Contemporary Mathematics

Higher dimensional Stokes structures are rare

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The purpose of this paper is to explore the geometry of the moduli of Stokes torsors. Stokes torsors are algebraic structures encoding the Stokes phenomenon for linear differential equations. By contrast to the Stokes matrices, Stokes torsors don't involve any choice and are available in any dimension. Moduli of *local* Stokes torsors were constructed in dimension 1 by Babbit and Varadarajan [**BV89**] following a method due to Deligne. In higher dimension, moduli of *local* Stokes torsors were constructed in [**Tey19**]. Although the works of Martinet and Ramis [**MR91**] and Loday-Richaud [**Lod94**] provide a concrete description of these moduli in dimension 1, very little is known in higher dimension. Our first result says that non trivial local Stokes torsors are rare in dimension > 1.

Theorem 1. *Let $\mathcal{N}$ be a good split meromorphic flat bundle in a neighbourhood of the origin in $\mathbb{C}^n$ for some $n \geqslant 2$. Suppose that the pole locus D of $\mathcal{N}$ has at least two components. Suppose that the eigenvalues of the monodromy of $\mathcal{N}$ along the components of D are generic. Then, there are no non trivial torsors under the Stokes sheaf of $\mathcal{N}$.*

Good meromorphic flat bundles are ubiquitous in the theory of linear systems of differential equations. According to a fundamental result of Kedlaya [**Ked10**][**Ked11**] and Mochizuki [**Moc09**][**Moc11**], any meromorphic flat bundle becomes good after a pull-back by a suitable composition of blow-ups above the pole locus. Note that Theorem 1 has no counterpart in dimension 1. In dimension 1 indeed, the moduli of torsors under the Stokes sheaf of $\mathcal{N}$ does not depend on the monodromy of $\mathcal{N}$. From [**Tey18**, 2.2.1], we deduce the following rigidity theorem refining [**Tey19**, Th. 3]

Theorem 2. *Let $\mathcal{N}$ be a good split meromorphic flat bundle in a neighbourhood of the origin 0 in $\mathbb{C}^n$ for some $n \geqslant 2$. Suppose that the pole locus D of $\mathcal{N}$ has at least two components. Suppose that the eigenvalues of the monodromy of $\mathcal{N}$ along the components of D are generic. Then, $\mathcal{N}$ itself is the only germ of meromorphic flat bundle at 0 formally isomorphic to $\mathcal{N}$ at 0.*

The local Stokes sheaf admits a global variant. In the global case, moduli of Stokes torsors were constructed in any dimension in [**Tey18**]. In this note, we show the following

Theorem 3. *Let X be a smooth complex algebraic variety. Let D be a normal crossing divisor in X. Let $\mathcal{M}$ be a rank 2 good meromorphic connection on X with* *poles along D. Then, the moduli of torsors under the Stokes sheaf of $\mathcal{M}$ along D is an affine space.*

As an application of Theorem 3, we construct non trivial examples of meromorphic connections with poles along the fibre of some abelian scheme. This answers a question asked to the author by Y. André. In general, we give a conjectural description of the global moduli of Stokes torsors predicting (at least in the case where D is smooth) that they are affine spaces.

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1. Recollection on Stokes torsors

1.1. Torsors. Let M be a manifold. Let $\mathcal{G}$ be a sheaf of groups on M. We recall that a torsor under $\mathcal{G}$ is a sheaf $\mathcal{F}$ on M endowed with a left action of $\mathcal{G}$ such that there exists a cover $\mathcal{U}$ by open subsets of M such that for every $U \in \mathcal{U}$, there exists an isomorphism of sheaves $\mathcal{F}_{|U} \simeq \mathcal{G}_{|U}$ commuting with the action of $\mathcal{G}$, where $\mathcal{G}$ acts on itself by left multiplication. It is a standard fact that the isomorphism classes of $\mathcal{G}$-torsors are in bijection with $H^1(M, \mathcal{G})$, the set of non abelian cohomology classes of $\mathcal{G}$.

1.2. Geometric setup. Let X be a smooth complex algebraic variety of dimension n. Let D be a normal crossing divisor in X. For a quasi-coherent sheaf $\mathcal{F}$ on X, we denote by $\mathcal{F}_{|D}$ the sheaf of germs of sections of $\mathcal{F}$ along D. Let $D_1, \dots, D_m$ be the irreducible components of D. For $I \subset [\![1, m]\!]$, set

$$D_I := \bigcap_{i \in I} D_i \text{ and } D_I^\circ := D_I \backslash \bigcup_{i \notin I} D_i$$

1.3. Functions with asymptotic expansion along D. For $i = 1, \dots, m$, let $\widetilde{X}_i \longrightarrow X$ be the real blow-up of X along D_i. Let $p : \widetilde{X} \longrightarrow X$ be the fibre product of the $\widetilde{X}_i$, $i = 1, \dots, m$ above X. For every subset $A \subset D$, put $\partial A := p^{-1}(A)$. Let $\iota_A : \partial A \longrightarrow \partial D$ be the canonical inclusion.

Let $\mathcal{A}$ be the sheaf of functions on ∂D admitting an asymptotic expansion along D [**Sab00**]. For a closed subset Z in D, let $\mathcal{A}_{\widehat{Z}}$ be the completion of $\mathcal{A}$ along the pull-back by p of the ideal sheaf of Z. Put $\mathcal{A}^{<Z} := \operatorname{Ker}(\mathcal{A} \longrightarrow \mathcal{A}_{\widehat{Z}})$. When $Z = D$, the sheaf $\mathcal{A}^{<D}$ can be concretely described locally as follows (see proposition 1.1.11 from [**Sab00**] for a proof). Let $(x_1, \dots, x_n)$ be local coordinates centred at $0 \in D$ such that D is defined around 0 by $x_1 \cdots x_l = 0$ for some $l \in [\![1, m]\!]$. Then, the germ of $\mathcal{A}^{<D}$ at $\theta \in \partial 0$ is given by those holomorphic functions u defined over the trace on $X \backslash D$ of a neighbourhood Ω of θ in $\widetilde{X}$, and such that for every compact $K \subset \Omega$, for every $N := (N_1, \dots, N_l) \in \mathbb{N}^l$, there exists a constant $C_{K,N} > 0$ satisfying

$$|u(x)| \leqslant C_{K,N} |x_1|^{N_1} \cdots |x_l|^{N_l} \text{ for every } x \in K \cap (X \backslash D)$$

