A BOUNDEDNESS THEOREM FOR NEARBY SLOPES OF HOLONOMIC $\mathcal{D}$-MODULES

by

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Abstract. — Using twisted nearby cycles, we define a new notion of slopes for complex holonomic $\mathcal{D}$-modules. We prove a boundedness result for these slopes, study their functoriality and use them to characterize regularity. For a family of (possibly irregular) algebraic connections $\mathcal{E}_t$ parametrized by a smooth curve, we deduce under natural conditions an explicit bound for the usual slopes of the differential equation satisfied by the family of irregular periods of the $\mathcal{E}_t$. This generalizes the regularity of the Gauss-Manin connection proved by Katz and Deligne.

Let V be a smooth algebraic variety over a finite field of characteristic $p > 0$, and let U be an open subset in V such that $D := V \backslash U$ is a normal crossing divisor. Let ℓ be a prime number different from p. Using restriction to curves, Deligne defined [**Del11**] a notion of ℓ-adic local system on U with bounded ramification along D. Such a definition is problematic to treat functoriality questions: the direct image of a local system is not a local system any more, duality does not commute with restriction in general. In this paper, we investigate the characteristic 0 aspect of this problem, that is the

Question 1. — *Let X be a complex manifold. Can one define a notion of holonomic $\mathcal{D}_X$-module with bounded irregularity which has good functoriality properties?*

In dimension 1, to bound the irregularity number of a $\mathcal{D}$-module with given generic rank amounts to bound its slopes. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module and let Z be a hypersurface of X. Mebkhout [**Meb90**] showed that the *irregularity complex* $\mathrm{Irr}_Z(\mathcal{M})$ of $\mathcal{M}$ along Z is a perverse sheaf endowed with a $\mathbb{R}_{>1}$ increasing locally finite filtration by sub-perverse sheaves $\mathrm{Irr}_Z(\mathcal{M})(r)$. If the support of the r-th graded piece of $(\mathrm{Irr}_Z(\mathcal{M})(r))_{r>1}$ is not empty, we say that $1/(r-1)$ is an *analytic slope of $\mathcal{M}$ along Z*[(1)].

The existence of a uniform bound in Z is not clear a priori. We thus formulate the following

(1) The transformation $r \longrightarrow 1/(r-1)$ is here so that in dimension 1, the analytic slopes correspond to the classical slopes defined via Newton polygons.

Conjecture 1. — *Locally on X, the set of analytic slopes of a holonomic $\mathcal{D}_X$-module is bounded.*

This statement means that for a holonomic $\mathcal{D}_X$-module $\mathcal{M}$, one can find for every point in X a neighbourhood U and a constant $C > 0$ such that the analytic slopes of $\mathcal{M}$ along any germ of hypersurface in U are $\leqslant C$. The main obstacle to the proof of conjecture 1 lies in the behavior of analytic slopes with respect to proper push-forward. On the other hand, Laurent defined *algebraic slopes* using his theory of micro-characteristic varieties [**Lau87**]. From Laurent and Mebkhout work [**LM99**], we know that modulo the transformation $r \longrightarrow 1/(r-1)$, the set of analytic slopes of a holonomic $\mathcal{D}$-module $\mathcal{M}$ along Z is equal to the set of algebraic slopes of $\mathcal{M}$ along Z. Since micro-characteristic varieties are invariant by duality, we deduce that analytic slopes are invariant by duality.

The aim of this paper is to define a third notion of slopes and to investigate some of its properties. The main idea lies in the observation that for a germ $\mathcal{M}$ of $\mathcal{D}_{\mathbb{C}}$-module at $0 \in \mathbb{C}$, *the slopes of $\mathcal{M}$ at 0 are encoded in the vanishing of certain nearby cycles.* We show in 2.3.1 that $r \in \mathbb{Q}_{\geqslant 0}$ is a slope for $\mathcal{M}$ at 0 if and only if one can find a germ N of meromorphic connection at 0 with slope r such that $\psi_0(\mathcal{M} \otimes N) \neq 0$.

We thus introduce the following definition. Let X be a complex manifold and let $\mathcal{M}$ be an object of the derived category $\mathcal{D}^b_{\mathrm{hol}}(X)$ of complexes of $\mathcal{D}_X$-modules with bounded and holonomic cohomology. Let $f \in \mathcal{O}_X$. We denote by ψ_f the nearby cycle functor(2) associated to f. We define the *nearby slopes of $\mathcal{M}$ associated to f* to be the set $\mathrm{Sl}^{\mathrm{nb}}_f(\mathcal{M})$ complement in $\mathbb{Q}_{\geqslant 0}$ of the set of rationals $r \geqslant 0$ such that for every germ N of meromorphic connection at 0 with slope r, we have

$$\psi_f(\mathcal{M} \otimes f^+ N) \simeq 0 \tag{0.0.1}$$

Let us observe that the left-hand side of (0.0.1) depends on N only via $\widehat{\mathcal{O}}_{\mathbb{C},0} \otimes_{\mathcal{O}_{\mathbb{C},0}} N$, and that nearby slopes are sensitive to the non reduced structure of $\operatorname{div} f$, whereas analytic and algebraic slopes only see the support of $\operatorname{div} f$.

Twisted nearby cycles appear for the first time in the algebraic context in [**Del07**]. Deligne proves in *loc. it.* that for a given function f, the set $\mathrm{Sl}^{\mathrm{nb}}_f(\mathcal{M})$ is finite.

The main result of this paper is an affirmative answer to conjecture 1 for nearby slopes, that is the

Theorem 1. — *Locally on X, the set of nearby slopes of a holonomic $\mathcal{D}$-module is bounded.*

This statement means that for a holonomic $\mathcal{D}_X$-module $\mathcal{M}$, one can find for every point in X a neighbourhood U and a constant $C > 0$ such that the nearby slopes of $\mathcal{M}$ associated to any $f \in \mathcal{O}_U$ are $\leqslant C$. For meromorphic connections with good formal structure, we show the following refinement:

(2)For general references on the nearby cycle functor, let us mention [**Kas83**],[**Mal83**],[**MS89**] and [**MM04**].

Theorem 2. — *Let $\mathcal{M}$ be a meromorphic connection with good formal structure. Let D be the pole locus of $\mathcal{M}$ and let $D_1, \ldots, D_n$ be the irreducible components of D. We denote by $r_i(\mathcal{M}) \in \mathbb{Q}_{\geqslant 0}$ the highest generic slope of $\mathcal{M}$ along D_i. Then, the nearby slopes of $\mathcal{M}$ are $\leqslant r_1(\mathcal{M}) + \cdots + r_n(\mathcal{M})$.*

The main tool used in the proof of theorem 1 is a structure theorem for formal meromorphic connections first conjectured in [**CS89**], studied by Sabbah [**Sab00**] and proved by Kedlaya [**Ked10**][**Ked11**] in the context of excellent schemes and analytic spaces, and independently by Mochizuki [**Moc09**][**Moc11b**] in the algebraic context.

Let us give some details on the strategy of the proof of theorem 1. A dévissage carried out in 3.1 allows one to suppose that $\mathcal{M}$ is a meromorphic connection. Using Kedlaya-Mochizuki theorem, one reduces further the proof to the case where $\mathcal{M}$ has good formal structure. We are thus left to prove theorem 2. We resolve the singularities of Z. The problem that occurs at this step is that a randomly chosen embedded resolution $p : \widetilde{X} \longrightarrow X$ will increase the generic slopes of $\mathcal{M}$ in a way that cannot be controlled. We show in 3.2.2 that a fine version of embedded resolution [**BM89**] allows to control the generic slopes of $p^+\mathcal{M}$ in terms of the sum $r_1(\mathcal{M}) + \cdots + r_n(\mathcal{M})$ and the multiplicities of p^*Z. A crucial tool for this is a theorem [**Sab00**, I 2.4.3] proved by Sabbah in dimension 2 and by Mochizuki [**Moc11a**, 2.19] in any dimension relating the good formal models appearing at a given point with the generic models on the divisor locus. Using a vanishing criterion 2.4.1, one finally proves (0.0.1) for $r > r_1(\mathcal{M}) + \cdots + r_n(\mathcal{M})$.

Let $\mathcal{M} \in \mathcal{D}^b_{\mathrm{hol}}(X)$ and let us denote by $\mathbb{D}\mathcal{M}$ the dual complex of $\mathcal{M}$. Nearby slopes satisfy the following functorialities

Theorem 3. — *(i) For every $f \in \mathcal{O}_X$, we have*

$$\mathrm{Sl}^{\mathrm{nb}}_f(\mathbb{D}\mathcal{M}) = \mathrm{Sl}^{\mathrm{nb}}_f(\mathcal{M})$$

(ii) Let $p : X \longrightarrow Y$ be a proper morphism and let $f \in \mathcal{O}_Y$ such that $p(X)$ is not contained in $f^{-1}(0)$. Then

$$\mathrm{Sl}^{\mathrm{nb}}_f(p_+\mathcal{M}) \subset \mathrm{Sl}^{\mathrm{nb}}_{fp}(\mathcal{M})$$

Let us observe that (ii) is a direct application of the compatibility of nearby cycles with proper direct image [**MS89**].

It is an interesting problem to try to compare nearby slopes and analytic slopes. This question won't be discussed in this paper, but we characterize regular holonomic $\mathcal{D}$-modules using nearby slopes.

Theorem 4. — *A complex $\mathcal{M} \in \mathcal{D}^b_{\mathrm{hol}}(X)$ is regular if and only if for every quasi-finite morphism $\rho : Y \longrightarrow X$ with Y a complex manifold, the set of nearby slopes of $\rho^+\mathcal{M}$ is contained in $\{0\}$.*

For an other characterization of regularity (harder to deal with in practice) using derived endomorphisms, we refer to [**Tey14**].

