal{C}}$ sends $A \in \mathrm{CAlg}_R$ to

$$
\begin{aligned}
\mathrm{Fun}_R\big((\mathcal{C}^{\omega})^{\mathrm{op}}, \mathrm{Perf}(A)\big) &\simeq \mathrm{Fun}_R\big((\Pi^{\Sigma}_{\infty}(X, P)^{\mathrm{op}} \otimes \mathrm{Perf}(R))^{\mathrm{op}}, \mathrm{Perf}(A)\big) \\
&\simeq \mathrm{Fun}_R\big(\Pi^{\Sigma}_{\infty}(X, P)^{\mathrm{op}} \otimes \mathrm{Perf}(R), \mathrm{Perf}(A)^{\mathrm{op}}\big)^{\mathrm{op}} \\
&\simeq \mathrm{Fun}\big(\Pi^{\Sigma}_{\infty}(X, P)^{\mathrm{op}}, \mathrm{Perf}(A)^{\mathrm{op}}\big)^{\mathrm{op}} \\
&\simeq \mathrm{Fun}\big(\Pi^{\Sigma}_{\infty}(X, P), \mathrm{Perf}(A)\big) \\
&\simeq \mathrm{Cons}^{\mathrm{hyp}}_{P,\omega}(X; \mathrm{Perf}(A)) \ .
\end{aligned}
$$

Therefore, the conclusion follows from [TV, Theorem 3.6 & Corollary 3.17]. $\qquad\square$

We can now state the main theorem of this section:

Theorem 7.8. *Let (X, P) be a conically stratified space with locally weakly contractible strata. Assume that:*

(1) *(X, P) is of finite stratified type;*

(2) *(X, P) is locally categorically compact.*

Then $\mathbf{Cons}_P(X)$ is a locally geometric derived stack, locally of finite presentation.

Remark 7.9.

(1) We do not know whether the assumptions of Theorem 7.8 imply that (X, P) is categorically compact. By contrast with Proposition 7.7, we are going to apply Lurie's representability theorem to prove Theorem 7.8.

(2) Theorem 7.8 covers *a priori* a larger class of stratified spaces (X, P) for which $\mathbf{Cons}_P(X)$ is a locally geometric derived stack than the one provided combining Theorem 7.5 and Proposition 7.7. Indeed, we do not know whether every conically smooth stratified space (X, P) which is also of finite stratified type is automatically finitary. Nevertheless, we formulate the following conjecture, that would extend Volpe's theorem:

Conjecture 7.10. *Let (X, P) be a conically smooth stratified space. If (X, P) is of finite stratified type, then (X, P) is also categorically compact.*

We do not know how to prove this conjecture at the moment. However we obtain a very close result below (see Lemma 7.11).

7.2. **Local finite presentation.** Recall that if $\{A_\alpha\}_{\alpha \in I}$ is a filtered diagram in CAlg_R with colimit A, the canonical map

$$\operatorname*{colim}_{\alpha \in I} \mathrm{Perf}(A_\alpha) \longrightarrow \mathrm{Perf}(A)$$

is an equivalence in Cat_∞ as shown in [SAG, Corollary 4.5.1.8]. Therefore, when $\Pi_\infty^\Sigma(X, P)$ is compact in Cat_∞, the functor $\mathbf{Cons}_P(X)$ is automatically locally of finite type. More generally, we have the following

Lemma 7.11. *Let (X, P) be a conically stratified space satisfying the assumptions of Theorem 7.8. Then the functor*

$$\mathrm{Fun}\left(\Pi_\infty^\Sigma(X, P), (-)^\omega\right) \colon \mathcal{P}\mathrm{r}_{\mathrm{st}}^{\mathrm{L}, \omega} \longrightarrow \mathrm{Cat}_\infty$$

commutes with filtered colimits. Hence $\mathbf{Cons}_P(X)$ is locally of finite presentation.

Proof. We proceed by induction on the cardinality n of P. When $n = 1$, the stratification is trivial and therefore $\Pi_\infty^\Sigma(X, P) = \Pi_\infty(X)$, which is compact in Cat_∞ by assumption. Hence, this case follows. If $n > 1$, take $p \in P$ a minimal element and consider $S := P \smallsetminus \{p\}$. Then both $P_{\leqslant p}$ and S have cardinality strictly less than n. Consider a filtered diagram $I \to \mathcal{P}\mathrm{r}_{\mathrm{st}}^{\mathrm{L}, \omega}$, noted $\alpha \mapsto \mathcal{E}_\alpha$ and let $\mathcal{E}$ be its colimit computed in $\mathcal{P}\mathrm{r}_{\mathrm{st}}^{\mathrm{L}, \omega}$. Consider the following diagram:

$$\begin{array}{ccccc}
\operatorname*{colim}_\alpha \mathrm{Cons}_{S,\omega}^{\mathrm{hyp}}(X_S; \mathcal{E}_\alpha) & \xrightarrow{i_{S,*}} & \operatorname*{colim}_\alpha \mathrm{Cons}_{P,\omega}^{\mathrm{hyp}}(X; \mathcal{E}_\alpha) & \xleftarrow{i_{p,*}} & \operatorname*{colim}_\alpha \mathrm{Cons}_{p,\omega}^{\mathrm{hyp}}(X_p; \mathcal{E}_\alpha) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Cons}_{S,\omega}^{\mathrm{hyp}}(X_S; \mathcal{E}) & \xrightarrow{i_{S,*}} & \mathrm{Cons}_{P,\omega}^{\mathrm{hyp}}(X; \mathcal{E}) & \xleftarrow{i_{p,*}} & \mathrm{Cons}_{p,\omega}^{\mathrm{hyp}}(X_p; \mathcal{E}) \ .
\end{array}$$

From Corollary 6.44, each square is commutative. The external vertical arrows are equivalences by inductive hypothesis. The horizontal functors exhibit the middle terms as recollements of the external ones as a result of Corollary 6.43. Therefore, the middle vertical arrow is an equivalence as a result of [HA, Proposition A.8.14]. We then conclude using the fact that the interior groupoïd functor commutes with filtered colimits. $\square$

7.3. **Cohesiveness, nilcompleteness and integrability.** For a review of definitions, let us refer to [SAG, 17.3.7]

Proposition 7.12. *Let (X, P) be a conically stratified space with locally weakly contractible strata. Then the derived stack $\mathbf{Cons}_P(X)$ is infinitesimally cohesive, nilcomplete and satisfies integrability.*

Proof. These statements follow from the analogous ones for Perf, the description of the category $\mathrm{Cons}_{P,\omega}^{\mathrm{hyp}}(X; \mathrm{Mod}_A)$ as $\mathrm{Fun}\left(\Pi_\infty^\Sigma(X, P), \mathrm{Perf}(A)\right)$ and the commutation of the interior groupoid functor with limits. Specifically, infinitesimally cohesiveness follows from [SAG, Theorem 16.2.0.1 and Proposition 16.2.3.1]. Nilcompleteness follows from [SAG, Propositions 2.7.3.2 and 19.2.1.5]. Integrability follows from [SAG, Remark 8.5.0.5 and Theorem 8.5.0.3]. $\square$

7.4. **Cotangent complex.** The next important step to verify the assumptions of Artin-Lurie's representability theorem is to check the existence of the cotangent complex for $\mathbf{Cons}_P(X)$.