1.4. Stokes torsors and the functor of relative Stokes torsors. Let $\mathcal{M}$ be a good meromorphic connection defined in a neighbourhood of D and with poles along D. We set

$$\partial\mathcal{M} = \mathcal{A} \otimes_{p_D^{-1}\mathcal{O}_{X|D}} p^{-1}\mathcal{M}$$

Let $\mathcal{D}_X$ be the sheaf of differential operators on X. The sheaf $\mathcal{A}$ is endowed with an action of $p^{-1}\mathcal{D}_{X|D}$. Hence, so does $\partial\mathcal{M}$. We can thus form the De Rham complex of $\mathcal{M}$ with coefficients in $\mathcal{A}$ as

$$\partial\mathcal{M} \longrightarrow \partial\mathcal{M} \otimes_{p^{-1}\mathcal{O}_{X|D}} p^{-1}\Omega^1_{X|D} \longrightarrow \cdots \longrightarrow \partial\mathcal{M} \otimes_{p^{-1}\mathcal{O}_{X|D}} p^{-1}\Omega^n_{X|D}$$

It is denoted by $\operatorname{DR}\partial\mathcal{M}$. Similarly, we denote by $\operatorname{DR}^{<D}\mathcal{M}$ the De Rham complex of $\mathcal{M}$ with coefficients in $\mathcal{A}^{<D}$.

Let Z be a closed subset of D. Let $\operatorname{St}^{<Z}_{\mathcal{M}}$ be the subsheaf of $\mathcal{H}^0 \operatorname{DR}\partial \operatorname{End}\mathcal{M}$ of sections asymptotic to the Identity along Z, that is of the form $\operatorname{Id}+f$ where f has coefficients in $\mathcal{A}^{<Z}$. The sheaf $\operatorname{St}^{<Z}_{\mathcal{M}}$ is a sheaf of complex unipotent algebraic groups on ∂Z. This is *the Stokes sheaf of $\mathcal{M}$ along Z*. In particular, $\operatorname{St}^{<D}_{\mathcal{M}}$ is a sheaf on ∂D and for every point $P \in D$, the sheaf $\operatorname{St}^{<P}_{\mathcal{M}}$ is a sheaf on ∂P. The sheaf $\operatorname{St}^{<P}_{\mathcal{M}}$ will also be denoted by $\operatorname{St}_{\mathcal{M}}$ when there is no ambiguity on the point under consideration. This abuse of language is done for consistency with the notations from [**Tey19**].

Since $\operatorname{St}^{<Z}_{\mathcal{M}}$ is a sheaf of complex algebraic groups, for every $R \in \mathbb{C}$-alg, the sheaf of R-points of $\operatorname{St}^{<Z}_{\mathcal{M}}$ is a well-defined sheaf of groups on ∂Z. It is denoted by $\operatorname{St}^{<Z}_{\mathcal{M}}(R)$. This is the Stokes sheaf of $\mathcal{M}$ along Z relative to R. Torsors under $\operatorname{St}^{<Z}_{\mathcal{M}}(R)$ are the *Stokes torsors along Z relative to R*. For every subset $A \subset Z$, let $H^1(\partial A, \operatorname{St}^{<Z}_{\mathcal{M}})$ be the functor

$$\begin{array}{ccc} \mathbb{C}\text{-alg} & \longrightarrow & \operatorname{Set} \\ R & \longrightarrow & H^1(\partial A, \operatorname{St}^{<Z}_{\mathcal{M}}(R)) \end{array}$$

From [**Tey19**, Th. 1], the functor $H^1(\partial P, \operatorname{St}^{<P}_{\mathcal{M}})$ is an affine scheme of finite type over $\mathbb{C}$ for every $P \in D$. From [**Tey18**, Th. 6], the functor $H^1(\partial D, \operatorname{St}^{<D}_{\mathcal{M}})$ is an affine scheme of finite type over $\mathbb{C}$.

From [**Tey18**, 1.6.2], the presheaf of functors $R^1 p_* \operatorname{St}^{<D}_{\mathcal{M}}$ defined as

$$\begin{array}{ccc} \operatorname{Open}(D) & \longrightarrow & \operatorname{Set} \\ U & \longrightarrow & H^1(\partial U, \operatorname{St}^{<D}_{\mathcal{M}}) \end{array}$$

is a sheaf of affine schemes of finite type over $\mathbb{C}$. Its stalk at $P \in D$ is $H^1(\partial P, \operatorname{St}^{<D}_{\mathcal{M}})$. As proved in [**Tey18**, 1.7.1], the sheaf $R^1 p_* \operatorname{St}^{<D}_{\mathcal{M}}$ is constructible. More precisely, for every $I \subset [\![1, m]\!]$, the sheaf $R^1 p_* \operatorname{St}^{<D}_{\mathcal{M}}$ is locally constant on D_I°.

2. The level filtration and applications

2.1. Geometric setup. In this section, we put $X = \mathbb{C}^n$ and D is given by $x_1 \cdots x_m = 0$. In particular, we have

$$\widetilde{X} \simeq ([0, +\infty[\times S^1)^m \times \mathbb{C}^{n-m}$$

and the map $p : \widetilde{X} \longrightarrow X$ reads

$$((r_k, z_k)_k, y) \longrightarrow ((r_k z_k)_k, y)$$

In particular, $\mathbb{T} := \partial 0$ is a torus. Let $i : \mathbb{T} \longrightarrow \widetilde{X}$ and $j : X\backslash D \longrightarrow \widetilde{X}$ be the canonical inclusions. Let $\pi : \mathbb{R}^m \longrightarrow \mathbb{T}$ be the canonical projection.