Let us give an application of the preceding results. Let U be a smooth complex algebraic variety and let $\mathcal{E}$ be an algebraic connection on U. We denote by $H^k_{\mathrm{dR}}(U, \mathcal{E})$ the k^{th} de Rham cohomology group of $\mathcal{E}$, and by $\mathcal{V}$ the local system of horizontal sections of $\mathcal{E}^{\mathrm{an}}$ on U^{an}. If $\mathcal{E}$ is regular, Deligne proved [**Del70**] that the canonical comparison morphism

$$H^k_{\mathrm{dR}}(U,\mathcal{E}) \longrightarrow H^k(U^{\mathrm{an}},\mathcal{V}) \tag{0.0.2}$$

is an isomorphism. If $\mathcal{E}$ is the trivial connection, this is due to Grothendieck [**Gro66**]. In the irregular case, (0.0.2) is no longer an isomorphism. It can happen that $H^k_{\mathrm{dR}}(U,\mathcal{E})$ is non zero and $H^k(U^{\mathrm{an}},\mathcal{V})$ is zero, which means that there are not enough topological cycles in U^{an}. The *rapid decay homology* $H^{\mathrm{rd}}_k(U,\mathcal{E}^*)$ needed to remedy this problem appears in dimension one in [**BE04**] and in higher dimension in [**Hie07**][**Hie09**]. It includes cycles drawn on a compactification of U^{an} taking into account the asymptotic at infinity of the solutions of the dual connection $\mathcal{E}^*$. By Hien duality theorem, we have a perfect pairing

$$\int : H^k_{\mathrm{dR}}(U,\mathcal{E}) \times H^{\mathrm{rd}}_k(U,\mathcal{E}^*) \longrightarrow \mathbb{C} \tag{0.0.3}$$

For $\omega \in H^k_{\mathrm{dR}}(U,\mathcal{E})$ and $\gamma \in H^{\mathrm{rd}}_k(U,\mathcal{E}^*)$, we call $\int_\gamma \omega$ a *k-period for* $\mathcal{E}$[3].

Let $f : X \longrightarrow S$ be a proper and generically smooth morphism, where X denotes an algebraic variety and S denotes a neighbourhood of 0 in $\mathbb{A}^1_{\mathbb{C}}$. Let U be the complement of a normal crossing divisor D of X such that for every $t \neq 0$ close enough to 0, D_t is a normal crossing divisor of X_t. Let $\mathcal{E}$ be an algebraic connection on U. Let us denote by $D_1,\dots,D_n$ the irreducible components of D meeting $f^{-1}(0)$ and let $r_i(\mathcal{E})$ be the highest generic slope of $\mathcal{E}$ along D_i.

As an application of theorem 2, we prove the following

Theorem 5. — *If $\mathcal{E}$ has good formal structure along D and if the fibers X_t, $t \neq 0$ of f are non characteristic at infinity*[4] *for $\mathcal{E}$, then the k-period vectors of the family $(\mathcal{E}_t)_{t\neq 0}$ are the analytic solutions of the system of differential equations associated to $\mathcal{H}^k f_+\mathcal{E}$. The slopes at 0 of this system are $\leqslant r_1(\mathcal{E}) + \cdots + r_n(\mathcal{E})$.*

In the case where $\mathcal{E}$ is the trivial connection, we recover that the periods of a proper generically smooth family of algebraic varieties are solutions of a regular singular differential equation with polynomial coefficients [**Kat70**][**Del70**].

The role played in this paper by nearby cycles has Verdier specialization [**Ver83**] and moderate nearby cycles as ℓ-adic counterparts. For a discussion of the problems arising in the ℓ-adic case, we refer to [**Tey15a**].

Conjecture 1 first appears in [**Tey15b**]. This paper grew out an attempt to prove it. I thank Pierre Deligne, Zoghman Mebkhout and Claude Sabbah for valuable comments on this manuscript and Marco Hien for mentioning [**HR08**], which inspired me a statement in the spirit of theorem 5 and reignited my interest for a proof of theorem 1. This work has been achieved with the support of Freie Universität/Hebrew University of Jerusalem joint post-doctoral program and ERC 226257 program. I thank Hélène Esnault and Yakov Varshavsky for their support.

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(3) note that this terminology is abusive, since we don't consider any rational structure on the cohomology spaces.

(4) this is for example the case if D is smooth and if the fibers of f are transverse to D.

1. Notations

We collect here a few definitions used all along this paper. The letter X will denote a complex manifold.

1.1. For a morphism $f : Y \longrightarrow X$ with Y a complex manifold, we denote by $f^+ : D^b_{\mathrm{hol}}(\mathcal{D}_X) \longrightarrow D^b_{\mathrm{hol}}(\mathcal{D}_Y)$ and $f_+ : D^b_{\mathrm{hol}}(\mathcal{D}_Y) \longrightarrow D^b_{\mathrm{hol}}(\mathcal{D}_X)$ the inverse image and direct image functors for $\mathcal{D}$-modules. We note $f^\dagger$ for $f^+[\dim Y - \dim X]$.

1.2. Let $\mathcal{M} \in \mathcal{D}^b_{\mathrm{hol}}(X)$ and $f \in \mathcal{O}_X$. From $\mathcal{H}^k\psi_f(\mathcal{M} \otimes f^+N) \simeq \psi_f(\mathcal{H}^k\mathcal{M} \otimes f^+N)$ for every k, we deduce

$$\mathrm{Sl}^{\mathrm{nb}}_f(\mathcal{M}) = \bigcup_k \mathrm{Sl}^{\mathrm{nb}}_f(\mathcal{H}^k\mathcal{M}) \tag{1.2.1}$$

Let us define $\mathrm{Sl}^{\mathrm{nb}}(\mathcal{M}) := \bigcup_{f\in\mathcal{O}_X} \mathrm{Sl}^{\mathrm{nb}}_f(\mathcal{M})$. The elements of $\mathrm{Sl}^{\mathrm{nb}}(\mathcal{M})$ are the *nearby slopes* of $\mathcal{M}$. For $S \subset \mathbb{Q}_{\geqslant 0}$, we denote by $\mathcal{D}^b_{\mathrm{hol}}(X)_S$ the full subcategory of $\mathcal{D}^b_{\mathrm{hol}}(X)$ of complexes whose nearby slopes are in S.

1.3. Let us denote by $\mathrm{DR} : D^b_{\mathrm{hol}}(\mathcal{D}_X) \longrightarrow D^b_c(X, \mathbb{C})$ the *de Rham functor*[5] and by $\mathrm{Sol} : D^b_{\mathrm{hol}}(\mathcal{D}_X) \longrightarrow D^b_c(X, \mathbb{C})$ the *solution functor* for holonomic $\mathcal{D}_X$-modules.

1.4. For every analytic subspace Z in X, we denote by $i_Z : Z \hookrightarrow X$ the canonical inclusion. The *local cohomology triangle* for Z and $\mathcal{M} \in \mathcal{D}^b_{\mathrm{hol}}(X)$ reads

$$R\Gamma_{[Z]}\mathcal{M} \longrightarrow \mathcal{M} \longrightarrow R\mathcal{M}(*Z) \xrightarrow{+1} \tag{1.4.1}$$

It is a distinguished triangle in $D^b_{\mathrm{hol}}(\mathcal{D}_X)$. The complex $R\Gamma_{[Z]}\mathcal{M}$ is *the local algebraic cohomology* of $\mathcal{M}$ along Z and $R\mathcal{M}(*Z)$ is the *localization* of $\mathcal{M}$ along Z.

1.5. Let $\mathcal{M}$ be a germ of meromorphic connection at the origin of $\mathbb{C}^n$. Let D be the pole locus of $\mathcal{M}$. For $x \in D$, we define $\widehat{\mathcal{M}}_x := \widehat{\mathcal{O}}_{\mathbb{C}^n,x} \otimes_{\mathcal{O}_{\mathbb{C}^n,x}} \mathcal{M}$. We say that $\mathcal{M}$ has *good formal structure* if

(1) D is a normal crossing divisor.
(2) For every $x \in D$, one can find coordinates $(x_1, \dots, x_n)$ centred at x with D defined by $x_1 \cdots x_i = 0$, and an integer $p \geqslant 1$ such that if ρ is the morphism $(x_1, \dots, x_n) \longrightarrow (x_1^p, \dots, x_i^p, x_{i+1}, \dots, x_n)$, we have a decomposition

$$\rho^+\widehat{\mathcal{M}}_x \simeq \bigoplus_{\varphi\in\mathcal{O}_{\mathbb{C}^n}(*D)/\mathcal{O}_{\mathbb{C}^n}} \mathcal{E}^\varphi \otimes \mathcal{R}_\varphi \tag{1.5.1}$$

where $\mathcal{E}^\varphi = (\widehat{\mathcal{O}}_{\mathbb{C}^n,x}(*D), d + d\varphi)$ and $\mathcal{R}_\varphi$ is a meromorphic connection with regular singularity along D.
(3) For all $\varphi \in \mathcal{O}_{\mathbb{C}^n}(*D)/\mathcal{O}_{\mathbb{C}^n}$ contributing to (1.5.1), we have $\mathrm{div}\,\varphi \leqslant 0$.

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[5] In this paper, we follow Hien's convention **[Hie09]** according to which for a holonomic module $\mathcal{M}$, the complex $\mathrm{DR}\,\mathcal{M}$ is concentrated in degrees $0, \dots, \dim X$.

Let us remark that classically, one asks for condition (3) to be also true for the differences of two φ intervening in (1.5.1). We won't impose this extra condition in this paper.