Notation 7.13. In proving the existence of the cotangent complex of $\mathbf{Cons}_P(X)$, we will crucially use the fact that for every $A \in \mathrm{CAlg}$, the ∞-category $\mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathrm{Mod}_A)$ is canonically tensor-enriched over Mod_A. We will denote by

$$\mathrm{Hom}_{\mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathrm{Mod}_A)}(-,-)\colon \mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathrm{Mod}_A)^{\mathrm{op}} \times \mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathrm{Mod}_A) \longrightarrow \mathrm{Mod}_A$$

the enriched Hom. Since $\otimes\colon \mathrm{Mod}_A \times \mathrm{Mod}_A \to \mathrm{Mod}_A$ is cocontinuous in each variable, Corollary 3.9 ensures the existence of a tensor product

$$(-) \otimes_A (-)\colon \mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathrm{Mod}_A) \otimes \mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathrm{Mod}_A) \longrightarrow \mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathrm{Mod}_A)$$

defined by applying the tensor product objectwise. If $F \in \mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathrm{Mod}_A)$ and $M \in \mathrm{Mod}_A$, we write $M \otimes F$ for $\Gamma^{*,\mathrm{hyp}}(M) \otimes F$.

Remark 7.14. Let (X, P) be a conically stratified space with locally weakly contractible strata. Let $f\colon A \to B$ be a morphism in CAlg. From Corollary 6.16, the adjunction

$$f^*\colon \mathrm{Mod}_A \leftrightarrows \mathrm{Mod}_B\colon f_*$$

induces an adjunction

$$f^* \circ -\colon \mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathrm{Mod}_A) \leftrightarrows \mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathrm{Mod}_B)\colon f_* \circ -$$

To simplifiy notations, for $F \in \mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathrm{Mod}_A)$ we write $f^*(F)$ instead of $f^* \circ F$ and similarly with f_*. From Corollary 6.16 again, the above adjunction can be described as the adjunction

$$f^* \circ -\colon \mathrm{Fun}\left(\Pi_\infty^\Sigma(X,P),\mathrm{Mod}_A\right) \leftrightarrows \mathrm{Fun}\left(\Pi_\infty^\Sigma(X,P),\mathrm{Mod}_B\right)\colon f_* \circ - \ .$$

In particular, for every $F \in \mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathrm{Mod}_A)$ and every $N \in \mathrm{Mod}_B$, the canonical map

$$f_*(N) \otimes_A F \longrightarrow f_*(N \otimes_B f^*(F))$$

is an equivalence.

Proposition 7.15. *Let (X, P) be a conically stratified space with locally weakly contractible strata. Assume that P is finite and that the strata of (X, P) have finitely many connected components. Then the derived stack $\mathbf{Cons}_P(X)$ admits a global cotangent complex.*

Proof. Let $\mathrm{Spec}(A)$ be a derived affine scheme over R. Let

$$x\colon \mathrm{Spec}(A) \longrightarrow \mathbf{Cons}_P(X)$$

be a morphism and let $F \in \mathrm{Cons}_{P,\omega}^{\mathrm{hyp}}(X;\mathrm{Mod}_A) \simeq \mathrm{Fun}\left(\Pi_\infty^\Sigma(X,P),\mathrm{Perf}(A)\right)$ be the corresponding A-linear family of hyperconstructible hypersheaves. Let

$$\Omega_x \mathbf{Cons}_P(X) := \mathrm{Spec}(A) \times_{\mathbf{Cons}_P(X)} \mathrm{Spec}(A)$$

be the loop stack at x. We view $\Omega_x \mathbf{Cons}_P(X)$ as a stack over $A \times A$. Consider the diagonal morphism

$$\delta_x\colon \mathrm{Spec}(A) \longrightarrow \Omega_x \mathbf{Cons}_P(X)$$

If $d_0 \in \mathrm{Der}_{\mathbf{Cons}_P(X)/R}(A; -)$ denotes the 0-derivation, there are canonical equivalences of functors over Mod_A :

$$\mathrm{Der}_{\mathbf{Cons}_P(X)/R}(A; -[-1]) \simeq \Omega_{d_0} \mathrm{Der}_{\mathbf{Cons}_P(X)/R}(A; -) \simeq \mathrm{Der}_{\Omega_x \mathbf{Cons}_P(X)/A \times A}(A; -)$$

In particular, $\mathbf{Cons}_P(X)$ admits a cotangent complex $x^* \mathbb{L}_{\mathbf{Cons}_P(X)/R}$ at x if and only if the morphism $\Omega_x \mathbf{Cons}_P(X) \to \mathrm{Spec}\, A \times \mathrm{Spec}\, A$ admits a cotangent complex $\delta_x^* \mathbb{L}_{\Omega_x \mathbf{Cons}_P(X)/A \times A}$ at δ_x. In this case, one has the relation

$$x^* \mathbb{L}_{\mathbf{Cons}_P(X)/R} \simeq \delta_x^* \mathbb{L}_{\Omega_x \mathbf{Cons}_P(X)/A \times A}[-1] \ .$$

We are thus left to check that the functor

$$\mathrm{Der}_{\Omega_x \mathbf{Cons}_P(X)/A \times A}(A, -) \colon \mathrm{Mod}_A \longrightarrow \mathcal{S}$$

is corepresentable. Unravelling the definitions, we find for every $M \in \mathrm{Mod}_A$ a natural equivalence

$$\mathrm{Der}_{\Omega_x \mathbf{Cons}_P(X)/A \times A}(A, M) \simeq \mathrm{Map}_{\mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathrm{Mod}_A)} \left(F, M \otimes_A F \right) .$$

Using Theorem 5.17, Corollary 6.10 and [GHN, Proposition 5.1], we can rewrite the above mapping space as

$$\int_{x \to y \in \Pi_\infty^\Sigma(X,P)} \mathrm{Map}_{\mathrm{Mod}_A}(F_x, M \otimes_A F_y) ,$$

where the integral denotes the limit over the functor

$$\mathrm{Tw}(\Pi_\infty^\Sigma(X,P)) \to \Pi_\infty^\Sigma(X,P)^{\mathrm{op}} \times \Pi_\infty^\Sigma(X,P) \to \mathcal{S}$$

defined as

$$x \to y \in \Pi_\infty^\Sigma(X,P) \longrightarrow \mathrm{Map}_{\mathrm{Mod}_A}(F_x, M \otimes_A F_y)$$

Since the stalks of F are dualizable, we can further simplify the above expression as

$$\int_{x \to y \in \Pi_\infty^\Sigma(X,P)} \mathrm{Map}_{\mathrm{Mod}_A}(F_x, M \otimes_A F_y) \simeq \int_{x \to y \in \Pi_\infty^\Sigma(X,P)} \mathrm{Map}_{\mathrm{Mod}_A}(F_x \otimes F_y^\vee, M)$$

$$\simeq \mathrm{Map}_{\mathrm{Mod}_A} \left(\int^{x \to y \in \Pi_\infty^\Sigma(X,P)} F_x \otimes F_y^\vee, M \right) .$$