2.2. Irregular values and truncation. In this paragraph, we follow part I, Chapter 2 from [**Moc11**]. We endow $\mathbb{Z}^m$ with the order given by $\mathbf{m} \leqslant \mathbf{m}'$ if and only if $m_i \leqslant m'_i$ for every $i = 1, \dots m$. For $a \in \mathcal{O}_{\mathbb{C}^n}(*D)/\mathcal{O}_{\mathbb{C}^n}$, we write $a = \sum_{\mathbf{m}\in\mathbb{Z}^m} a_{\mathbf{m}} z^{\mathbf{m}}$ and denote by $\operatorname{ord} a$ the minimum of

$$\{\mathbf{m} \in \mathbb{Z}^m_{\leqslant 0} \text{ such that } a_{\mathbf{m}} \neq 0\}$$

when it exists. Let $\mathcal{I}$ be a good set of irregular values with poles contained in D. By definition, $\mathcal{I}$ is a finite subset of $\mathcal{O}_{\mathbb{C}^n,0}(*D)/\mathcal{O}_{\mathbb{C}^n,0}$ such that

– For every non zero $a \in \mathcal{I}$, $\operatorname{ord} a$ exists and $a_{\operatorname{ord} a}$ is invertible in a neighbourhood of 0.

– For every distinct $a, b \in \mathcal{I}$, $\operatorname{ord} a - b$ exists and $(a-b)_{\operatorname{ord} a-b}$ is invertible in a neighbourhood of 0.

– The set $\Phi(\mathcal{I}) := \{\operatorname{ord} a - b, a, b \in \mathcal{I} \text{ distinct}\}$ is totally ordered.

The elements of $\Phi(\mathcal{I})$ are the *levels of* $\mathcal{I}$. In particular, the set $\{\operatorname{ord} a, a \in \mathcal{I}\}$ is totally ordered. Let $\mathbf{m}(0) \in \mathbb{Z}^m_{\leqslant 0}$ be its minimum. Let $(\mathbf{m}(0), \dots, \mathbf{m}(L), \mathbf{m}(L+1))$ be an auxiliary sequence for $\mathcal{I}$. This means that $\mathbf{m}(i+1) = \mathbf{m}(i) + (0, \dots, 1, \dots, 0)$ with 1 located in position $h_i \leqslant m$, that $\Phi(\mathcal{I}) \subset \{\mathbf{m}(0), \dots, \mathbf{m}(L+1)\}$ and that $\mathbf{m}(L+1) = 0$ by convention. We set for every $a \in \mathcal{I}$ and every $i = 0, \dots, L+1$,

$$\xi_{\mathbf{m}(i)}(a) := \sum_{\mathbf{n} \ngeqslant \mathbf{m}(i)} a_{\mathbf{n}} z^{\mathbf{n}}$$

and $a_{\geqslant \mathbf{m}(i)} := a - \xi_{\mathbf{m}(i)}(a)$.

2.3. Good unramified split bundle. For every $a \in \mathcal{I}$, set

$$\mathcal{E}^a = (\mathcal{O}_{\mathbb{C}^n,0}(*D), d - da)$$

We fix once for all a germ of split unramified good meromorphic flat bundle of rank r with poles along D

$$\mathcal{N} := \bigoplus_{a\in\mathcal{I}} \mathcal{E}^a \otimes \mathcal{R}_a$$

where the $\mathcal{R}_a$ are regular. For $i = 0, \dots, L+1$, we set $\mathcal{I}(i) := \xi_{\mathbf{m}(i)}(\mathcal{I})$ and

$$\mathcal{N}(i) := \bigoplus_{a\in\mathcal{I}} \mathcal{E}^{\xi_{\mathbf{m}(i)}(a)} \otimes \mathcal{R}_a$$

The levels of $\mathcal{N}(i)$ belong to $\{\mathbf{m}(0), \dots, \mathbf{m}(i-1)\}$. For $\alpha \in \mathcal{I}(i)$, we set

$$\mathcal{N}_\alpha := \bigoplus_{a\in\mathcal{I}, \xi_{\mathbf{m}(i)}(a)=\alpha} \mathcal{E}^a \otimes \mathcal{R}_a$$

The levels of $\mathcal{N}_\alpha$ belong to $\{\mathbf{m}(i), \dots, \mathbf{m}(L+1)\}$.

For $a \in \mathcal{I}$, let $\mathcal{L}_a(\mathcal{N})$ be the local system of flat sections for $\mathcal{R}_a$ on $\mathbb{C}^n\backslash D$. We put $\widetilde{\mathcal{L}}_a(\mathcal{N}) = i^* j_* \mathcal{L}_a(\mathcal{N})$.

2.4. The level filtration. In this subsection, we abuse notations by noting $\mathrm{St}_{\mathcal{N}}$ for $\mathrm{St}_{\mathcal{N}}^{<0}$. We recall the definition of the level filtration on $\mathrm{St}_{\mathcal{N}}$. It is a straightforward generalization of [**BV89**, II 3.2.1]. We include it for the reader's convenience due to a lack of reference in the higher dimensional case. For $i = 0, \dots, L+1$, let us set

$$\mathrm{St}_{\mathcal{N}}^i := \{g \in \mathrm{St}_{\mathcal{N}} \,|e^a(g - \mathrm{id}) \text{ has rapid decay for every } a \text{ with } \operatorname{ord} a > \mathbf{m}(i-1)\}$$

The sheaf $\mathrm{St}_{\mathcal{N}}^i$ is a sheaf of normal algebraic subgroups of $\mathrm{St}_{\mathcal{N}}$. Let us define three diagonal matrices $M := \mathrm{Diag}(e^a, a \in \mathcal{I})$, $M_{\leqslant} := \mathrm{Diag}(e^{\xi_{\mathbf{m}(i)}(a)}, a \in \mathcal{I})$ and $M_{\geqslant} := \mathrm{Diag}(e^{a_{\geqslant \mathbf{m}(i)}}, a \in \mathcal{I})$. The sheaf $\mathrm{St}_{\mathcal{N}}^i$ admits the following Stokes theoretic description:

Lemma 2.4.1. *The map*

$$\begin{array}{rcl} \varphi : \mathrm{St}_{\mathcal{N}(i)} & \longrightarrow & \mathrm{St}_{\mathcal{N}} \\ s & \longrightarrow & e^{M_{\geqslant}} s e^{-M_{\geqslant}} \end{array}$$

induces an isomorphism between $\mathrm{St}_{\mathcal{N}(i)}$ *and* $\mathrm{St}_{\mathcal{N}}^i$.