1.6. Let $\mathcal{M}$ be a meromorphic connection on X such that the pole locus D of $\mathcal{M}$ has only a finite number of irreducible components $D_1, \dots, D_n$. For every $i = 1, \dots, n$, we denote by $r_{D_i}(\mathcal{M})$ the highest generic slope of $\mathcal{M}$ along D_i. We define the *divisor of highest generic slopes of* $\mathcal{M}$ by

$$r_{D_1}(\mathcal{M})D_1 + \cdots + r_{D_n}(\mathcal{M})D_n \in Z(X)_{\mathbb{Q}}$$

2. Preliminaries on nearby cycles in the case of good formal structure

2.1. Let n be an integer and take $i \in \mathbb{N}^{[\![1,n]\!]}$. The *support of* i is the set of $k \in [\![1,n]\!]$ such that $i_k \neq 0$. If $E \subset [\![1,n]\!]$, we define i_E by $i_{Ek} = i_k$ for $k \in E$ and $i_{Ek} = 0$ if $k \notin E$.

2.2. Let R be a regular $\mathbb{C}((t))$-differential module, and take $\varphi \in \mathbb{C}[t^{-1}]$. For every $n \geqslant 1$, we define $\rho : t \longrightarrow t^p = x$ and

$$\mathrm{El}(\rho, \varphi, R) := \rho_+(\mathcal{E}^{\varphi} \otimes R)$$

If R is the trivial rank 1 module, we will use the notation $\mathrm{El}(\rho, \varphi)$. In general, $\mathrm{El}(\rho, \varphi, R)$ has slope $\operatorname{ord} \varphi/p$. The $\mathbb{C}((x))$-modules of type $\mathrm{El}(\rho, \varphi, R)$ for variable (ρ, φ, R) are called *elementary modules*. From [**Sab08**, 3.3], we know that every $\mathbb{C}((x))$-differential module can be written as a direct sum of elementary modules.

2.3. Dimension 1. — In this paragraph, we work in a neighbourhood of the origin $0 \in \mathbb{C}$. Let x be a coordinate on $\mathbb{C}$. Take $p \geqslant 1$ and define $\rho : x \longrightarrow t = x^p$.

Proposition 2.3.1. — *Let $\mathcal{M}$ be a germ of holonomic $\mathcal{D}$-module at the origin. Let $r > 0$ be a rational number. The following conditions are equivalent*

(1) The rational r is not a slope for $\mathcal{M}$ at 0.

(2) For every germ N of meromorphic connection of slope r/p, we have

$$\psi_\rho(\mathcal{M} \otimes \rho^+ N) \simeq 0$$

Proof. — Since ψ is not sensitive to localization and formalization, one can work formally at 0 and suppose that $\mathcal{M}$ and N are differential $\mathbb{C}((x))$-modules.

Let us prove (2) $\Longrightarrow$ (1) by contraposition. Define $\rho' : u \longrightarrow u^{p'} = x$, $\varphi(u) \in \mathbb{C}[u^{-1}]$ with $q = \operatorname{ord} \varphi(u)$ and R a $\mathbb{C}((u))$-regular module such that $\mathrm{El}(\rho', \varphi(u), R)$ is a non zero elementary factor 2.2 of $\mathcal{M}$ with slope $r = q/p$. Define

$$N := \rho_+ \mathrm{El}(\rho', -\varphi(u)) = \mathrm{El}(\rho\rho', -\varphi(u))$$

The module N has slope $q/pp' = r/p$. A direct factor of $\psi_\rho(\mathcal{M} \otimes \rho^+ N)$ is

$$\begin{aligned} \psi_\rho(\rho'_+(\mathcal{E}^{\varphi} \otimes R) \otimes \rho^+ N)) &\simeq \psi_\rho(\rho'_+(\mathcal{E}^{\varphi} \otimes R) \otimes \rho^+ \mathrm{El}(\rho\rho', -\varphi(u))) \\ &\simeq \psi_\rho(\rho'_+(\mathcal{E}^{\varphi} \otimes R \otimes (\rho\rho')^+ \mathrm{El}(\rho\rho', -\varphi(u))) \\ &\simeq \psi_{\rho\rho'}(\mathcal{E}^{\varphi} \otimes R \otimes (\rho\rho')^+ \mathrm{El}(\rho\rho', -\varphi(u))) \end{aligned}$$

where the last identification comes from the compatibility of ψ with proper direct image. By [**Sab08**, 2.4], we have

$$(\rho\rho')^+ \operatorname{El}(\rho\rho', -\varphi(u)) \simeq \bigoplus_{\zeta^{pp'}=1} \mathcal{E}^{-\varphi(\zeta u)}$$

So $\psi_{\rho\rho'}R$ is a direct factor of $\psi_\rho(\mathcal{M} \otimes \rho^+ N)$ of rank $np(\operatorname{rg} R) > 0$, and (2) $\Longrightarrow$ (1) is proved.

Let us prove (1) $\Longrightarrow$ (2). Let N be a $\mathbb{C}((t))$-differential module of slope r/p. Then ρ^+N has slope r. Thus, the slopes of $\mathcal{M} \otimes \rho^+ N$ are > 0. Hence, it is enough to show the following

Lemma 2.3.2. — *Let M be a $\mathbb{C}((x))$-differential module whose slopes are > 0. Then, we have $\psi_\rho M \simeq 0$.*

By Levelt-Turrittin decomposition, we are left to study the case where M is a direct sum of modules of type $\mathcal{E}^\varphi \otimes R$, where $\varphi \in \mathbb{C}[x^{-1}]$ and where R is a regular $\mathbb{C}((x))$-module. The hypothesis on the slopes of M implies $\varphi \neq 0$, and the expected vanishing is standard. □

2.4. A vanishing criterion. — Let $\mathcal{M}$ be a germ of meromorphic connection at the origin $0 \in \mathbb{C}^n$. We suppose that $\mathcal{M}$ has good formal structure at 0. Let D be the pole locus of $\mathcal{M}$. Let ρ_p be a ramification of degree p along the components of D as in (1.5.1).

Proposition 2.4.1. — *Let $f \in \mathcal{O}_{\mathbb{C}^n,0}$. Let us define $Z := \operatorname{div} f$ and suppose that $|Z| \subset D$. We suppose that for every irreducible component E of $|Z|$, we have*

$$r_E(\mathcal{M}) \leqslant r v_E(f)$$

Then for every germ N of meromorphic connection at 0 with slopes $> r$, we have

$$\psi_f(\mathcal{M} \otimes f^+ N) \simeq 0 \tag{2.4.2}$$

in a neighbourhood of 0.

Proof. — Let us choose local coordinates $(x_1, \dots, x_n)$ and $a \in \mathbb{N}^n$ such that f is the function $x \longrightarrow x^a$. Take N with slopes $> r$. One can always suppose that N is a $\mathbb{C}((t))$-differential module and $p = qk$ where $\rho' : t \longrightarrow t^k$ decomposes N.

The morphism ρ_p is a finite cover away from D, so the canonical adjunction morphism

$$\rho_{p+}\rho_p^+ \mathcal{M} \longrightarrow \mathcal{M} \tag{2.4.3}$$

is surjective away from D. So the cokernel of (2.4.3) has support in D. From [**Meb04**, 3.6-4], we know that both sides of (2.4.3) are localized along D. So (2.4.3) is surjective. We thus have to prove

$$\psi_{f\rho_p}(\rho_p^+ \mathcal{M} \otimes (f\rho_p)^+ N) \simeq 0 \tag{2.4.4}$$

Since $|Z| \subset D$, we have $f\rho_p = \rho' f\rho_q$. So the left hand side of (2.4.4) is a direct sum of k copies of

$$\psi_{f\rho_q}(\rho_p^+\mathcal{M} \otimes (f\rho_p)^+ N) \tag{2.4.5}$$

We thus have to prove that (2.4.5) is 0 in a neighbourhood of 0. We have

$$(f\rho_p)^+ N \simeq (f\rho_q)^+ \rho'^+ N$$

with $\rho'^+ N$ decomposed with slopes $> rk$. The zero locus of $f\rho_q$ is $|Z|$, and if E is an irreducible component of $|Z|$, the highest generic slope of $\rho_p^+\mathcal{M}$ along E is

$$r_E(\rho_p^+\mathcal{M}) = p \cdot r_E(\mathcal{M}) \leqslant rk \cdot q \cdot v_E(f) = rk \cdot v_E(f\rho_q)$$

Hence we can suppose that $\rho_p = \mathrm{id}$ and that N is decomposed.

Take

$$N = \mathcal{E}^{P(t)/t^m} \otimes R$$

with $P(t) \in \mathbb{C}[t]$ satisfying $P(0) \neq 0$, with $m > r$ and with R regular. Since ψ is insensitive to formalization, one can suppose

$$\mathcal{M} = \mathcal{E}^{\varphi(x)} \otimes \mathcal{R}$$

with $\varphi(x)$ as in 1.5 (3) and $\mathcal{R}$ regular. By Sabbah-Mochizuki theorem, the multiplicity of $-\operatorname{div} \varphi(x)$ along a component D' of D is a generic slope of $\mathcal{M}$ along D'. Thus, one can write $\varphi(x) = g(x)/x^b$ where $g(0) \neq 0$ and where the b_i are such that if $i \in \operatorname{Supp} a$, we have $b_i \leqslant ra_i < ma_i$. We thus have to prove the

Lemma 2.4.6. — *Take $g, h \in \mathcal{O}_{\mathbb{C}^n,0}$ such that $g(0) \neq 0$ and $h(0) \neq 0$. Let $\mathcal{R}$ be a regular meromorphic connection with poles contained in $x_1 \cdots x_n = 0$. Take $a, b \in \mathbb{N}^{[\![1,n]\!]}$ such that $A := \operatorname{Supp} a$ is non empty and $b_i < a_i$ for every $i \in A$. Then*

$$\psi_{x^a}(\mathcal{E}^{g(x)/x^b + h(x)/x^a} \otimes \mathcal{R}) \simeq 0$$

in a neighbourhood of 0.