Hence, $\mathrm{Der}_{\Omega_x \mathbf{Cons}_P(X)/A \times A}(A, -)$ is corepresentable and we have

$$\delta_x^* \mathbb{L}_{\Omega_x \mathbf{Cons}_P(X)/A \times A} \simeq \int^{x \to y \in \Pi_\infty^\Sigma(X,P)} F_x \otimes F_y^\vee .$$

We have to check that the above A-module is eventually connective. For every $x, y \in \Pi_\infty^\Sigma(X,P)$, the A-module $F_x \otimes F_y^\vee$ is perfect. In particular, there exists an integer $n(x,y)$ such that

$$F_x \otimes F_y^\vee \in (\mathrm{Mod}_A)_{\geqslant n(x,y)} .$$

On the other hand, P is finite and each stratum of (X, P) has finitely many connected components. Since F is hyperconstructible on (X, P), we deduce that

$$N := \max_{x,y \in \Pi_\infty^\Sigma(X,P)} n(x,y)$$

is a well defined integer. It follows that the functor

$$\mathrm{Tw}(\Pi_\infty^\Sigma(X,P))^{\mathrm{op}} \longrightarrow \mathrm{Mod}_A$$

given by $(x \to y) \mapsto F_x \otimes F_y^\vee$ takes values in $(\mathrm{Mod}_A)_{\geqslant N}$. Since $(\mathrm{Mod}_A)_{\geqslant N}$ is closed under arbitrary colimits inside Mod_A, we finally deduce that

$$\delta_x^* \mathbb{L}_{\Omega_x \mathbf{Cons}_P(X)/A \times A} \in (\mathrm{Mod}_A)_{\geqslant N} .$$

We are left to check the cotangent complex is global. In the above situation, let $f \colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be morphism and let $y := x \circ f$ for the induced morphism classifying the B-linear family

of hyperconstructible hypersheaves $f^*(F)$. Then for every $N \in \mathrm{Mod}_B$, we have:

$$\mathrm{Hom}_{\mathrm{Mod}_B}(\delta_y^* \mathbb{L}_{\Omega_x \mathbf{Cons}_P(X)/A \times A}, N) \simeq \mathrm{Hom}_{\mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathrm{Mod}_B)}(f^*(F), N \otimes f^*(F))$$
$$\simeq \mathrm{Hom}_{\mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathrm{Mod}_A)}(F, f_*(N \otimes f^*(F)))$$
$$\simeq \mathrm{Hom}_{\mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathrm{Mod}_A)}(F, f_*(N) \otimes F) \qquad \text{by Remark 7.14}$$
$$\simeq \mathrm{Hom}_{\mathrm{Mod}_A}(\delta_x^* \mathbb{L}_{\Omega_x \mathbf{Cons}_P(X)/A \times A}, f_*(N))$$
$$\simeq \mathrm{Hom}_{\mathrm{Mod}_B}(f^* \delta_x^* \mathbb{L}_{\Omega_x \mathbf{Cons}_P(X)/A \times A}, N).$$

Thus, the conclusion follows from the Yoneda lemma. $\qquad \square$

7.5. **Representability results.** Before giving the proof of Theorem 7.8, let us fix the following couple of notation:
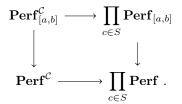
Notation 7.16.

(1) For integers $a \leqslant b$, we let $\mathbf{Perf}_{[a,b]}$ be the substack of $\mathbf{Perf}$ parametrizing perfect complexes of tor-amplitude contained in $[a,b]$. Notice that [SAG, Proposition 6.1.4.5] implies that this is an open substack of $\mathbf{Perf}$.

(2) Given an ∞-category $\mathcal{C}$, we write $\mathbf{Perf}^{\mathcal{C}}$ (resp. $\mathbf{Perf}^{\mathcal{C}}_{[a,b]}$) for the derived stack sending $S \in$ dAff to the maximal ∞-groupoid contained in $\mathrm{Fun}(\mathcal{C}, \mathrm{Perf}(S))$ (resp. $\mathrm{Fun}(\mathcal{C}, \mathrm{Perf}(S)_{[a,b]})$).

Lemma 7.17. *Let $\mathcal{C}$ be an ∞-category with only a finite number of equivalence classes of objects. Let $a \leqslant b$ be integers. Then the induced morphism of derived stacks*

$$\mathbf{Perf}^{\mathcal{C}}_{[a,b]} \longrightarrow \mathbf{Perf}^{\mathcal{C}}$$

is representable by an open immersion.

Proof. Choose a finite set S of objects of $\mathcal{C}$ representing all equivalence classes of objects in $\mathcal{C}$. Then the diagram of derived stacks

$$
\begin{array}{ccc}
\mathbf{Perf}^{\mathcal{C}}_{[a,b]} & \longrightarrow & \displaystyle\prod_{c \in S} \mathbf{Perf}_{[a,b]} \\
\downarrow & & \downarrow \\
\mathbf{Perf}^{\mathcal{C}} & \longrightarrow & \displaystyle\prod_{c \in S} \mathbf{Perf} \;.
\end{array}
$$

is cartesian. Since S is finite, the right vertical arrow is an open immersion, as already observed in Notation 7.16-(1). Hence, so is the left vertical arrow, whence the conclusion. $\qquad \square$

Remark 7.18. If $\mathcal{C}$ is the homotopy type of a topological space X in Lemma 7.17, we will note $\mathbf{LC}^{\mathrm{hyp}}_{[a,b]}(X)$ instead of $\mathbf{Perf}^{\Pi_\infty(X)}_{[a,b]}$.

We are now ready to prove the main result of this section:

Proof of Theorem 7.8. Let $a \leqslant b$ be integers and define $\mathbf{Cons}_P(X)_{[a,b]}$ as $\mathbf{Perf}^{\Pi_\infty^\Sigma(X,P)}_{[a,b]}$. Lemma 7.17 shows that the canonical map

$$\mathbf{Cons}_P(X)_{[a,b]} \to \mathbf{Cons}_P(X)$$

is representable by an open immersion. We now apply Artin-Lurie's representability theorem to each $\mathbf{Cons}_P(X)_{[a,b]}$:

(1) *Truncatedness*: if A is discrete, the mapping spaces in $\operatorname{Perf}(A)_{[a,b]}$ are $(b-a)$-truncated, and therefore $\operatorname{Perf}(A)_{[a,b]}$ is a $(b-a+1)$-truncated ∞-category. Thus, the same goes for $\operatorname{Fun}\left(\Pi_\infty^\Sigma(X,P),\operatorname{Perf}(A)_{[a,b]}\right)$, whence the conclusion.

(2) *Locally of finite presentation*: this is the content of Lemma 7.11.

(3) *Integrability, Nilcompleteness* and *infinitesimal cohesiveness*: this has been verified in Proposition 7.12.

(4) *Existence of cotangent complex*: this follows from Proposition 7.15.