Proof. The statement is local. Hence, it is enough to work on an open set $\mathcal{S}$ contained in a product of strict open intervals. For such an open set, a choice of fundamental matrix F of flat sections for $\bigoplus_{a \in \mathcal{I}} \mathcal{R}_a$ yields a commutative diagram with injective arrows

$$\begin{tikzcd} \Gamma(\mathcal{S}, \mathrm{St}_{\mathcal{N}(i)}) \arrow[r, "\varphi"] \arrow[dr, "\widetilde{\iota}"'] & \Gamma(\mathcal{S}, \mathrm{St}_{\mathcal{N}}) \arrow[d, "\iota"] \\ & \mathrm{GL}_r \end{tikzcd} \tag{2.4.2}$$

where ι is given by $s \longrightarrow e^{-M} F^{-1} s F e^{M}$ and where $\widetilde{\iota}$ is given by $s \longrightarrow e^{-M_{\leqslant}} F^{-1} s F e^{M_{\leqslant}}$. By definition, $\iota(\Gamma(\mathcal{S}, \mathrm{St}_{\mathcal{N}}))$ is the subgroup of elements $g \in \mathrm{GL}_r$ such that for every $a, b \in \mathcal{I}$,

$$\begin{cases} g_{aa} = \mathrm{id} \\ g_{ab} = 0 \qquad \text{if } a \neq b \text{ and } a \nless_{\mathcal{S}} b \end{cases}$$

Hence, $\widetilde{\iota}(\Gamma(\mathcal{S}, \mathrm{St}_{\mathcal{N}(i)}))$ is the subgroup of elements of $g \in GL_r$ such that for every $a, b \in \mathcal{I}$,

$$\begin{cases} g_{aa} = \mathrm{id} \\ g_{ab} = 0 \qquad \text{if } \xi_{\mathbf{m}(i)}(a) \neq \xi_{\mathbf{m}(i)}(b) \text{ and } \xi_{\mathbf{m}(i)}(a) \nless_{\mathcal{S}} \xi_{\mathbf{m}(i)}(b) \\ g_{ab} = 0 \qquad \text{if } a \neq b \text{ and } \xi_{\mathbf{m}(i)}(a) = \xi_{\mathbf{m}(i)}(b) \end{cases}$$

Note that if $a, b \in \mathcal{I}$ with $\xi_{\mathbf{m}(i)}(a) \neq \xi_{\mathbf{m}(i)}(b)$, then

$$a \nless_{\mathcal{S}} b \text{ if and only if } \xi_{\mathbf{m}(i)}(a) \nless_{\mathcal{S}} \xi_{\mathbf{m}(i)}(b)$$

Thus, $\widetilde{\iota}(\Gamma(\mathcal{S}, \mathrm{St}_{\mathcal{N}(i)}))$ is the subgroup of elements $g \in \iota(\Gamma(\mathcal{S}, \mathrm{St}_{\mathcal{N}}))$ such that for every $a, b \in \mathcal{I}$,

$$g_{ab} = 0 \text{ if } \xi_{\mathbf{m}(i)}(a) = \xi_{\mathbf{m}(i)}(b) \text{ and } a \neq b$$

Let $s \in \Gamma(\mathcal{S}, \mathrm{St}_{\mathcal{N}}^i)$, and let $a, b \in \mathcal{I}$ with $a \neq b$. If $\xi_{\mathbf{m}(i)}(a) = \xi_{\mathbf{m}(i)}(b)$, then

$$\iota(s)_{ab} = e^{b-a} F_a^{-1} s_{ab} F_b = F_a^{-1}(e^{b_{\geqslant \mathbf{m}(i)} - a_{\geqslant \mathbf{m}(i)}} s_{ab}) F_b$$

By definition, $e^{b_{\geqslant \mathbf{m}(i)} - a_{\geqslant \mathbf{m}(i)}} s_{ab}$ has rapid decay. Since F_a and F_b have moderate growth at 0, we deduce that the constant matrix $\iota(s)_{ab}$ has rapid decay. Hence,

$\iota(s)_{ab} = 0$. Thus $\mathrm{St}^i_{\mathcal{N}} \subset \varphi(\mathrm{St}_{\mathcal{N}(i)})$. On the other hand, let $s \in \varphi(\mathrm{St}_{\mathcal{N}(i)})$ and let c with $\operatorname{ord} c > \mathbf{m}(i-1)$. We have to show that for every $a, b \in \mathcal{I}$ with $a \neq b$,

$$e^c s_{ab} = e^{c+a-b} F_a \iota(s)_{ab} F_b^{-1}$$

has rapid decay. We can suppose $\xi_{\mathbf{m}(i)}(a) <_{\mathcal{S}} \xi_{\mathbf{m}(i)}(b)$. In particular $a <_{\mathcal{S}} b$. Since the leading term of $c+a-b$ is the leading term of $a-b$, the exponential e^{c+a-b} has rapid decay on $\mathcal{S}$. Thus, so does $e^c s_{ab}$. Hence, $s \in \mathrm{St}^i_{\mathcal{N}}$ and we deduce $\mathrm{St}^i_{\mathcal{N}} = \varphi(\mathrm{St}_{\mathcal{N}(i)})$. □

2.5. Quotients of the level filtration.

LEMMA 2.5.1. *There is a split exact sequence of sheaves of algebraic groups*

$$1 \longrightarrow \mathrm{St}_{\mathcal{N}(i)} \xrightarrow{\varphi} \mathrm{St}_{\mathcal{N}} \xrightarrow{\psi} \prod_{\alpha \in \mathcal{I}(i)} \mathrm{St}_{\mathcal{N}_\alpha} \longrightarrow 1$$

In particular, $\mathrm{Gr}^i \mathrm{St}_{\mathcal{N}} := \mathrm{St}^{i+1}_{\mathcal{N}} / \mathrm{St}^i_{\mathcal{N}} \simeq \prod_{\alpha \in \mathcal{I}(i)} \mathrm{St}_{\mathcal{N}(i+1)_\alpha}$

PROOF. Let us define

$$\begin{aligned} \psi : \mathrm{St}_{\mathcal{N}} &\longrightarrow \prod_{\alpha \in \mathcal{I}(i)} \mathrm{St}_{\mathcal{N}_\alpha} \\ s &\longrightarrow \times (s_{ab})_{\substack{\xi_{\mathbf{m}(i)}(a)=\alpha \\ \xi_{\mathbf{m}(i)}(b)=\alpha}} \end{aligned}$$