□

2.5. Proof of 2.4.6. — We define $\mathcal{M} := \mathcal{E}^{g(x)/x^b + h(x)/x^a} \otimes \mathcal{R}$. Since A is not empty, a change of variable allows one to suppose $h = 1$. If $\operatorname{Supp} b \subset A$, a change of variable shows that 2.4.6 is a consequence of 2.6.1. Let $i \in \operatorname{Supp} b$ be an integer such that $i \notin A$. Using x_i, a change of variable allows one to suppose $g = 1$. Let $p_1, \ldots, p_n \in \mathbb{N}^*$ such that $a_j p_j$ is independent from j for every $j \in A$ and $p_j = 1$ if $j \notin A$. Let ρ_p be the morphism $x \longrightarrow x^p$. Like in (2.4.3), we see that

$$\rho_{p+}\rho_p^+\mathcal{M} \longrightarrow \mathcal{M}$$

is surjective. We are thus left to prove that 2.4.6 holds for multi-indices a such that a_j does not depend on j for every $j \in A$. Let us denote by $\mathbb{1}_A$ the characteristic function of A. From [**Sab05**, 3.3.13], it is enough to prove

$$\psi_{x^{\mathbb{1}_A}}(\mathcal{E}^{1/x^b + 1/x^a} \otimes \mathcal{R}) \simeq 0$$

Using the fact that $\mathcal{R}$ is a successive extension of regular modules of rank 1, one can suppose that $\mathcal{R} = x^c$, where $c \in \mathbb{C}^{[\![1,n]\!]}$. Let

$$\begin{array}{ccc} \mathbb{C}^n & \overset{i}{\hookrightarrow} & \mathbb{C}^n \times \mathbb{C} \\ & \searrow^{x^{\mathbb{1}_A}} & \downarrow \\ & & \mathbb{C} \end{array}$$

be the inclusion given by the graph of $x \longrightarrow x^{\mathbb{1}_A}$. Let t be a coordinate on the second factor of $\mathbb{C}^n \times \mathbb{C}$. We have to prove

$$\psi_t(i_+(x^c\mathcal{E}^{1/x^b+1/x^a})) \simeq 0$$

Define $\delta := \delta(t - x^{\mathbb{1}_A}) \in i_+(x^c\mathcal{E}^{1/x^b+1/x^a})$ and let $(V_k)_{k\in\mathbb{Z}}$ be the Kashiwara-Malgrange filtration on $\mathcal{D}_{\mathbb{C}^n\times\mathbb{C}}$ relative to t. For $d \in \mathbb{N}^{[\![1,n]\!]}$ such that $x^d = 0$ is the pole locus of $x^c\mathcal{E}^{1/x^b+1/x^a}$, the family of sections x^d generates $x^c\mathcal{E}^{1/x^b+1/x^a}$. For such d, the family $s := x^d\delta$ generates $i_+(x^c\mathcal{E}^{1/x^b+1/x^a})$. We are left to prove $s \in V_{-1}s$. One can always suppose that $1 \in A$.

$$x_1\partial_1 s = (d_1 + c_1)s - \frac{b_1}{x^b}s - \frac{a_1}{x^a}s - x^{\mathbb{1}_A}\partial_t s$$

We define $M \in \mathbb{N}^{[\![1,n]\!]}$ by $M_k = \max(a_k, b_k)$ for every $k \in [\![1,n]\!]$. We thus have

$$x^M x_1\partial_1 s = (d_1 + c_1)x^M s - b_1 x^{M-b}s - a_1 x^{M-a}s - x^M x^{\mathbb{1}_A}\partial_t s \tag{2.5.1}$$

We have $M = a + b_{A^c} = \mathbb{1}_A + (a - \mathbb{1}_A) + b_{A^c} = \mathbb{1}_A + b + m$ with $m \in \mathbb{N}^{[\![1,n]\!]}$. So

$$x^{M-b}s = x^m ts \in V_{-1}s$$

Moreover, we have

$$x^M x_1\partial_1 s = x_1\partial_1 x^M s - M_1 x^M s = x_1\partial_1 x^{m+b}ts - M_1 x^{m+b}ts \in V_{-1}s$$

and

$$x^M x^{\mathbb{1}_A}\partial_t s = x^{m+b}\partial_t x^{2\times\mathbb{1}_A}s = x^{m+b}\partial_t t^2 s = 2x^{m+b}ts + x^{m+b}t(t\partial_t)s \in V_{-1}s$$

So (2.5.1) gives

$$x^{M-a}s \in V_{-1}s \tag{2.5.2}$$

Let us recall that i is such that $i \notin A$ and $i \in \operatorname{Supp} b$. In particular $(M-a)_i = b_i \neq 0$ and $\partial_i\delta = 0$. Applying $x_i\partial_i$ to (2.5.2), we obtain

$$(d_i + c_i + b_i)x^{M-a}s - b_i\frac{x^{M-a}}{x^b}s \in V_{-1}s$$

so from (2.5.2), we deduce $x^{M-a-b}s \in V_{-1}s$. We have $M - a - b = -b_A$, so by multiplying $x^{M-a-b}s$ by x^{b_A}, we get $s \in V_{-1}s$.

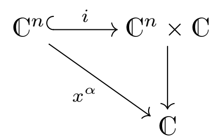
2.6. The aim of this paragraph is to prove the following

Lemma 2.6.1. — *Let* $\alpha, a \in \mathbb{N}^{[\![1,n]\!]}$ *such that* $\operatorname{Supp}\alpha$ *is not empty and* $\operatorname{Supp}\alpha \subset \operatorname{Supp} a$. *Let* $\mathcal{R}$ *be a regular meromorphic connection with poles contained in* $x_1 \cdots x_n = 0$. *We have*

$$\psi_{x^\alpha}(\mathcal{E}^{1/x^a} \otimes \mathcal{R}) \simeq 0$$

Proof. — Let $p_1, \ldots, p_n$ be integers such that $\alpha_i p_i$ does not depend of i for every $i \in \operatorname{Supp}\alpha$ (we denote by m this integer) and $p_i = 1$ if $i \neq \operatorname{Supp}\alpha$. Let ρ_p be the morphism $x \longrightarrow x^p$. Like in (2.4.3), the morphism $\rho_{p+}\rho_p^+\mathcal{M} \longrightarrow \mathcal{M}$ is surjective. We are left to prove 2.6.1 for α such that α_i does not depend of i for every $i \in \operatorname{Supp}\alpha$. From [**Sab05**, 3.3.13], one can suppose $\alpha_i = 1$ for every $i \in \operatorname{Supp}\alpha$. So $\alpha \leqslant a$.

One can suppose $\mathcal{R} = x^b$ where $b \in \mathbb{N}^{[\![1,n]\!]}$. Let



be the inclusion given by the graph of $x \longrightarrow x^\alpha$. Let t be a coordinate on the second factor of $\mathbb{C}^n \times \mathbb{C}$. We have to show

$$\psi_t(i_+(x^b\mathcal{E}^{1/x^a})) \simeq 0$$

Define $\delta := \delta(t - x^\alpha) \in i_+(x^b\mathcal{E}^{1/x^a})$. For $c \in \mathbb{N}^{[\![1,n]\!]}$ such that $\operatorname{Supp} c \subset \operatorname{Supp} a \cup \operatorname{Supp} b$, the family of sections x^c generates $x^b\mathcal{E}^{1/x^a}$. For such c, the family $s := x^c\delta$ generates $i_+(x^b\mathcal{E}^{1/x^a})$. It is thus enough to show $s \in V_{-1}s$. Let us choose $i \in \operatorname{Supp}\alpha$. We have

$$x_i\partial_i s = (c_i + b_i)s - \frac{a_i}{x^a}s - x^\alpha\partial_t s$$

We have $\alpha \leqslant a$. Define $a = \alpha + a'$. From

$$x^\alpha x_i\partial_i s = x_i\partial_i x^\alpha s - x^\alpha s = x_i\partial_i ts - ts \in V_{-1}s$$

we deduce that $a_i s + x^{a'}x^{2\alpha}\partial_t s \in V_{-1}s$. We also have $x^{2\alpha}\partial_t s = \partial_t x^{2\alpha}s = \partial_t t^2 s = 2ts + t(t\partial_t)s \in V_{-1}s$. Since $a_i \neq 0$, we deduce $s \in V_{-1}s$ and 2.6.1 is proved.

□

3. Proof of theorem 1

3.1. Dévissage to the case of meromorphic connections. — Suppose that theorem 1 is true for meromorphic connections for every choice of ambient manifold. Let us show that theorem 1 is true for $\mathcal{M} \in \mathcal{D}^b_{\mathrm{hol}}(X)$. We argue by induction on $\dim X$. The case where X is a point is trivial. Let us suppose that $\dim X > 0$. We define $Y := \operatorname{Supp}\mathcal{M}$ and we argue by induction on $\dim Y$.

Let us suppose that Y is a strict closed subset of X. We denote by $i : Y \longrightarrow X$ the canonical inclusion. Let $\pi : \widetilde{Y} \longrightarrow Y$ be a resolution of the singularities of Y [**AHV29**] and $p := i\pi$. The regular locus $\operatorname{Reg} Y$ of Y is a dense open subset in Y and

π is an isomorphism above $\operatorname{Reg} Y$. By Kashiwara theorem, we deduce that the cone $\mathcal{C}$ of the adjunction morphism

$$p_+p^\dagger\mathcal{M} \longrightarrow \mathcal{M}$$

has support in $\operatorname{Sing} Y$, with $\operatorname{Sing} Y$ a strict closed subset in Y. Let $x \in X$ and let B be a neighbourhood of x with compact closure $\overline{B}$. Then, $p^{-1}(\overline{B})$ is compact. Since $\dim \widetilde{Y} < \dim X$, theorem 1 is true for $p^\dagger\mathcal{M} \in \mathcal{D}^b_{\mathrm{hol}}(\widetilde{Y})$. Let (U_i) be a finite family of open sets in $\widetilde{Y}$ covering $p^{-1}(\overline{B})$ and such that for every i, the set $\mathrm{Sl}^{\mathrm{nb}}((p^\dagger\mathcal{M})_{|U_i})$ is bounded by a rationnal r_i. Define $R = \max_i r_i$.