$\square$

Corollary 7.19. *Under the assumptions of Theorem 7.8, let* $x\colon \operatorname{Spec}(A) \to \mathbf{Cons}_P(X)$ *be a morphism and let* $F \in \operatorname{Fun}(\Pi_\infty^\Sigma(X,P),\operatorname{Perf}(A))$ *be the corresponding A-linear family of hyperconstructible hypersheaves. Then*

$$x^*\mathbb{T}_{\mathbf{Cons}_P(X)} \simeq \operatorname{Hom}_{\operatorname{Cons}_P^{\mathrm{hyp}}(X;\operatorname{Mod}_A)}(F,F)[1] \ .$$

Proof. Since $\mathbf{Cons}_P(X)$ is a derived geometric stack of finite presentation, [SAG, Proposition 17.4.2.3-(1)] shows that its cotangent complex at x is perfect. Therefore,

$$x^*\mathbb{T}_{\mathbf{Cons}_P(X)} \simeq \left(x^*\mathbb{L}_{\mathbf{Cons}_P(X)}\right)^\vee \ .$$

Using the formula obtained in Proposition 7.15, we now obtain:

$$x^*\mathbb{T}_{\mathbf{Cons}_P(X)} \simeq \operatorname{Hom}_{\operatorname{Mod}_A}\left(\int^{x\to y\in\Pi_\infty^\Sigma(X,P)} F_x \otimes F_y^\vee[-1], A\right)$$

$$\simeq \int_{x\to y\in\Pi_\infty^\Sigma(X,P)} F_x^\vee \otimes F_y[1]$$

$$\simeq \int_{x\to y\in\Pi_\infty^\Sigma(X,P)} \operatorname{Hom}_{\operatorname{Mod}_A}(F_x,F_y)[1]$$

$$\simeq \operatorname{Hom}_{\operatorname{Cons}_P^{\mathrm{hyp}}(X;\operatorname{Mod}_A)}(F,F)[1] \ ,$$

where the last equivalence is due to [GHN, Proposition 5.1]. $\square$

7.6. **Perverse sheaves.** As corollary of Theorem 7.8, we can obtain a general representability result for the stack of perverse sheaves. To state the result, let us begin by fixing some notation.

Notation 7.20. Let (X,P) be a conically stratified space with locally weakly contractible strata. Let $\mathcal{E}$ be a presentable ∞-category. For every $p \in P$, we can factor the inclusion $j_p\colon X_p \to X$ as

$$X_p \xrightarrow{i_p} X_{\geqslant p} \xrightarrow{j_{\geqslant p}} X \ ,$$

where i_p is a closed immersion and $j_{\geqslant p}$ is an open immersion. Then

$$i_{p,*}\colon \operatorname{Sh}^{\mathrm{hyp}}(X_p;\mathcal{E}) \longrightarrow \operatorname{Sh}^{\mathrm{hyp}}(X_{\geqslant p};\mathcal{E})$$

commutes with colimits so in particular admits a right adjoint $i_p^{!,\mathrm{hyp}}$. We set

$$j_p^{!,\mathrm{hyp}} := i_p^{!,\mathrm{hyp}} \circ j_{\geqslant p}^{*,\mathrm{hyp}} \colon \operatorname{Sh}^{\mathrm{hyp}}(X;\mathcal{E}) \longrightarrow \operatorname{Sh}^{\mathrm{hyp}}(X_p;\mathcal{E}) \ .$$

Lemma 7.21. *Let* (X,P) *be conically stratified space with locally weakly contractible strata. Let* $\mathcal{E}$ *be a presentable stable ∞-category. For every $p \in P$, the functor $j_p^{!,\mathrm{hyp}}$ restricts to a functor*

$$j_p^{!,\mathrm{hyp}}\colon \operatorname{Cons}_P^{\mathrm{hyp}}(X;\mathcal{E}) \longrightarrow \operatorname{LC}^{\mathrm{hyp}}(X_p;\mathcal{E}) \ .$$

If furthermore (X,P) is locally categorically compact and P is finite, then $j_p^{!,\mathrm{hyp}}$ is cocontinuous.

Proof. We can assume without loss of generality that $X = X_{\geqslant p}$. In this case, we simply write $j\colon X_p \hookrightarrow X$ for the natural inclusion, and $i\colon X_{>p} \hookrightarrow X$ for the inclusion of the open complementary. For every $F \in \mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E})$, we have the following fiber sequence:

$$(7.22) \qquad\qquad j_* j^{!,\mathrm{hyp}}(F) \longrightarrow F \longrightarrow i_* i^{*,\mathrm{hyp}}(F) ,$$

computed in $\mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E})$. Assume now that F is P-hyperconstructible. Then the same goes for $i^{*,\mathrm{hyp}}(F)$, and Proposition 6.35 implies that $i_* i^{*,\mathrm{hyp}}(F)$ is hyperconstructible on (X, P) as well. Thus, Corollary 5.20 implies that $j_* j^{!,\mathrm{hyp}}(F)$ belongs to $\mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E})$. In particular, $j^{!,\mathrm{hyp}}(F) \simeq j^{*,\mathrm{hyp}} j_* j^{!,\mathrm{hyp}}(F)$ belongs to $\mathrm{LC}^{\mathrm{hyp}}(X_p; \mathcal{E})$.

We are left to show that $j_p^{!,\mathrm{hyp}}$ commutes with filtered colimits of P-hyperconstructible sheaves when (X, P) is locally categorically compact. Under this assumption, Proposition 6.42 guarantees that the functor

$$i_*\colon \mathrm{Cons}_{P_{>p}}^{\mathrm{hyp}}(X_{>p}; \mathcal{E}) \longrightarrow \mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E})$$

commutes with filtered colimits. Since j_* commutes with colimits and is fully faithful, the conclusion follows from observing that the right and the middle terms of (7.22) commute with filtered colimits. □

Changing coefficients commutes with the exceptional inverse image:

Lemma 7.23. *Let (X, P) be a conically stratified space with locally weakly contractible strata. Let $f\colon \mathcal{E} \to \mathcal{E}'$ be an exact functor between presentable ∞-categories. Assume that the conditions of Theorem 5.17 are satisfied. Then for every $p \in P$, the commutative diagram*

$$\begin{array}{ccc}
\mathrm{LC}^{\mathrm{hyp}}(X_p; \mathcal{E}) & \xrightarrow{\ f\ } & \mathrm{LC}^{\mathrm{hyp}}(X_p; \mathcal{E}') \\
\downarrow{\scriptstyle j_{p,*}} & & \downarrow{\scriptstyle j_{p,*}} \\
\mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E}) & \xrightarrow{\ f\ } & \mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E}')
\end{array}$$

is vertically right adjointable. That is, the Beck-Chevalley transformation

$$\gamma\colon f \circ j_p^{!,\mathrm{hyp}} \longrightarrow j_p^{!,\mathrm{hyp}} \circ f$$

is an equivalence.