From the local description of $\mathrm{St}_{\mathcal{N}(i)}$ given in the proof of 2.4.1, we see that the only a priori non obvious thing to prove is the fact that ψ is a group homomorphism. Let $\mathcal{S}$ be an open set of $\mathbb{T}$, let $s, t \in \Gamma(\mathcal{S}, \mathrm{St}_{\mathcal{N}})$, let $\alpha \in \mathcal{I}(i)$ and let $a, b \in \mathcal{I}$ such that $\xi_{\mathbf{m}(i)}(a) = \xi_{\mathbf{m}(i)}(b) = \alpha$. Let us denote by ψ_α the component of ψ associated to α. Then

$$(\psi_\alpha(st))_{ab} = \sum_{c \in \mathcal{I}} s_{ac} t_{cb} = \sum_{\substack{c \in \mathcal{I} \\ a \leqslant_{\mathcal{S}} c \leqslant_{\mathcal{S}} b}} s_{ac} t_{cb}$$

If $\xi_{\mathbf{m}(i)}(c) \neq \alpha$, the leading coefficient of $c - a$ is that of $\xi_{\mathbf{m}(i)}(c) - \xi_{\mathbf{m}(i)}(a) = \xi_{\mathbf{m}(i)}(c) - \alpha$. Hence, $a \leqslant_{\mathcal{S}} c$ if and only if $\alpha <_{\mathcal{S}} \xi_{\mathbf{m}(i)}(c)$. Similarly, $c \leqslant_{\mathcal{S}} b$ if and only if $\xi_{\mathbf{m}(i)}(c) <_{\mathcal{S}} \alpha$. Hence, for $\xi_{\mathbf{m}(i)}(c) \neq \alpha$, the condition $a \leqslant_{\mathcal{S}} c \leqslant_{\mathcal{S}} b$ is empty. Thus

$$(\psi_\sigma(st))_{ab} = \sum_{\substack{c \in \mathcal{I} \\ \xi_{\mathbf{m}(i)}(c)=\alpha}} s_{ac} t_{cb} = (\psi_\alpha(s)\psi_\alpha(t))_{ab}$$

□

2.6. Action of the fundamental group in the local one level case. We consider in this paragraph the case where $\mathcal{N}$ has a unique level $\mathbf{m}$ and we suppose that $\mathcal{N}$ is not regular, that is $\mathbf{m} \in \mathbb{Z}^m_{\leqslant 0} \backslash \{0\}$. Let $\iota : C \longrightarrow \mathbb{C}^n$ be a smooth curve passing through 0 and not contained in D. Let $\widetilde{C}$ be the real-blow up of C at 0. Let S^1_C be the boundary of $\widetilde{C}$. Following [**Sab12**, 8.b p120], the map ι lifts as a map $\widetilde{\iota} : \widetilde{C} \longrightarrow \widetilde{X}$. Let us suppose that the restricted map $S^1_C \longrightarrow \mathbb{T}$ is injective. Let $H_{\mathbf{m}}$ be the hyperplane of $\mathbb{R}^m$ defined by $\sum_{i=1}^m m_i x_i = 0$. For an interval I of S^1_C, set $\mathbb{T}(H, I) := \pi(H) + I$ and

$$\mathbb{T}(\mathbf{m}, I) := \mathbb{T}(H_{\mathbf{m}}, I)$$

For $x \in \mathbb{T}$, the translation t_x by x provides an isomorphism

$$\pi_1(\mathbb{T}(\mathbf{m}, \pi(0)), \pi(0)) \longrightarrow \pi_1(\mathbb{T}(\mathbf{m}, x), x)$$

Hence, $\pi_1(\mathbb{T}(\mathbf{m},\pi(0)),\pi(0))$ acts on $\mathrm{St}_{\mathcal{N}_C} \simeq (\mathrm{St}_{\mathcal{N}})_{|S^1_C}$ via the parallel transport. We deduce that $\pi_1(\mathbb{T}(\mathbf{m},\pi(0)),\pi(0))$ acts on $H^1(S^1_C,\mathrm{St}_{\mathcal{N}_C})$. To simplify notations, we denote by $H^1(S^1_C,\mathrm{St}_{\mathcal{N}_C})^{\pi_1}$ the invariants for this action.

LEMMA 2.6.1. *For every cover $\underline{I}$ of S^1_C adapted to $\mathcal{N}_C$, the morphisms in the commutative triangle*

$$\begin{tikzcd} H^1(\mathbb{T},\mathrm{St}_{\mathcal{N}}) \arrow[rr, "\mathbf{res}_C"] & & H^1(S^1_C,\mathrm{St}_{\mathcal{N}_C})^{\pi_1} \\ & Z^1(\mathbb{T}(\mathbf{m},\underline{I}),\mathrm{St}_{\mathcal{N}}) \arrow[ul, "f"] \arrow[ur] & \end{tikzcd} \tag{2.6.2}$$

are isomorphisms. In particular, $H^1(\mathbb{T},\mathrm{St}_{\mathcal{N}})$ is an affine space.

For a single level connection in dimension 1, an *adapted cover* is a finite cover $\underline{I}$ by open intervals such that for $I_1, I_2, I_3 \in \underline{I}$ distinct, we have $I_1 \cap I_2 \cap I_3 = \varnothing$, and such that for every $I \in \underline{I}$ and every pair of distinct irregular values $\{a,b\}$, the interval I contains exactly one Stokes direction associated to $a-b$.