By induction hypothesis applied to $\mathcal{C}$, one can suppose at the cost of taking a smaller B containing x that the set $\mathrm{Sl}^{\mathrm{nb}}(\mathcal{C}_{|B})$ is bounded by a rational R'. Take $f \in \mathcal{O}_B$. We have a distinguished triangle

$$\psi_f(p_+p^\dagger\mathcal{M} \otimes f^+N) \longrightarrow \psi_f(\mathcal{M} \otimes f^+N) \longrightarrow \psi_f(\mathcal{C} \otimes f^+N) \xrightarrow{+1} \tag{3.1.1}$$

By projection formula and compatibility of ψ with proper direct image, (3.1.1) is isomorphic to

$$p_+\psi_{fp}(p^\dagger\mathcal{M} \otimes (pf)^+N) \longrightarrow \psi_f(\mathcal{M} \otimes f^+N) \longrightarrow \psi_f(\mathcal{C} \otimes f^+N) \xrightarrow{+1}$$

So we have the desired vanishing on B for $r > \max(R, R')$.

We are left with the case where $\dim \operatorname{Supp} \mathcal{M} = \dim X$. Let Z be a hypersurface containing $\operatorname{Sing} \mathcal{M}$. We have a triangle

$$R\Gamma_{[Z]}\mathcal{M} \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}(*Z) \xrightarrow{+1}$$

By applying the induction hypothesis to $R\Gamma_{[Z]}\mathcal{M}$, we are left to prove theorem 1 for $\mathcal{M}(*Z)$. The module $\mathcal{M}(*Z)$ is a meromorphic connection, which concludes the reduction step.

3.2. The case of meromorphic connections. — At the cost of taking an open cover of X, let us take a resolution of turning points $p : \widetilde{X} \longrightarrow X$ for $\mathcal{M}$ as given by Kedlaya-Mochizuki theorem. Let D be the pole locus of $\mathcal{M}$. Since p is an isomorphism above $X \backslash D$, the cone of

$$p_+p^+\mathcal{M} \longrightarrow \mathcal{M} \tag{3.2.1}$$

has support in the pole locus D of $\mathcal{M}$. From [**Meb04**, 3.6-4], the left hand side of (3.2.1) is localized along D. So (3.2.1) is an isomorphism. We thus have a canonical isomorphism

$$p_+\psi_{fp}(p^+\mathcal{M} \otimes (fp)^+N) \simeq \psi_f(\mathcal{M} \otimes f^+N)$$

Since p is proper, we see as in 3.1 that we are left to prove theorem 1 for $p^+\mathcal{M}$. We thus suppose that $\mathcal{M}$ has a good formal structure. At the cost of taking an open cover, we can suppose that D has only a finite number of irreducible components. Let S be the divisor of highest generic slopes 1.6 of $\mathcal{M}$. Let $S_1, \dots, S_m$ be the irreducible components of $|S|$. Let us prove that $\mathrm{Sl}^{\mathrm{nb}}(\mathcal{M})$ is bounded by $\deg S$. It is is a local statement. Let $f \in \mathcal{O}_X$ and define $Z := \operatorname{div} f$. Let us denote by $|Z|$ (resp. $|S|$) the support of Z (resp. S) and let us admit for a moment the validity of the following

Proposition 3.2.2. — *Locally on X, one can find a proper birationnal morphism $\pi : \widetilde{X} \longrightarrow X$ such that*
(1) π is an isomorphism above $X \backslash |Z|$.
(2) $\pi^{-1}(|Z|) \cup \pi^{-1}(|S|)$ is a normal crossing divisor.
(3) for every valuation v_E measuring the vanishing order along an irreducible component E of $\pi^{-1}(|Z|)$, we have

$$v_E(S) \leqslant (\deg S) v_E(f)$$

Let us suppose that 3.2.2 is true. At the cost of taking an open cover, let us take a morphism $\pi : \widetilde{X} \longrightarrow X$ as in 3.2.2. Since condition (1) is true, the cone of the canonical comparison morphism

$$\pi_+ \pi^+ \mathcal{M} \longrightarrow \mathcal{M} \tag{3.2.3}$$

has support in $|Z|$. Since $f^+ N$ is localized along $|Z|$, we deduce that (3.2.3) induces an isomorphism

$$(\pi_+ \pi^+ \mathcal{M}) \otimes f^+ N \xrightarrow{\sim} \mathcal{M} \otimes f^+ N$$

Applying ψ_f and using the fact that π is proper, we see that it is enough to prove

$$\psi_{f\pi}(\pi^+ \mathcal{M} \otimes (f\pi)^+ N) \simeq 0 \tag{3.2.4}$$

for every germ N of meromorphic connection at the origin with slope $r > \deg S$. Since $(f\pi)^+ N$ is localized along $\pi^{-1}(|Z|)$, the left-hand side of (3.2.4) is

$$\psi_{f\pi}((\pi^+ \mathcal{M})(*\pi^{-1}(|Z|)) \otimes (f\pi)^+ N) \tag{3.2.5}$$

The vanishing of (3.2.5) is a local statement on $\widetilde{X}$. Since (2) and (3) are true, 2.4.1 asserts that it is enough to show that for every irreducible component E of $\pi^{-1}(|Z|)$, we have

$$r_E((\pi^+ \mathcal{M})(*\pi^{-1}(|Z|))) \leqslant (\deg S) v_E(f\pi)$$

Let us notice that $v_E(f\pi) = v_E(f)$. Let P be a point in the smooth locus of E. Let φ as in (1.5.1) for $\mathcal{M}$ at the point $Q := \pi(P)$. For $i = 1, \dots, n$, let $t_i = 0$ be an equation of S_i in a neighbourhood of Q. Modulo a unit in $\mathcal{O}_{X,Q}$, we have $\varphi = 1/t_1^{r_1} \cdots t_n^{r_n}$ where $r_i \in \mathbb{Q}_{\geqslant 0}$. If $u = 0$ is a local equation for E in a neighbourhood of P, we have modulo a unit in $\mathcal{O}_{\widetilde{X},P}$

$$\varphi\pi = \frac{1}{u^{r_1 v_E(t_1)} \dots u^{r_n v_E(t_n)}}$$

So the slope of $\mathcal{E}^{\varphi\pi}(*\pi^{-1}(|Z|))$ along E is $r_1 v_E(t_1) + \cdots + r_n v_E(t_n)$. By Sabbah-Mochizuki theorem, r_i is a slope of $\mathcal{M}$ generically along S_i, so $r_i \leqslant r_{S_i}(\mathcal{M})$. We deduce that

$$r_E(\pi^+ \mathcal{M}(*\pi^{-1}(|Z|))) \leqslant \sum_i r_{S_i}(\mathcal{M}) v_E(t_i) = v_E(S) \leqslant (\deg S) v_E(f)$$

This concludes the proof of theorem 1 and theorem 2.

3.3. Proof of 3.2.2. — At the cost of taking an open cover of X, let us take a finite sequence of blow-up

$$\pi_n : X_n \xrightarrow{p_{n-1}} X_{n-1} \xrightarrow{p_{n-2}} \cdots \longrightarrow X_1 \xrightarrow{p_0} X_0 = X \tag{3.3.1}$$

given by 3.15 and 3.17 of [**BM89**] for Z relatively to the normal crossing divisor $|S|$. Let $|Z|_i$ be the strict transform of $|Z|$ in X_i and let C_i be the center of p_i. We define inductively $H_0 = |S|$ and $H_{i+1} = p_i^{-1}(H_i) \cup p_i^{-1}(C_i)$ for $i = 1, \dots, n$, where p_i^{-1} denotes the set theoretic inverse image. In particular H_{i+1} is a closed subset of X_{i+1}. We will endow it with its canonical reduced structure. Then, (3.3.1) satisfies

(i) C_i is a smooth closed subset of $|Z|_i$.
(ii) C_i is nowhere dense in $|Z|_i$.
(iii) C_i and H_i have normal crossing for every i.
(iv) $|Z|_n \cup H_n$ is a normal crossing divisor.

Since C_i and the components of H_i are reduced and smooth, condition (iii) means that locally on X_i, one can find coordinates $(x_1, \dots, x_k)$ such that H_i is given by the equation $x_1 \cdots x_l = 0$ and the ideal of C_i is generated by some x_j for $j = 1, \dots, k$. Using condition (i), we see by induction that $\pi_n^{-1}(|Z|) \cup \pi_n^{-1}(|S|) = |Z|_n \cup H_n$. Proposition 3.2.2 is thus a consequence of

Proposition 3.3.2. — *Let*

$$\pi_n : X_n \xrightarrow{p_{n-1}} X_{n-1} \xrightarrow{p_{n-2}} \cdots \longrightarrow X_1 \xrightarrow{p_0} X_0 = X$$

be a sequence of blow-up satisfying (i),(ii) *and* (iii). *For every irreducible component* E *of* $\pi_n^{-1}(|Z|)$, *we have*

$$v_E(S) \leqslant (\deg S) v_E(f) \tag{3.3.3}$$

Proof. — Let $S_1, \dots, S_m$ be the irreducible components of $|S|$ and let $Z_1, \dots, Z_{m'}$ be the irreducible components of Z. Note that some Z_i can be in $|S|$. We define $a_i = v_{Z_i}(f) > 0$ and let Z_{ji} (resp. S_{ji}) be the strict transform of Z_j (resp. S_j) in X_i.

We argue by induction on n. If $n = 0$, E is one of the Z_i and then (3.3.3) is obvious. We suppose that (3.3.3) is true for a composite of n blow-up and we prove that (3.3.3) is true for a composite of $n+1$ blow-up.