Proof. We can without loss of generality assume that $X = X_{\geqslant p}$. In this case, we simply write $j\colon X_p \hookrightarrow X$ for the canonical closed immersion and $i\colon X_{>p} \hookrightarrow X$ for the inclusion of the open complementary. Since j_* is fully faithful, it is enough to prove that $j_*(\gamma)$ is an equivalence. Fix $F \in \mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E})$ and consider the following commutative diagram whose upper and bottom rows are fiber sequences:

$$\begin{array}{ccccc}
f j_* j^{!,\mathrm{hyp}}(F) & \longrightarrow & f(F) & \longrightarrow & f i_* i^{*,\mathrm{hyp}}(F) \\
\downarrow{\scriptstyle \alpha} & & \| & & \downarrow{\scriptstyle \beta} \\
j_* f j^{!,\mathrm{hyp}}(F) & \longrightarrow & f(F) & \longrightarrow & i_* f i^{*,\mathrm{hyp}}(F) \\
\downarrow{\scriptstyle j_*(\gamma)} & & \| & & \downarrow{\scriptstyle \delta} \\
j_* j^{!,\mathrm{hyp}} f(F) & \longrightarrow & f(F) & \longrightarrow & i_* i^{*,\mathrm{hyp}} f(F) .
\end{array}$$

The morphisms α and β are tautologically equivalences. Hence the middle row is a fiber sequence as well. Finally, δ is an equivalence as recalled in Section 2.4. It follows that $j_*(\gamma)$ is an equivalence as well, whence the conclusion. □

Definition 7.24. Let (X, P) be a conically stratified space with locally weakly contractible strata. Let $\mathcal{E}$ be a presentable stable ∞-category equipped with a t-structure $\tau = (\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$. Let

$$\mathfrak{p} \colon P \longrightarrow \mathbb{Z}$$

be a function. We define

$${}^{\mathfrak{p}}\operatorname{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E})_{\geqslant 0} := \left\{ F \in \operatorname{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E}) \mid \forall p \in P \ , \ \pi_i(j_p^{*, \mathrm{hyp}}(F)) = 0 \text{ for every } i < \mathfrak{p}(p) \right\} ,$$

and

$${}^{\mathfrak{p}}\operatorname{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E})_{\leqslant 0} := \left\{ F \in \operatorname{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E}) \mid \forall p \in P \ , \ \pi_i(j_p^{!, \mathrm{hyp}}(F)) = 0 \text{ for every } i > \mathfrak{p}(p) \right\}$$

where the homotopy groups are computed with respect to the naturally induced t-structure on $\operatorname{Sh}^{\mathrm{hyp}}(X; \mathcal{E})$.

It is a standard fact that the above definition gives rise to a t-structure ${}^{\mathfrak{p}}\tau$ on the stable ∞-category $\operatorname{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E})$. We write ${}^{\mathfrak{p}}\operatorname{Perv}_P^{\mathrm{hyp}}(X; \mathcal{E})$ for the heart of this t-structure. Following [DPSb], we give the following definition:

Definition 7.25. Let $S = \operatorname{Spec}(A) \in \operatorname{dAff}_R$ be a derived affine. Let (X, P) be a conically stratified space with locally weakly contractible strata. Let $\mathfrak{p} \colon P \to \mathbb{Z}$ be a function. We say that an A-linear family of hyperconstructible hypersheaves $F \in \operatorname{Cons}_{P,\omega}^{\mathrm{hyp}}(X; \operatorname{Mod}_A)$ is ${}^{\mathfrak{p}}\tau$-*flat relative to S* if for every $M \in \operatorname{Mod}_A^{\heartsuit}$, one has

$$M \otimes_A F \in {}^{\mathfrak{p}}\operatorname{Perv}_P^{\mathrm{hyp}}(X; \operatorname{Mod}_A) .$$

Warning 7.26. Recall that A lies in $\operatorname{Mod}_A^{\heartsuit}$ if and only if the canonical morphism $A \to \pi_0(A)$ is an equivalence, that is when A is *underived*. In that case, a ${}^{\mathfrak{p}}\tau$-flat object automatically lies in ${}^{\mathfrak{p}}\operatorname{Perv}_P^{\mathrm{hyp}}(X; \operatorname{Mod}_A)$. The converse holds when A is a field but is false otherwise.

Construction 7.27. Let $i \colon Z \hookrightarrow X$ be a closed immersion of topological spaces. Since the functor $i^{*, \mathrm{hyp}} \colon \operatorname{Sh}^{\mathrm{hyp}}(X; \operatorname{Mod}_A) \to \operatorname{Sh}^{\mathrm{hyp}}(Z; \operatorname{Mod}_A)$ is strong monoidal, its right adjoint i_* is automatically lax monoidal. In particular, for every $M \in \operatorname{Mod}_A$ and $F \in \operatorname{Sh}^{\mathrm{hyp}}(Z; \operatorname{Mod}_A)$ there is an induced morphism

$$M \otimes_A i_*(F) \longrightarrow i_*(M \otimes_A F) .$$

Observe that this is an isomorphism for $M = A$. Since i is a closed immersion, i_* commutes with colimits and therefore the above morphism is an equivalence. In turn, this induces a natural Beck-Chevalley transformation

$$\alpha_{M,F}^{!, \mathrm{hyp}} \colon M \otimes_A i^{!, \mathrm{hyp}}(F) \longrightarrow i^{!, \mathrm{hyp}}(M \otimes_A F) .$$

Lemma 7.28. *Let (X, P) be a conically stratified space with locally weakly contractible strata. Assume that (X, P) is locally categorically compact and that P is finite. Let $S = \operatorname{Spec}(A)$ be a derived affine. Then for every $M \in \operatorname{Mod}_A$, every $F \in \operatorname{Cons}_P^{\mathrm{hyp}}(X; \operatorname{Mod}_A)$ and every $p \in P$, the natural transformation*

$$\alpha_{M,F}^! \colon M \otimes_A j_p^{!, \mathrm{hyp}}(F) \to j_p^{!, \mathrm{hyp}}(M \otimes_A F)$$

is an equivalence.

Proof. Lemma 7.21 guarantees that $j_p^{!, \mathrm{hyp}}$ commutes with colimits. Since the transformation $\alpha_{M,F}^!$ is an equivalence for $M = A$, it follows that it is an equivalence for every $M \in \operatorname{Mod}_A$. $\quad\square$

The following elementary lemma is an immediate consequence of the Tor spectral sequence [HA, 7.2.1.19].

Lemma 7.29. *Let $S = \operatorname{Spec}(A)$ be a derived affine. Let $F \in \operatorname{Mod}_A$ and assume that $\pi_i(F) \simeq 0$ for $i \ll 0$. Let $k \in \mathbb{Z}$. Then the following are equivalent :*

(1) F is k-connective, that is it belongs to $(\mathrm{Mod}_A)_{\geqslant k}$;

(2) for every $M \in \mathrm{Mod}_A^\heartsuit$, $M \otimes_A F$ is k-connective;

(3) $\pi_0(A) \otimes_A F$ is k-connective.

The following proposition provides a useful reformulation of $^{\mathfrak{p}}\tau$-flatness.