PROOF. We first show that f is surjective. Let $\mathcal{T} \in H^1(\mathbb{T},\mathrm{St}_{\mathcal{N}})$. Let $x = (x_1,\dots,x_m) \in \pi(H_{\mathbf{m}})$. If C is the curve $t \longrightarrow (c_1(t),\dots,c_n(t))$, let C_x be the curve given by $t \longrightarrow (x_1c_1(t),\dots,x_mc_m(t),\dots,c_n(t))$. By construction $S^1_{C_x} = x + S^1_C$. Since $\mathcal{N}$ has only one level, the same holds for $\mathcal{N}_{C_x}$. Since $\underline{I}$ is an adapted cover for $\mathcal{N}_C$, the translated cover $x+\underline{I}$ is an adapted cover for $\mathcal{N}_C$. From [**BV89**], the restriction $\mathcal{T}_{x+S^1_C} \in H^1(x+S^1_C,\mathrm{St}_{\mathcal{N}_{C_x}})$ of $\mathcal{T}$ to $x+S^1_C$ admits a unique trivialisation $t_I(x)$ on each $x+I \in \underline{I}$, and

$$Z^1(\underline{I},\mathrm{St}_{\mathcal{N}_C}) \xrightarrow{\sim} H^1(S^1_C,\mathrm{St}_{\mathcal{N}_C}) \tag{2.6.3}$$

Since $\mathrm{St}_{\mathcal{N}}$ has no non trivial section on $x+I$ for every $x \in \pi(H_{\mathbf{m}})$ and every $I \in \underline{I}$, the sections $(t_I(x))_{x\in\pi(H_{\mathbf{m}})}$ with I fixed glue into a section of $\mathcal{T}$ on $\mathbb{T}(\mathbf{m},I)$. Hence, the cocycle corresponding to $\mathcal{T}_{S^1_C}$ via (2.6.3) extends uniquely into a cocycle for $\mathcal{T}$ relative to the cover $\mathbb{T}(\mathbf{m},\underline{I})$. In particular, f is surjective.

Let us consider the commutative diagram

$$\begin{tikzcd} Z^1(\mathbb{T}(\mathbf{m},\underline{I}),\mathrm{St}_{\mathcal{N}}) \arrow[r, "f"] \arrow[d] & H^1(\mathbb{T},\mathrm{St}_{\mathcal{N}}) \arrow[d] \\ Z^1(\underline{I},\mathrm{St}_{\mathcal{N}_C}) \arrow[r, "\sim"] & H^1(S^1_C,\mathrm{St}_{\mathcal{N}_C}) \end{tikzcd} \tag{2.6.4}$$

Every section of $\mathrm{St}_{\mathcal{N}}$ on a connected open set is determined by its germ at a point. Hence, the left vertical arrow of (2.6.4) is injective. We deduce that f is bijective. Tautologically, the image of the left vertical arrow is formed by those $g \in Z^1(\underline{I},\mathrm{St}_{\mathcal{N}_C})$ extending to $\mathbb{T}(\mathbf{m},\underline{I})$. These are exactly the invariants under the action of $\pi_1(\mathbb{T}(\mathbf{m},\pi(0)),\pi(0))$. Taking the invariants under $\pi_1(\mathbb{T}(\mathbf{m},\pi(0)),\pi(0))$

thus yields a commutative diagram

$$\begin{tikzcd} Z^1(\mathbb{T}(\mathbf{m},\underline{I}),\mathrm{St}_{\mathcal{N}}) \arrow[r, "f"] \arrow[d, "\wr"'] & H^1(\mathbb{T},\mathrm{St}_{\mathcal{N}}) \arrow[d, "\mathbf{res}_C"] \\ Z^1(\underline{I},\mathrm{St}_{\mathcal{N}_C})^{\pi_1} \arrow[r, "\sim"] & H^1(S^1_C,\mathrm{St}_{\mathcal{N}_C})^{\pi_1} \end{tikzcd} \tag{2.6.5}$$

We deduce that $\mathbf{res}_C$ is bijective. To conclude, we observe that $\mathrm{St}_{\mathcal{N}}$ being a sheaf of unipotent algebraic groups, the scheme $\Gamma(U,\mathrm{St}_{\mathcal{N}})$ is an affine space for every open subset $U\subset\mathbb{T}$. Since $Z^1(\mathbb{T}(\mathbf{m},\underline{I}),\mathrm{St}_{\mathcal{N}})$ is a product of such schemes, it is also an affine space. □

2.7. Proof of Theorem 1. Let $\mathcal{N}$ be a good split meromorphic flat bundle in a neighbourhood of the origin in $\mathbb{C}^n$ for some $n\geqslant 2$ as in paragraph 2.3. Suppose that the pole locus D of $\mathcal{N}$ has at least two components. Suppose that the eigenvalues of the monodromy of $\mathcal{N}$ along the components of D are generic. We want to show that $H^1(\mathbb{T},\mathrm{St}_{\mathcal{N}})$ is reduced to the trivial torsor. We argue recursively on the number of levels of $\mathcal{N}$. The notations from section 2 will be in use. Suppose that $\mathcal{N}$ has only one level $\mathbf{m}$. Let Δ be the diagonal curve of $\mathbb{C}^n$. Let $\underline{I}$ be a cover of S^1_Δ adapted to $\mathcal{N}_\Delta$. Fix $I_1,I_2\in\underline{I}$, put $J=I_1\cap I_2$ and suppose without loss of generality that $\pi(0)\in J$. Note that $\mathbb{T}(\mathbf{m},J)$ is homotopic to the product of $m-1$ circles. By assumption, $m>1$. Hence, $\pi_1(\mathbb{T}(\mathbf{m},J),\pi(0))$ is non trivial. Pick $\gamma\in\pi_1(\mathbb{T}(\mathbf{m},J),\pi(0))$ non trivial and let $\mathbf{n}=(n_1,\dots,n_m)\in\mathbb{Z}^m$ be the coordinates of γ in the canonical basis of $\pi_1(\mathbb{T},\pi(0))$. For $a\in\mathcal{I}$ and $i=1,\dots,m$, let $\Sigma_a(i)$ be the set of eigenvalues for the monodromy of $\mathcal{L}_a(\mathcal{N})$ around D_i. Then, the eigenvalues for the monodromy of $\widetilde{\mathcal{L}}_b(\mathcal{N})^*\otimes\widetilde{\mathcal{L}}_a(\mathcal{N})$ along γ are the

$$\{\prod_{i=1}^m\left(\frac{\lambda_i}{\mu_i}\right)^{n_i},\lambda_i\in\Sigma_a(i),\mu_i\in\Sigma_b(i),i=1,\dots,m\}$$

Since $\mathbf{n}\neq 0$, the zero locus of the polynomial

$$\prod_{i=1}^m x_i^{n_i}-\prod_{i=1}^m y_i^{n_i}$$

is a strict closed subset in $\mathbb{C}^{2m}$. Hence, for a generic choice of the $\Sigma_a(i)$, the restriction of $\widetilde{\mathcal{L}}_b(\mathcal{N})^*\otimes\widetilde{\mathcal{L}}_a(\mathcal{N})=\widetilde{\mathcal{L}}_{a-b}(\mathrm{End}\,\mathcal{N})$, $a\neq b$ to $\mathbb{T}(\mathbf{m},J)$ cannot have non zero global sections. Hence, the Identity is the only section of $\mathrm{St}_{\mathcal{N}}$ on $\mathbb{T}(\mathbf{m},J)$. Thus, Theorem 1 in the one level case is a consequence of lemma 2.6.1.