Let $\mathcal{C}_n$ be the set of irreducible components of

$$\bigcup_{i=0}^{n-1} (p_{n-1} \cdots p_i)^{-1}(C_i)$$

Each element $E \in \mathcal{C}_n$ will be endowed with its reduced structure. Condition (i) implies that the irreducible components of $\pi_n^* Z$ are the Z_{in} and the elements of $\mathcal{C}_n$. Condition (ii) implies that none of the Z_{in} belongs to $\mathcal{C}_n$. Thus, we have

$$\pi_n^* Z = \operatorname{div} f\pi_n = a_1 Z_{1n} + \cdots + a_{m'} Z_{m'n} + \sum_{E \in \mathcal{C}_n} v_E(f) E$$

On the other hand, we have

$$\pi_n^* S = r_{S_1}(\mathcal{M}) S_{1n} + \cdots + r_{S_m}(\mathcal{M}) S_{mn} + \sum_{E \in \mathcal{C}_n} v_E(S) E$$

Let us consider the last blow-up $p_n : X_{n+1} \longrightarrow X_n$. Let us denote by P the exceptionnal divisor of p_n and let E_{n+1} be the strict transform of $E \in \mathcal{C}_n$ in X_{n+1}. We have

$$p_n^* Z_{in} = Z_{in+1} + \alpha_i P \qquad \text{with } \alpha_i \in \mathbb{N}$$

Since

$$H_n = \bigcup_{j=0}^{m} S_{jn} \cup \bigcup_{E \in \mathcal{C}_n} E$$

we deduce from condition (iii) and smoothness of C_n that

$$p_n^* E = E_{n+1} + \epsilon_E P \qquad \text{with } \epsilon_E \in \{0, 1\}$$

and

$$p_n^* S_{in} = S_{in+1} + \epsilon_i P \qquad \text{with } \epsilon_i \in \{0, 1\}$$

Hence, we have

$$\pi_n^* Z = \sum a_i Z_{in+1} + \sum_{E \in \mathcal{C}_n} v_E(f) E_{n+1} + (\sum a_i \alpha_i + \sum_{E \in \mathcal{C}_n} \epsilon_E v_E(f)) P$$

and

$$\pi_n^* S = \sum r_{S_i}(\mathcal{M}) S_{in+1} + \sum_{E \in \mathcal{C}_n} v_E(S) E_{n+1} + (\sum r_{S_i}(\mathcal{M}) \epsilon_i + \sum_{E \in \mathcal{C}_n} \epsilon_E v_E(S)) P$$

Formula (3.3.3) is true for the Z_{in+1}. By induction hypothesis, formula (3.3.3) is true for E_{n+1}, where $E \in \mathcal{C}_n$. We are left to prove that (3.3.3) is true for P. Conditions (i) and (ii) imply that one of the α_i is non zero, so

$$\begin{aligned}(\deg S)\left(\sum a_i \alpha_i + \sum \epsilon_E v_E(f)\right) &\geqslant (\deg S) + (\deg S) \sum \epsilon_E v_E(f) \\ &\geqslant \sum r_{S_i}(\mathcal{M}) \epsilon_i + \sum \epsilon_E (\deg S) v_E(f) \\ &\geqslant \sum r_{S_i}(\mathcal{M}) \epsilon_i + \sum \epsilon_E v_E(S)\end{aligned}$$

□

4. Duality

We prove theorem 3 (i). Let us denote by $\mathbb{D}$ the duality functor for $\mathcal{D}$-modules. There is a canonical comparison morphism

$$\mathbb{D}(\mathcal{M} \otimes f^+ N) \longrightarrow \mathbb{D}\mathcal{M} \otimes f^+ \mathbb{D} N \tag{4.0.4}$$

On a punctured neighbourhood of $0 \in \mathbb{C}$, the module N is isomorphic to a finite sum of copies of the trivial connection. Thus, there is a neighbourhood U of Z such that the restriction of (4.0.4) to $U \backslash Z$ is an isomorphism. Hence, the cone of (4.0.4) has support in Z. We deduce that

$$(\mathbb{D}(\mathcal{M} \otimes f^+ N))(*Z) \xrightarrow{\sim} \mathbb{D}\mathcal{M} \otimes f^+((\mathbb{D} N)(*0))$$

We have $(\mathbb{D}N)(*0) \simeq N^*$, where $*$ is the duality functor for meromorphic connection. Note that $*$ is a slope preserving involution. Since nearby cycles are insensitive to localization and commute with duality for $\mathcal{D}$-modules, we have

$$\psi_f(\mathbb{D}\mathcal{M} \otimes f^+N^*) \simeq \mathbb{D}(\psi_f(\mathcal{M} \otimes f^+N))$$

and theorem 3 (i) is proved.

5. Regularity and nearby cycles

The aim of this section is to prove theorem 4.

5.1. We will use the following

Lemma 5.1.1. — *Let F be a germ of closed analytic subspace at the origin $0 \in \mathbb{C}^n$. Let $Y_1, \dots, Y_k$ be irreducible closed analytic subspaces of $\mathbb{C}^n$ containing 0 and such that $F \cap Y_i$ is a strict closed subset of Y_i for every i. Then, there exists a germ of hypersurface Z at the origin containing F and such that $Z \cap Y_i$ has codimension 1 in Y_i for every i.*

Proof. — Denote by $\mathcal{I}_F$ (resp. $\mathcal{I}_{Y_i}$) the ideal sheaf of F (resp. Y_i). By irreducibility, $\mathcal{I}_{Y_i,0}$ is a prime ideal in $\mathcal{O}_{\mathbb{C}^n,0}$. The hypothesis say $\mathcal{I}_F \nsubseteq \mathcal{I}_{Y_i}$ for every i. From [**Mat80**, 1.B], we deduce

$$\mathcal{I}_F \nsubseteq \bigcup_i \mathcal{I}_{Y_i}$$

Any function $f \in \mathcal{I}_F$ not in $\bigcup_i \mathcal{I}_{Y_i}$ defines a hypersurface as wanted. □

5.2. We say that a holonomic module $\mathcal{M}$ is *smooth* if the support $\operatorname{Supp}\mathcal{M}$ of $\mathcal{M}$ is smooth equidimensional and if the characteristic variety of $\mathcal{M}$ is equal to the conormal of $\operatorname{Supp}\mathcal{M}$ in X. We denote by $\operatorname{Sing}\mathcal{M}$ the complement of the smooth locus of $\mathcal{M}$. It is a strict closed subset of $\operatorname{Supp}\mathcal{M}$.

Let $x \in X$ and let us define F as the union of $\operatorname{Sing}\mathcal{M}$ with the irreducible components of $\operatorname{Supp}\mathcal{M}$ passing through x which are not of maximal dimension. Define $Y_1, \dots, Y_k$ to be the irreducible components of $\operatorname{Supp}\mathcal{M}$ of maximal dimension passing through x. From 5.1.1, one can find a hypersurface Z passing through x such that

(1) $Z \cap \operatorname{Supp}\mathcal{M}$ has codimension 1 in $\operatorname{Supp}\mathcal{M}$.
(2) The cohomology modules of $\mathcal{H}^k\mathcal{M}$ are smooth away from Z.
(3) $\dim \operatorname{Supp} R\Gamma_{[Z]}\mathcal{M} < \dim \operatorname{Supp}\mathcal{M}$.

5.3. The direct implication of theorem 4 is a consequence of the preservation of regularity by inverse image and the following

Proposition 5.3.1. — *We have $\mathcal{D}^b_{\mathrm{hol}}(X)_{\mathrm{reg}} \subset \mathcal{D}^b_{\mathrm{hol}}(X)_{\{0\}}$.*

Proof. — Take $\mathcal{M} \in \mathcal{D}^b_{\mathrm{hol}}(X)_{\mathrm{reg}}$. We argue by induction on $\dim X$. The case where X is a point is trivial. By arguing on $\dim \operatorname{Supp}\mathcal{M}$ as in 3.1, we are left to prove 5.3.1

in the case where $\mathcal{M}$ is a regular meromorphic connection. Let D be the pole locus of $\mathcal{M}$. Take $f \in \mathcal{O}_X$ and let N with slope > 0. To prove

$$\psi_f(\mathcal{M} \otimes f^+ N) \simeq 0$$

one can suppose using embedded desingularization that $D + \operatorname{div} f$ is a normal crossing divisor. We then conclude with 2.4.1. □

5.4. To prove the reverse implication of theorem 4, we argue by induction on $\dim X \geqslant 1$. The case of curves follows from 2.3.1. We suppose that $\dim X \geqslant 2$ and we take $\mathcal{M} \in \mathcal{D}^b_{\mathrm{hol}}(X)_{\{0\}}$. We argue by induction on $\dim \operatorname{Supp} \mathcal{M}$. The case where $\operatorname{Supp} \mathcal{M}$ is punctual is trivial.

Suppose that $0 < \dim \operatorname{Supp} \mathcal{M} < \dim X$. Since $\operatorname{Supp} \mathcal{M}$ is a strict closed subset of X, one can always locally write $X = X' \times D$ where $D$is the unit disc of $\mathbb{C}$ and where the projection $X' \times D \longrightarrow X'$ is finite on $\operatorname{Supp} \mathcal{M}$. Let $i : X' \times D \longrightarrow X' \times \mathbb{P}^1$ be the canonical immersion. There is a commutative diagram

$$\begin{array}{ccc} \operatorname{Supp} \mathcal{M} & \longrightarrow & X' \times \mathbb{P}^1 \\ & \searrow & \downarrow p \\ & & X' \end{array} \tag{5.4.1}$$

The oblique arrow of (5.4.1) is finite, and p is proper. So the horizontal arrow is proper. Thus, $\operatorname{Supp} \mathcal{M}$ is a closed subset in $X' \times \mathbb{P}^1$. Hence, $\mathcal{M}$ can be extended by 0 to $X' \times \mathbb{P}^1$. We still denote by $\mathcal{M}$ this extension. It is an object of $\mathcal{D}^b_{\mathrm{hol}}(X' \times \mathbb{P}^1)_{\{0\}}$ and we have to show that it is regular.