Proposition 7.30. *Let* (X, P) *be a conically stratified space with locally weakly contractible strata. Assume that* (X, P) *is locally categorically compact and that* P *is finite. Let* $S = \mathrm{Spec}(A)$ *be a derived affine. Let* $F \in \mathrm{Cons}_{P,\omega}^{\mathrm{hyp}}(X; \mathrm{Mod}_A)$. *Then the following statements are equivalent:*

(1) *F is $^{\mathfrak{p}}\tau$-flat relative to S;*

(2) *F belongs to* $\mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathrm{Mod}_A)_{\geqslant 0}$ *and for every* $p \in P$, *the stalks of* $j_p^{!,\mathrm{hyp}}(F)$ *have tor-amplitude contained in* $(-\infty, \mathfrak{p}(p)]$.

(3) *for every* $p \in P$, *the stalks of* $j_p^{*,\mathrm{hyp}}(F)$ *are* $\mathfrak{p}(p)$-*connectives and the stalks of* $j_p^{!,\mathrm{hyp}}(F)$ *have tor-amplitude contained in* $(-\infty, \mathfrak{p}(p)]$.

Proof. Let $p \in P$. From Lemma 7.29, the stalks of $j_p^{*,\mathrm{hyp}}(F)$ are $\mathfrak{p}(p)$-connective if and only if for every $M \in \mathrm{Mod}_A^\heartsuit$, the stalks of

$$M \otimes_A j_p^{*,\mathrm{hyp}}(F) \simeq j_p^{*,\mathrm{hyp}}(M \otimes_A F)$$

are $\mathfrak{p}(p)$-connective, where the equivalence follows from the fact that $j_p^{*,\mathrm{hyp}}$ is strongly monoidal.

On the other hand, by definition of tor-amplitude, we see that the stalks of $j_p^{!,\mathrm{hyp}}(F)$ have tor-amplitude contained in $(-\infty, \mathfrak{p}(p)]$ if and only if for every $M \in \mathrm{Mod}_A^\heartsuit$ the stalks of

$$M \otimes_A j_p^{!,\mathrm{hyp}}(F) \simeq j_p^{!,\mathrm{hyp}}(M \otimes_A F)$$

are concentrated in homological degrees $(-\infty, \mathfrak{p}(p)]$, where the equivalence follows from Lemma 7.28. This finishes the proof of Proposition 7.30. □

Remark 7.31. Observe that condition (3) in Proposition 7.30 is stable under base change. Hence, the $^{\mathfrak{p}}\tau$-flatness condition defines a substack $^{\mathfrak{p}}\mathbf{Perv}_P(X)$ of $\mathbf{Cons}_P(X)$ referred to as the *derived stack of perverse sheaves on* (X, P).

Theorem 7.32. *Let* (X, P) *be a conically stratified space with locally weakly contractible strata. Assume that:*

(1) *(X, P) is of finite stratified type;*

(2) *(X, P) is locally categorically compact.*

Then,

$$^{\mathfrak{p}}\mathbf{Perv}_P(X) \longrightarrow \mathbf{Cons}_P(X)$$

is representable by an open immersion. In particular, $\mathbf{Perv}_P(X)$ *is locally geometric locally of finite presentation.*

Proof. Proposition 7.30 ensures that the following diagram of derived stacks

$$
\begin{array}{ccc}
^{\mathfrak{p}}\mathbf{Perv}_P(X) & \xrightarrow{\;(j_p^{*,\mathrm{hyp}}, j_p^{!,\mathrm{hyp}})_{p \in P}\;} & \prod_{p \in P} \mathbf{LC}_{\geqslant \mathfrak{p}(p)}^{\mathrm{hyp}}(X_p) \times \mathbf{LC}_{(-\infty, \mathfrak{p}(p)]}^{\mathrm{hyp}}(X_p) \\
\downarrow & & \downarrow \\
\mathbf{Cons}_P(X) & \xrightarrow{\;(j_p^{*,\mathrm{hyp}}, j_p^{!,\mathrm{hyp}})_{p \in P}\;} & \prod_{p \in P} \mathbf{LC}^{\mathrm{hyp}}(X_p) \times \mathbf{LC}^{\mathrm{hyp}}(X_p)
\end{array}
$$

is cartesian. Here $\mathbf{LC}^{\mathrm{hyp}}_{\geqslant \mathfrak{p}(p)}(X_p)$ (resp. $\mathbf{LC}^{\mathrm{hyp}}_{(-\infty,\mathfrak{p}(p)]}(X_p)$) denotes the substack of $\mathbf{LC}^{\mathrm{hyp}}(X_p)$ parametrizing locally hyperconstant hypersheaves whose stalks are perfect and $\mathfrak{p}(p)$-connective (resp. have tor-amplitude within $(-\infty, \mathfrak{p}(p)]$). The first part of Theorem 7.32 thus follows from Notation 7.16-(1) and Lemma 7.17. We then deduce the representability of $^{\mathfrak{p}}\mathbf{Perv}_P(X)$ from Theorem 7.8. $\qquad\square$

7.7. Cohomological Hall algebras.
As an application, let us sketch the construction of a cohomological Hall algebra associated to perverse sheaves on a smooth projective curve over $\mathbb{C}$. From now on, we will assume that X is a smooth curve over $\mathbb{C}$ equipped with n marked points $\{p_1, p_2, \ldots, p_n\}$. We see X as a stratified space over the poset $P = \{0 < 1\}$ equipped with the middle perversity function, that is

$$\mathfrak{p}(0) = 0 \ , \qquad \mathfrak{p}(1) = -1 \ .$$

Lemma 7.33. *Let k be a field. Let (X, P) as above. Let $F, G \in {}^{\mathfrak{p}}\mathrm{Perv}^{\mathrm{hyp}}_P(X; \mathrm{Mod}_k)$. Then*

$$\mathrm{Hom}_{\mathrm{Cons}^{\mathrm{hyp}}_P(X; \mathrm{Mod}_k)}(F, G) \in \mathrm{Mod}_k$$

is concentrated in homological degrees $[-2, 0]$.

Proof. Let us denote by $\mathbb{D}\colon \mathrm{Cons}^{\mathrm{hyp}}_P(X; \mathrm{Mod}_k) \to \mathrm{Cons}^{\mathrm{hyp}}_P(X; \mathrm{Mod}_k)$ the Verdier duality functor. Then, there is a canonical equivalence

$$\mathrm{Hom}_{\mathrm{Cons}^{\mathrm{hyp}}_P(X; \mathrm{Mod}_k)}(F, G) \simeq R\Gamma(X, \mathbb{D}(F \otimes \mathbb{D}G)) \ .$$