Suppose that $\mathcal{N}$ has at least two levels. Let $(\mathbf{m}(0),\dots,\mathbf{m}(L),\mathbf{m}(L+1))$ be an auxiliary sequence for $\mathcal{I}$. Then, there is an index i such that $\mathcal{N}(i)$ has only one level and such that the number of levels of $\mathrm{St}_{\mathcal{N}_\alpha}$ is strictly less than number of levels of $\mathcal{N}$ for every $\alpha\in\mathcal{I}(i)$. Since the $\mathcal{N}_\alpha$ are direct summands of $\mathcal{N}$, they have generic monodromy eigenvalues along the irreducible components of D. By recursion hypothesis applied to the $\mathcal{N}_\alpha$, we obtain that the right term in the exact sequence of pointed sets

$$H^1(\mathbb{T},\mathrm{St}_{\mathcal{N}(i)})\longrightarrow H^1(\mathbb{T},\mathrm{St}_{\mathcal{N}})\longrightarrow\prod\nolimits_{\alpha\in\mathcal{I}(i)}H^1(\mathbb{T},\mathcal{N}_\alpha)$$

deduced from lemma 2.5.1 is trivial. Hence $H^1(\mathbb{T},\mathrm{St}_{\mathcal{N}})\simeq H^1(\mathbb{T},\mathrm{St}_{\mathcal{N}(i)})$. Note that the eigenvalues of the monodromy of $\mathcal{N}(i)$ along the irreducible components of

D are generic. Hence, the one level case treated above implies that $H^1(\mathbb{T}, \mathrm{St}_{\mathcal{N}(i)})$ is a point. This finishes the proof of Theorem 1.

3. Moduli of Stokes torsors and the relative Stokes group

3.1. Roadmap. The goal of this section is to describe the global moduli of Stokes torsors constructed in [**Tey18**] when the irregular values have only one level. To do this, we compare a relative version of the Stokes group from [**MR91**][**Lod94**] with the relative non abelian cohomology of the Stokes sheaf. For the problem raised by this comparison in the multi-level case, we refer to the discussion in 3.6. Note that over a smooth base (corresponding in this paper to the case where D is smooth), relative Stokes groups appeared in the one level case in [**JMU81**] and in more generality in [**Boa02**]. In particular, over a smooth base, they were already considered in the multi-level case in [**Boa14**].

3.2. Loday-Richaud's theorem. In this subsection, the notations from section 1 will be in use and we will suppose that X is a neighbourhood of the origin 0 in the affine line. Let $\mathcal{M}$ be an unramified meromorphic connection on X with poles at 0. Let $\mathcal{I}$ be the set of irregular values of $\mathcal{M}$ at 0. Let $p : \widetilde{X} \longrightarrow X$ be the real blow-up of X at 0. For $a, b \in \mathcal{I}$, the function $G_{a,b} := (a-b)/|a-b|$ induces a C^∞-function $\partial G_{a,b}$ on $\partial 0$. The *anti-Stokes directions of* (a,b) are the directions of $z \in \partial 0$ such that $\partial G_{a,b}(z) \in \mathbb{R}^-$. Let $H^<$ be the set of all anti-Stokes directions, for $a, b \in \mathcal{I}$ distinct. For $z \in H^<$, we put

$$\mathbb{S}to_{\mathcal{M},z} = \{g \in \mathrm{St}_{\mathcal{M},z} \text{ such that for every } a, b \in \mathcal{I} \text{ distinct}, g_{ab} = 0 \text{ unless } z \in H^<_{ab}\}$$

and

$$\mathbb{S}to_{\mathcal{M}} := \prod_{z \in H^<} \mathbb{S}to_{\mathcal{M},z}$$

The group $\mathbb{S}to_{\mathcal{M}}$ is the *Stokes group of* $\mathcal{M}$. For a possibly ramified connection $\mathcal{M}$, we define the Stokes group of $\mathcal{M}$ via Galois descent from the unramified case. Note that an element of the Stokes group $g = (g_z)_{z \in H^<}$ induces a well-defined Stokes torsor. To see this, let $N \geqslant 1$ be an integer and let $(z_i)_{i \in \mathbb{Z}/N\mathbb{Z}}$ be the anti-Stokes directions of $\mathcal{M}$ ordered in an ascending order with respect to the clockwise orientation on $\partial 0$. Pick $i \in \mathbb{Z}/N\mathbb{Z}$. Since $\mathrm{St}_{\mathcal{M}}$ is constructible, g_{z_i} extends into a section $\widetilde{g}_{z_i}$ of $\mathrm{St}_{\mathcal{M}}$ on a small interval $]z_i - \epsilon, z_i + \epsilon[$ for $\epsilon > 0$. At the cost of shrinking ϵ, we can suppose that

(1) The intervals $I_i = (]z_i - \epsilon, z_{i+1} + \epsilon[)_{i \in \mathbb{Z}/N\mathbb{Z}}$ contain exactly two anti-Stokes directions, namely z_i and z_{i+1}.

(2) We have $I_i \cap I_j \cap I_k = \varnothing$ for every $i, j, k \in \mathbb{Z}/N\mathbb{Z}$ distinct.

In particular, $(\widetilde{g}_{z_i})_{i \in \mathbb{Z}/N\mathbb{Z}}$ defines a 1-cocycle for $\mathrm{St}_{\mathcal{M}}$ with respect to the cover $(I_i)_{i \in \mathbb{Z}/N\mathbb{Z}}$. Let $\mathrm{LR}_0(g)$ be the associated Stokes torsor on $\partial 0$. It is independent of ϵ for ϵ small enough. The following theorem is due to Loday-Richaud [**Lod94**, II.2.1]

THEOREM 4. *The map*

$$\mathrm{LR}_0 : \mathbb{S}to_{\mathcal{M}} \longrightarrow H^1(\partial 0, \mathrm{St}_{\mathcal{M}}) \tag{3.2.1}$$

is an isomorphism.