Let Z be a divisor in X' given by the equation $f = 0$ and let $\rho : Y \longrightarrow X'$ be a finite morphism. Since p is smooth, the analytic space Y' making the following diagram

$$\begin{array}{ccc} Y' & \xrightarrow{\rho'} & X' \times \mathbb{P}^1 \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[\rho]{} & X' \end{array}$$

cartesian is smooth. Moreover ρ' is finite. By base change [**HTT00**, 1.7.3], projection formula and compatibility of ψ with proper direct image, we have for every germ N of meromorphic connection with slope > 0

$$\begin{aligned} \psi_f(\rho^+ p_+ \mathcal{M} \otimes f^+ N) &\simeq \psi_f(p'_+ \rho'^+ \mathcal{M} \otimes f^+ N) \\ &\simeq \psi_f(p'_+(\rho'^+ \mathcal{M} \otimes (fp')^+ N)) \\ &\simeq p'_+ \psi_{fp'}(\rho'^+ \mathcal{M} \otimes (fp')^+ N) \\ &\simeq 0 \end{aligned}$$

By induction hypothesis $p_+ \mathcal{M}$ is regular. Let $Y_1, \dots, Y_n$ be the irreducible components of $\operatorname{Supp} \mathcal{M}$ with maximal dimension. Since $\operatorname{Sing} \mathcal{M} \cap Y_i$ is a strict closed subset of Y_i and since a finite morphism preserves dimension, $p(\operatorname{Sing} \mathcal{M}) \cap p(Y_i)$ is a strict closed subset of the irreducible closed set $p(Y_i)$. In a neighbourhood of a given point of

$p(\operatorname{Sing}\mathcal{M})$, one can find from 5.2 a hypersurface Z containing $p(\operatorname{Sing}\mathcal{M})$ such that $Z \cap p(Y_i)$ has codimension 1 in $p(Y_i)$ for every i. So $p^{-1}(Z)$ contains $\operatorname{Sing}\mathcal{M}$ and

$$\dim p^{-1}(Z) \cap Y_i = \dim Z \cap p(Y_i) = \dim p(Y_i) - 1 = \dim Y_i - 1$$

Since Irr^*_Z is compatible with proper direct image [**Meb04**, 3.6-6], we have

$$\operatorname{Irr}^*_Z p_+\mathcal{M} \simeq Rp_* \operatorname{Irr}^*_{p^{-1}(Z)} \mathcal{M} \simeq 0$$

Since p is finite over $\operatorname{Supp}\mathcal{M}$, we have

$$Rp_* \operatorname{Irr}^*_{p^{-1}(Z)} \mathcal{M} \simeq p_* \operatorname{Irr}^*_{p^{-1}(Z)} \mathcal{M}$$

So for every $x \in p^{-1}(Z)$, the germ of $\operatorname{Irr}^*_{p^{-1}(Z)} \mathcal{M}$ at x is a direct factor of the complex $(p_* \operatorname{Irr}^*_Z p_+\mathcal{M})_{p(x)} \simeq 0$. Thus $\operatorname{Irr}^*_{p^{-1}(Z)} \mathcal{M} \simeq 0$. From [**Meb04**, 4.3-17], We deduce that $\mathcal{M}(*p^{-1}(Z))$ is regular.

To show that $\mathcal{M}$ is regular, we are left to prove that $R\Gamma_{[p^{-1}(Z)]}\mathcal{M}$ is regular. From 5.3, the nearby slopes of all quasi-finite inverse images of $\mathcal{M}(*p^{-1}(Z))$ are contained in $\{0\}$. Thus, this is also the case for $R\Gamma_{[p^{-1}(Z)]}\mathcal{M}$. By construction of Z,

$$\dim \operatorname{Supp} R\Gamma_{[p^{-1}(Z)]}\mathcal{M} < \dim \operatorname{Supp}\mathcal{M}$$

We conclude by applying the induction hypothesis to $R\Gamma_{[p^{-1}(Z)]}\mathcal{M}$.

Let us suppose that $\operatorname{Supp}\mathcal{M}$ has dimension $\dim X$, and let Z be a hypersurface as in 5.2. Then $\mathcal{M}(*Z)$ is a meromorphic connection with poles along Z. Let us show that $\mathcal{M}(*Z)$ is regular. By [**Meb04**, 4.3-17], it is enough to prove regularity generically along Z. Hence, one can suppose that Z is smooth. By Malgrange theorem [**Mal96**], one can suppose that Z is smooth and that $\mathcal{M}(*Z)$ has good formal structure along Z. Let $(x_1, \dots, x_n, t)$ be coordinates centred at $0 \in Z$ such that Z is given by $t = 0$ and let $\rho : (x, u) \longrightarrow (x, u^p)$ be as in 1.5 for $\mathcal{M}(*Z)$. Let $\mathcal{E}^{g(x,u)/u^k} \otimes \mathcal{R}$ be a factor of $\rho^+(\widehat{\mathcal{M}}_0(*Z))$ where $g(0,0) \neq 0$ and where $\mathcal{R}$ is a regular meromorphic connection with poles along Z. For a choice of k-th root in a neighbourhood of $g(0,0)$, we have

$$\psi_{u/\sqrt[k]{g}}(\rho^+\mathcal{M} \otimes (u/\sqrt[k]{g})^+\mathcal{E}^{-1/u^k}) \simeq 0$$

Since nearby cycles commute with formalization, we deduce

$$\psi_u(\rho^+(\widehat{\mathcal{M}}_0(*Z)) \otimes \mathcal{E}^{-g/u^k}) \simeq \psi_u(\rho^+\widehat{\mathcal{M}}_0 \otimes \mathcal{E}^{-g/u^k}) \simeq 0$$

Thus $\psi_u\mathcal{R} \simeq 0$, so $\mathcal{R} \simeq 0$. Hence, the only possibly non zero factor of $\rho^+(\widehat{\mathcal{M}}_0(*Z))$ is the regular factor. So $\mathcal{M}(*Z)$ is regular. We obtain that $\mathcal{M}$ is regular by applying the induction hypothesis to $R\Gamma_{[Z]}\mathcal{M}$.

6. Slopes and irregular periods

6.1. The main reference for what follows is [**Sab00**, II]. Let X be a smooth complex manifold of dimension d and let D be a normal crossing divisor in X. Define $U := X\backslash D$ and let $j : U \longrightarrow X$ be the canonical inclusion. Let $\mathcal{M}$ be a meromorphic connection on X with poles along D. We denote by $p : \widetilde{X} \longrightarrow X$ the real blow-up of X along D and $\widetilde{\imath} : U \longrightarrow \widetilde{X}$ the canonical inclusion.

Let $\mathcal{A}^{<D}_{\widetilde{X}}$ be the sheaf of differentiable functions on $\widetilde{X}$ whose restriction to U are

holomorphic and whose asymptotic development along $p^{-1}(D)$ is zero, and let $\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}}$ be the sheaf of differentiable functions on $\widetilde{X}$ whose restriction to U are holomorphic with moderate growth along $p^{-1}(D)$. We define the *de Rham complex with rapid decay* by

$$\mathrm{DR}^{<D}_{\widetilde{X}}\,\mathcal{M} := \mathcal{A}^{<D}_{\widetilde{X}} \otimes_{p^{-1}\mathcal{O}_X} p^{-1}\,\mathrm{DR}_X\,\mathcal{M}$$

and the *moderate de Rham complex* by

$$\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}}\,\mathcal{M} := \mathcal{A}^{\mathrm{mod}}_{\widetilde{X}} \otimes_{p^{-1}\mathcal{O}_X} p^{-1}\,\mathrm{DR}_X\,\mathcal{M}$$

6.2. With the notations in 6.1, if $\mathcal{M}$ has good formal structure along D, we define [**Hie09**, Prop 2]

$$H^{\mathrm{rd}}_k(X,\mathcal{M}) := H^{2d-k}(\widetilde{X}, \mathrm{DR}^{<D}_{\widetilde{X}}\,\mathcal{M})$$

The left-hand side is the space of *cycles with rapid decay for* $\mathcal{M}$. For a topological description justifying the terminology, we refer to [**Hie09**, 5.1].

6.3. Proof of theorem 5. — We first prove the assertion concerning the slopes of $\mathcal{H}^k f_+\mathcal{E}$. We denote by $j : U \longrightarrow X$ the canonical immersion, $d := \dim X$ and $\mathrm{Sl}_0(\mathcal{H}^k f_+\mathcal{E})$ the slopes of $\mathcal{H}^k f_+\mathcal{E}$ at 0. We will also use the letter f for the restriction of f to U. From [**HTT00**, 4.7.2], we have a canonical identification

$$(f_+\mathcal{E})^{\mathrm{an}} \simeq (f_+(j_+\mathcal{E}))^{\mathrm{an}} \xrightarrow{\ \sim\ } f^{\mathrm{an}}_+(j_+\mathcal{E})^{\mathrm{an}} \tag{6.3.1}$$

We deduce

$$\mathrm{Sl}_0(\mathcal{H}^k f_+\mathcal{E}) = \mathrm{Sl}_0(\mathcal{H}^k f^{\mathrm{an}}_+(j_+\mathcal{E})^{\mathrm{an}})$$

Let x be a local coordinate on S centred at the origin. From 2.3.1, we have

$$\mathrm{Sl}_0(\mathcal{H}^k f^{\mathrm{an}}_+(j_+\mathcal{E})^{\mathrm{an}}) = \mathrm{Sl}^{\mathrm{nb}}_x(\mathcal{H}^k f^{\mathrm{an}}_+(j_+\mathcal{E})^{\mathrm{an}})$$