By Poincaré-Verdier duality, showing that the above complex is concentrated in homological degrees $[-2, 0]$ amounts to show that $R\Gamma_c(X, F \otimes \mathbb{D}G)$ is concentrated in homological degrees $[0, 2]$. Since $\mathbb{D}G$ is perverse, we are thus left to show that if $F, G \in {}^{\mathfrak{p}}\mathrm{Perv}^{\mathrm{hyp}}_P(X; \mathrm{Mod}_k)$, then $\Gamma_c(X, F \otimes G)$ is concentrated in homological degrees $[0, 2]$. By definition $F \otimes G$ is concentrated in homological degrees $[0, 2]$ and both $\pi_1(F \otimes G)$ and $\pi_0(F \otimes G)$ are punctually supported. Since X is a curve over $\mathbb{C}$, the only potential non zero terms of the spectral sequence

$$E^2_{p,q}\colon \pi_p\Gamma_c(X, \pi_q(F \otimes G)) \Longrightarrow \pi_{p+q}\Gamma_c(X, F \otimes G)$$

are thus

$$E^2_{0,0}, \ E^2_{0,1}, \ E^2_{0,2}, \ E^2_{-1,2}, \ E^2_{-2,2} \ .$$

This implies the sought-after range. $\qquad\square$

Following [PS, §4.1], there is a canonical simplicial object

$$\mathcal{S}_\bullet{}^{\mathfrak{p}}\mathbf{Perv}_P(X)\colon \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathrm{dSt}_R \ ,$$

which satisfies the 2-Segal condition. Here $\mathcal{S}_0{}^{\mathfrak{p}}\mathbf{Perv}_P(X) \simeq \mathrm{Spec}(R)$ is the final object in dSt_R, while

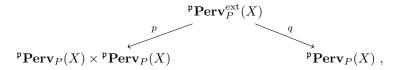$$\mathcal{S}_1{}^{\mathfrak{p}}\mathbf{Perv}_P(X) \simeq {}^{\mathfrak{p}}\mathbf{Perv}_P(X)$$

is the derived stack of $\mathfrak{p}$-perverse sheaves we just constructed. Moreover,

$$^{\mathfrak{p}}\mathbf{Perv}^{\mathrm{ext}}_P(X) \coloneqq \mathcal{S}_2{}^{\mathfrak{p}}\mathbf{Perv}_P(X)$$

can be explicitly described as the derived stack sending a derived affine $S = \mathrm{Spec}(A)$ to the ∞-groupoid consisting of fiber sequences in $\mathrm{Cons}^{\mathrm{hyp}}_{P,\omega}(X; \mathrm{Mod}_A)$ of the form

$$F_1 \longrightarrow F_2 \longrightarrow F_3 \ ,$$

where we further ask that F_1, F_2 and F_3 are $^\flat\tau$-flat relative to S. The simplicial structure of $\mathcal{S}_\bullet{}^\flat\mathbf{Perv}_P(X)$ induces a canonical correspondence

$$^\flat\mathbf{Perv}_P^{\mathrm{ext}}(X)$$

$$^\flat\mathbf{Perv}_P(X) \times {}^\flat\mathbf{Perv}_P(X) \xleftarrow{\ p\ } \qquad \qquad \xrightarrow{\ q\ } {}^\flat\mathbf{Perv}_P(X)\ ,$$

where p sends a fiber sequence as above to its extremes (F_3, F_1) and q sends it to its middle term F_2. We have:

Lemma 7.34.

    (1) The map q is representable by proper algebraic spaces.

    (2) The map p is derived lci.

Proof. We start by point (1). Notice that this question is insensitive to the derived structure, so we will pass to the truncations. Direct inspection reveals that q is 0-truncated; since source and target are Artin stacks, it follows that q is representable by algebraic spaces. Observe now that if X is a smooth (possibly open) $\mathbb{C}$-analytic curve equipped with a trivial stratification, then the statement is known (see e.g. the proof of [PS, Theorem 4.9]). Since the collection of representable maps by proper algebraic spaces is closed under fiber products, we can reduce ourselves to the case where X is an $\mathbb{C}$-analytic disk D, equipped with the stratification over $P = \{0 < 1\}$ whose closed stratum consists exactly with the origin of D. Observe that $^\flat\mathbf{Perv}_P(\mathrm{D})$ can be also described in terms of representations of the following quiver Q:

$$\overset{0}{\bullet} \longrightarrow \overset{1}{\bullet} \, \circlearrowright$$

More specifically, the stable ∞-category $\mathsf{Rep}(Q)$ of $\mathbb{C}$-linear Q-representations, but equip it with the t-structure τ defined by declaring that an object $F \in \mathsf{Rep}(Q)$ is connective if and only if

$$F(0) \in (\mathrm{Mod}_k)_{\geqslant 0} \qquad \text{and} \qquad F(1) \in (\mathrm{Mod}_k)_{\geqslant 2}\ .$$

Then [DPSb] defines a derived moduli stack $\mathbf{Rep}(Q, \tau)$ parametrizing τ-flat families of finite dimensional Q-representations. Explicitly, $\mathbf{Rep}(Q, \tau)$ sends a derived affine $S = \mathrm{Spec}(A)$ to the ∞-groupoid of A-linear Q-representations F such that both $F(0)$ and $F(1)[-2]$ are projective A-modules of finite rank. Inspection reveals that $\mathbf{Rep}(Q, \tau)$ and $^\flat\mathbf{Perv}_P(\mathrm{D})$ are equivalent. We now use the valuative criterion of properness [Stacks, Tag 0A40] to verify that q is proper. Let therefore A be a discrete valuation ring with fraction field K and consider the lifting problem

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{\ t\ } & {}^\flat\mathbf{Perv}_P^{\mathrm{ext}}(\mathrm{D}) \\ \downarrow & \nearrow & \downarrow{q} \\ \mathrm{Spec}(A) & \xrightarrow{\ s\ } & {}^\flat\mathbf{Perv}_P(\mathrm{D})\ . \end{array}$$

let F be the A-linear family of perverse sheaves classified by s, and let

$$F' \longrightarrow F_K \longrightarrow F''$$

be the extension of K-linear families of perverse sheaves classified by t, where $F_K := K \otimes_A F$. Write furthermore

$$M := F(0) \qquad \text{and} \qquad N := F(1)\ .$$

Notice that we can review this extension as an extension of perverse sheaves with coefficients in Mod_A (whose stalks will no longer be perfect). Define $\widetilde{F}''$ to be the image of $F \to F_K \to F''$, computed in the abelian category ${}^\mathfrak{p}\mathrm{Perv}_P^{\mathrm{hyp}}(X; \mathrm{Mod}_A)$. By construction, $\widetilde{F}'' = (\widetilde{M}'', \widetilde{N}'', \widetilde{\alpha}'', \widetilde{\beta}'')$ is a subobject of $F'' = (M'', N'', \alpha'', \beta'')$. This means that $\widetilde{M}''$ and $\widetilde{N}''$ are subobjects of M'' and N'', and hence they are torsion-free. Since A is a valuation ring, they are automatically free [Stacks, Tag 0539]. Besides, $M \to \widetilde{M}''$ and $N \to \widetilde{N}''$ are surjective, and hence $\widetilde{M}''$ and $\widetilde{N}''$ have finite rank. Thus, $\widetilde{F}' := \ker(F \to \widetilde{F}'')$ is flat and of finite rank as well. This provides a solution of the previous lifting problem. The uniqueness follows at once from the uniqueness of the epi-mono factorization for the morphism $F \to F_K \to F''$. Thus, q is proper.

We now prove point (2). The computation of the relative cotangent complex for the map p done in [PS, Proposition 3.6], reduces us to check that for every field k and every pair (F_3, F_1) of k-linear families of perverse sheaves, the object

$$\mathrm{Hom}_{\mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathrm{Mod}_k)}(F_3, F_1) \in \mathrm{Mod}_k$$

is concentrated in homological degrees $[-2, 0]$. This follows from Lemma 7.33. □

At this point, following the discussion in [PS, §4.2], we are led to:

Corollary 7.35. *Let X be a smooth projective curve over $\mathbb{C}$ equipped with an algebraic stratification over $P = \{0 < 1\}$. Let $\mathfrak{p}$ be the middle perversity function. Then $\mathrm{Coh}^{\mathrm{b}}({}^\mathfrak{p}\mathbf{Perv}_P(X))$ carries a canonical $\mathbb{E}_1$-monoidal structure of Hall type. Similarly, the Borel-Moore homology and G-theory of ${}^\mathfrak{p}\mathbf{Perv}_P(X)$ carry Hall multiplications.*

Remark 7.36. These (categorical, cohomological, K-theoretical) Hall algebras generalize the Betti Hall algebras of [PS, M] (a.k.a. the Hall algebras of the character variety), which in turn can be seen as a generalization of Schiffmann-Vasserot CoHA of the commuting variety [SV]. Exploiting the microlocal nature of perverse sheaves, it is natural to expect that these "perverse CoHAs" admit a presentation as a Jacobi algebra, generalizing the one of [M]. Moreover, following [DPSa] it is natural to wonder whether there exists a torsion pair on ${}^\mathfrak{p}\mathrm{Perv}_P^{\mathrm{hyp}}(X)$ giving rise to a "perverse geometric Yangian". We will come back to these questions in a future work.

References

[AF]   A. Andreotti and T. Frankel, *The Lefschetz theorem on hyperplane sections*, Ann. of Math. (2) **69** (1959), 713–717.

[AFTa] D. Ayala, J. Francis, and H. L. Tanaka, *Factorization homology of stratified spaces*, Selecta Math. (N.S.) **23** (2017), no. 1, 293–362.

[AFTb] ———, *Local structures on stratified spaces*, Adv. Math. **307** (2017), 903–1028.

[AGV]   J. Ayoub, M. Gallauer, and A. Vezzani, *The six-functor formalism for rigid analytic motives*, arXiv:2010.15004 (2020).

[APT]   D. Arinkin, T. Pantev, and B. Toën, *Relative shifted symplectic forms*, In preparation, 2022.

[BGH]   C. Barwick, S. Glasman, and P. J. Haine, *Exodromy*, arXiv:1807.03281 (2018).

[B]   E. Bishop, *Mappings of partially analytic spaces*, Amer. J. Math. **83** (1961), 209–242.

[BZNP] D. Ben-Zvi, D. Nadler, and A. Preygel, *Integral transforms for coherent sheaves*, J. Eur. Math. Soc. (JEMS) **19** (2017), no. 12, 3763–3812.

[C]   D.-C. Cisinski, *Higher categories and homotopical algebra*, Cambridge Studies in Advanced Mathematics, vol. 180, Cambridge University Press, 2019.

[D]   S. Douteau, *Homotopy theory of stratified spaces*, Algebr. Geom. Topol. **21** (2021), no. 1, 507–541.

[DPSa] D.-E. Diaconescu, M. Porta, and F. Sala, *Cohomological hall algebras and their representations via torsion pairs*, arXiv:2207.08926 (2022).

[DPSb] ———, *Moduli of flat objects*, In preparation, 2022.

[GHN]   D. Gepner, R. Haugseng, and T. Nikolaus, *Lax colimits and free fibrations in ∞-categories*, Doc. Math. **22** (2017), 1225–1266.

[H]   P. J. Haine, *On the Homotopy Theory of Stratified Spaces*, ProQuest LLC, Ann Arbor, MI, 2021, Thesis (Ph.D.)–Massachusetts Institute of Technology.

[HL]   J. V. S. Holstein and A. Lazarev, *Categorical Koszul duality*, Adv. Math. **409** (2022), Paper No. 108644, 52.

[Ha]   J. V. S. Holstein, *Morita cohomology*, Math. Proc. Cambridge Philos. Soc. **158** (2015), no. 1, 1–26.

[Hb]   _____, *Morita cohomology and homotopy locally constant sheaves*, Math. Proc. Cambridge Philos. Soc. **158** (2015), no. 1, 27–35.

[HPT]   P. J. Haine, M. Porta, and J.-B. Teyssier, *The homotopy-invariance of constructible sheaves of spaces*, arXiv:2010.06473, Accepted for publication in Homotopy, Homology and Applications.

[KS]   R. C. Kirby and L. C. Siebenmann, *On the triangulation of manifolds and the Hauptvermutung*, Bull. Amer. Math. Soc. **75** (1969), 742–749.

[L]   D. Lejay, *Constructible hypersheaves via exit paths*, arXiv:2102.12325 (2021).

[HTT]   Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009.

[HA]   Jacob Lurie, *Higher algebra*, Preprint, August 2017.

[SAG]   Jacob Lurie, *Spectral algebraic geometry*, Preprint, 2018.

[M]   V. Mistry, *On the cohomological hall algebra of a character variety*, arXiv:2209.00680 (2022).

[N]   R. Narasimhan, *Imbedding of holomorphically complete complex spaces*, Amer. J. Math. **82** (1960), 917–934.

[NS]   N. Nitsure and C. Sabbah, *Moduli of pre-𝒟-modules, perverse sheaves and the Riemann-Hilbert morphism. I*, Math. Ann. **306** (1996), no. 1, 47–73.

[NV]   G. Nocera and M. Volpe, *Whitney stratifications are conically smooth*, arXiv:2105.09243 (2021).

[PS]   M. Porta and F. Sala, *Two-dimensional categorified hall algebras*, arXiv:1903.07253 (2019), To appear in Journal of European Mathematical Society.

[PY]   Mauro Porta and Tony Yue Yu, *Higher analytic stacks and GAGA theorems*, Advances in Mathematics **302** (2016), 351–409.

[R]   R. Remmert, *Sur les espaces analytiques holomorphiquement séparables et holomorphiquement convexes*, C. R. Acad. Sci. Paris **243** (1956), 118–121.

[Stacks] The Stacks Project Authors, *Stacks Project*, http://stacks.math.columbia.edu, 2013.

[SV]   O. Schiffmann and E. Vasserot, *Cherednik algebras, W-algebras and the equivariant cohomology of the moduli space of instantons on* $\mathbf{A}^2$, Publ. Math. Inst. Hautes Études Sci. **118** (2013), 213–342.

[T]   D. Treumann, *Exit paths and constructible stacks*, Compos. Math. **145** (2009), no. 6, 1504–1532.

[TV]   Bertrand Toën and Michel Vaquié, *Moduli of objects in dg-categories*, Ann. Sci. École Norm. Sup. (4) **40** (2007), no. 3, 387–444.

[V]   M. Volpe, *Verdier duality on conically smooth stratified spaces*, arXiv:2206.02728 (2022).

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