Since $\mathrm{Sl}^{\mathrm{nb}}_x(\mathcal{H}^k f^{\mathrm{an}}_+(j_+\mathcal{E})^{\mathrm{an}}) \subset \mathrm{Sl}^{\mathrm{nb}}_x(f^{\mathrm{an}}_+(j_+\mathcal{E})^{\mathrm{an}})$, we deduce from theorem 2 and theorem 3

$$\mathrm{Sl}_0(\mathcal{H}^k f_+\mathcal{E}) \subset \mathrm{Sl}^{\mathrm{nb}}_{f(x)}((j_+\mathcal{E})^{\mathrm{an}}) \subset [0, r_1 + \cdots + r_n]$$

We are thus left to relate $\mathrm{Sol}(\mathcal{H}^k f^{\mathrm{an}}_+(j_+\mathcal{E})^{\mathrm{an}})$ to the periods of $\mathcal{E}_t$, for $t \neq 0$ close enough to 0. Such a relation appears for a special type of rank 1 connections in [**HR08**]. We prove more generally the following

Proposition 6.3.2. — *For every k, we have a canonical isomorphism*

$$R^k f^{\mathrm{an}}_* \,\mathrm{Sol}(j_+\mathcal{E})^{\mathrm{an}} \xrightarrow{\ \sim\ } R^k (f^{\mathrm{an}} p)_*\,\mathrm{DR}^{<D}_{\widetilde{X}}(j_+\mathcal{E}^*)^{\mathrm{an}} \tag{6.3.3}$$

For $t \neq 0$ close enough to 0, the fiber of the right-hand side of (6.3.3) *at t is canonically isomorphic to* $H^{\mathrm{rd}}_{2d-2-k}(U_t, \mathcal{E}^*_t) := H^{\mathrm{rd}}_{2d-2-k}(X^{\mathrm{an}}_t, (j_{t+}\mathcal{E}^*_t)^{\mathrm{an}})$.

Proof. — Set $\mathcal{M} := (j_+\mathcal{E}^*)^{\mathrm{an}}$. Hien duality for the De Rham cohomology of $\mathcal{E}$ on U is induced by a canonical isomorphism of sheaves

$$\mathrm{DR}^{<D}_{\widetilde{X^{\mathrm{an}}}}\,\mathcal{M}^* \simeq R\mathcal{H}om(\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X^{\mathrm{an}}}}\,\mathcal{M}, \widetilde{\iota}_!\mathbb{C})$$

We thus have

$$\begin{aligned} Rp_* \operatorname{DR}^{<D}_{\widetilde{X^{\mathrm{an}}}} \mathcal{M}^* &\simeq Rp_* R\mathcal{H}om(\operatorname{DR}^{\mathrm{mod}}_{\widetilde{X^{\mathrm{an}}}} \mathcal{M}, \widetilde{\iota}_! \mathbb{C}) \\ &\simeq R\mathcal{H}om(Rp_* \operatorname{DR}^{\mathrm{mod}}_{\widetilde{X^{\mathrm{an}}}} \mathcal{M}, \mathbb{C}) \\ &\simeq R\mathcal{H}om(\operatorname{DR}_{X^{\mathrm{an}}} \mathcal{M}, \mathbb{C}) \\ &\simeq \operatorname{Sol} \mathcal{M} \end{aligned}$$

The second isomorphism comes from Poincaré-Verdier duality and the fact that $\widetilde{\iota}_! \mathbb{C}[2 \dim X]$ is the dualizing sheaf of $\widetilde{X^{\mathrm{an}}}$. The third isomorphism comes from the projection formula and the canonical identification [**Sab00**, II 1.1.8]

$$Rp_* \mathcal{A}^{\mathrm{mod}}_{\widetilde{X^{\mathrm{an}}}} \simeq \mathcal{O}_{X^{\mathrm{an}}}(*D)$$

The last isomorphism comes from the duality theorem for $\mathcal{D}$-modules [**Meb79**][**KK81**]. By applying Rf^{an}_*, we obtain for every k and every $t \neq 0$ close enough to 0 the following commutative diagram

$$\begin{array}{ccc}
(R^k f^{\mathrm{an}}_* \operatorname{Sol} \mathcal{M})_t & \xrightarrow{\sim} & (R^k (f^{\mathrm{an}} p)_* \operatorname{DR}^{<D}_{\widetilde{X}} \mathcal{M}^*)_t \\
\downarrow{\scriptstyle (1)} & & \downarrow{\scriptstyle (6)} \\
H^k(X^{\mathrm{an}}_t, (\operatorname{Sol} \mathcal{M})_t) & & H^k(X^{\mathrm{an}}_t, (\operatorname{DR}^{<D}_{\widetilde{X}} \mathcal{M}^*)_t) \\
\downarrow{\scriptstyle (2)} & & \\
H^k(X^{\mathrm{an}}_t, \operatorname{Sol} \mathcal{M}_t) & & \downarrow{\scriptstyle (7)} \\
\downarrow{\scriptstyle (3)} & & \\
H^{-k}(X^{\mathrm{an}}_t, \mathbb{D} \operatorname{Sol} \mathcal{M}_t)^* & & H^k(X^{\mathrm{an}}_t, \operatorname{DR}^{<D_t}_{\widetilde{X}_t} \mathcal{M}^*_t) \\
\downarrow{\scriptstyle (4)} & & \downarrow{\scriptstyle \wr} \\
H^{2d-2-k}(X^{\mathrm{an}}_t, \operatorname{DR} \mathcal{M}_t)^* & & H^{\mathrm{rd}}_{2d-2-k}(X^{\mathrm{an}}_t, \mathcal{M}^*_t) \\
\downarrow{\scriptstyle (5)} & & \downarrow{\scriptstyle \mid} \\
H^{2d-2-k}(U_t, \operatorname{DR} \mathcal{E}_t)^* & \xrightarrow[(8)]{} & H^{\mathrm{rd}}_{2d-2-k}(U_t, \mathcal{E}^*_t)
\end{array}$$

By proper base change theorem, the morphisms (1) and (6) are isomorphisms. The morphism (2) is an isomorphism by non charactericity hypothesis. The morphism (3) is an isomorphism by Poincaré-Verdier duality. The morphism (4) is an isomorphism by duality theorem for $\mathcal{D}$-modules. The morphism (5) is an isomorphism by GAGA and exactness of j_{t*} where $j_t : U_t \longrightarrow X_t$ is the inclusion morphism. The morphism (8) is an isomorphism by Hien duality theorem. We deduce that (7) is an isomorphism.
□

Let $\mathbf{e} := (e_1, \ldots, e_n)$ be a local trivialization of $\mathcal{H}^k(f_+ \mathcal{E})(*0)$ in a neighbourhood of 0. One can suppose that f is smooth above $S^* := S \backslash \{0\}$. Set $U^* := U \backslash \{f^{-1}(0)\}$. From

[**DMSS00**, 1.4], we have an isomorphism of left $\mathcal{D}_S$-modules

$$\mathcal{H}^k(f_+\mathcal{E})_{|S*} \simeq R^{k+d-1} f_* \operatorname{DR}_{U*/S*} \mathcal{E}$$

where the right hand side is endowed with the Gauss-Manin connection as defined in [**KO68**]. We deduce that $(\mathbf{e}_t)_{t\neq 0}$ is an algebraic family of bases for the family of spaces $(H^{k+d-1}_{\mathrm{dR}}(X_t, \mathcal{E}_t))_{t\neq 0}$.

At the cost of shrinking S, Kashiwara perversity theorem [**Kas75**] shows that the only possibly non zero terms of the hypercohomology spectral sequence

$$E_2^{pq} = \mathcal{H}^p \operatorname{Sol} \mathcal{H}^{-q}(f_+\mathcal{E})^{\mathrm{an}}_{|S*} \Longrightarrow \mathcal{H}^{p+q} \operatorname{Sol}(f_+\mathcal{E})^{\mathrm{an}}_{|S*}$$

sit on the line $p = 0$. Hence, at the cost of shrinking S again, we have

$$\operatorname{Sol} \mathcal{H}^k(f_+\mathcal{E})^{\mathrm{an}}_{|S*} \simeq \mathcal{H}^0 \operatorname{Sol} \mathcal{H}^k(f_+\mathcal{E})^{\mathrm{an}}_{|S*} \simeq \mathcal{H}^{-k} \operatorname{Sol}(f_+\mathcal{E})^{\mathrm{an}}_{|S*} \tag{6.3.4}$$

Since Sol is compatible with proper direct image, we deduce from (6.3.1) and (6.3.4)

$$\operatorname{Sol} \mathcal{H}^k(f_+\mathcal{E})^{\mathrm{an}}_{|S*} \simeq R^{-k+d-1} f_* \operatorname{Sol}(j_+\mathcal{E})^{\mathrm{an}} \tag{6.3.5}$$

Let $s : \mathcal{H}^k(f_+\mathcal{E})^{\mathrm{an}} \longrightarrow \mathcal{O}_{S^{\mathrm{an}}}$ be a local section of $\operatorname{Sol} \mathcal{H}^k(f_+\mathcal{E})^{\mathrm{an}}$ over an open subset of $S^{*\,\mathrm{an}}$. From (6.3.5) and 6.3.2, there exists a unique continuous family $(\gamma_t)_{t\neq 0}$ of elements of the spaces $(H^{\mathrm{rd}}_{2d-2-k}(U_t, \mathcal{E}_t^*))_{t\neq 0}$ inducing s, that is

$$s(e) : t \longrightarrow \int_{\gamma_t} e_t$$

for every $e \in \mathcal{H}^k(f_+\mathcal{E})_{|S*}$. Hence, the vector function

$$t \longrightarrow \left(\int_{\gamma_t} e_{1t}, \dots, \int_{\gamma_t} e_{nt} \right)$$

satisfies the system of differential equations corresponding to $\mathcal{H}^k(f_+\mathcal{E})$, and theorem 5 is proved.

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