3.3. Relative Stokes groups. In this subsection, we introduce a relative version of the Stokes group. The notations from section 1 will be in use. Let X be a smooth complex algebraic variety and let D be a normal crossing divisor in X. Let $\mathcal{M}$ be a good meromorphic connection on X with poles along D. Let $\mathcal{I}$ be the sheaf of irregular values of $\mathcal{M}$. We first suppose that $\mathcal{M}$ is unramified. In particular, $\mathcal{I}$ is a subsheaf of $\mathcal{O}_X(*D)/\mathcal{O}_X$. Let $p:\widetilde{X}\longrightarrow X$ be the real blow-up of X along D. For $a,b\in\mathcal{I}$, the function $G_{a,b}:=(a-b)/|a-b|$ induces a C^∞-function $\partial G_{a,b}$ on ∂D. The *anti-Stokes hyperplanes of* (a,b) are the irreducible components of

$$H_{ab}^{<}:=\{z\in\partial D \text{ such that } \partial G_{a,b}(z)\in\mathbb{R}^-\}$$

The set $H_{ab}^{<}$ is a smooth C^∞-hypersurface in ∂D. Let $H^{<}$ be the union of all the $H_{ab}^{<}$, $a,b\in\mathcal{I}$ distinct. Let Z be a closed subset of D. Let $\mathbb{S}to_{\mathcal{M}}^{<Z}$ be the sheaf on $\partial Z\cap H^{<D}$ whose germ at z is

$$\mathbb{S}to_{\mathcal{M},z}^{<Z}=\{g\in \mathrm{St}_{\mathcal{M},z}^{<Z} \text{ such that for every } a,b\in\mathcal{I} \text{ distinct}, g_{ab}=0 \text{ unless } z\in H_{ab}^{<}\}$$

We call $p_*\mathbb{S}to_{\mathcal{M}}^{<Z}$ the *relative Stokes group of* $\mathcal{M}$ *along* Z. Note that when Z is a point $P\in D$, the sheaf $\mathbb{S}to_{\mathcal{M}}^{<P}$ will also be denoted by $\mathbb{S}to_{\mathcal{M}}$ when there is no ambiguity on the point under consideration.

For a possibly ramified connection $\mathcal{M}$, we define the relative Stokes group of $\mathcal{M}$ via Galois descent from the unramified case.

Lemma 3.3.1. *If D is smooth, the sheaf $p_*\mathbb{S}to_{\mathcal{M}}^{<D}$ is a local system on D.*

Proof. For every $a,b\in\mathcal{I}$, the Stokes hyperplanes of (a,b) are parallel to the anti-Stokes hyperplanes of (a,b). Hence, $H_{ab}^{<}$ does not meet any Stokes hyperplane of (a,b). Thus, for any $z\in\partial D$ and any $g\in\mathbb{S}to_{\mathcal{M},z}^{<D}$, the germ g extends uniquely on a small product $\Delta\times I$ containing z, where Δ is a disc in D centred at $p(z)$ and where I is an interval of S^1. Both Δ and I depend only on z and not on g. This concludes the proof of lemma 3.3.1. □

3.4. Comparison over a point in the one level case.

Proposition 3.4.1. *Let X be a smooth complex algebraic variety and let D be a normal crossing divisor in X. Let $P\in D$. Let $\mathcal{M}$ be a good meromorphic connection on X with poles along D. Suppose that $\mathcal{M}$ has a single level. Then, there is a canonical isomorphism*

$$\mathrm{LR}_P:\Gamma(\partial P,\mathbb{S}to_{\mathcal{M}})\longrightarrow H^1(\partial P,\mathrm{St}_{\mathcal{M}}) \tag{3.4.2}$$

Proof. By Galois descent, we can suppose that $\mathcal{M}$ is unramified. Let $\mathbf{m}\in\mathbb{Z}_{\leqslant 0}^m\backslash\{0\}$ be the level of $\mathcal{M}$. We are going to construct a cover of ∂P similar to that used by Loday-Richaud in Theorem 4. Let $(x_1,\dots,x_n)$ be local coordinates centred at P such that D is defined by $x_1\cdots x_m=0$ in a neighbourhood of P in X. For $a,b\in\mathcal{I}$ distinct, we have

$$a-b=f_{ab}x^{\mathbf{m}}$$

with $f_{ab}(0)\neq 0$. Put $-g_{ab}^{-1}=f_{ab}/|f_{ab}|$. Then, the map $\partial G_{a,b}$ reads

$$((r_k,z_k)_{1\leqslant k\leqslant m},(x_k)_{m+1\leqslant k\leqslant n})\longrightarrow -g_{ab}(r_kz_k,x_k)^{-1}z^{\mathbf{m}} \tag{3.4.3}$$

If $\phi_m:\partial P\longrightarrow S^1$ is the map $z\longrightarrow z^{\mathbf{m}}$, then

$$H_{ab}^{<}\cap\partial P=\{z\in\partial D \text{ with } \phi_m(z)=g_{ab}(0)\}$$

Let us re-index the g_{ab} with $\mathbb{Z}/N\mathbb{Z}$ in such a way that the $(g_i(0))_{i\in\mathbb{Z}/N\mathbb{Z}}$ are in an ascending order with respect to the clockwise orientation on S^1. Then, at the cost of shrinking ϵ, the connected components of the

$$\phi_m^{-1}(]g_i(0)-\epsilon, g_{i+1}(0)+\epsilon[), i \in \mathbb{Z}/N\mathbb{Z}$$

form a cover $\mathcal{S} = (S_i)_{i\in\mathbb{Z}/N'\mathbb{Z}}$ of ∂P by open multi-sectors such that

(1) The open set S_i contains exactly two anti-Stokes hyperplanes.

(2) The open set $S_i \cap S_{i+1}$ contains exactly one anti-Stokes hyperplane for every $i \in \mathbb{Z}/N'\mathbb{Z}$.

(3) We have $S_i \cap S_j \cap S_k = \varnothing$ for every $i, j, k \in \mathbb{Z}/N'\mathbb{Z}$ distinct.

At the cost of shrinking ϵ, a section $g \in \Gamma(\partial P, \mathbb{S}to_{\mathcal{M}})$ extends uniquely into a 1-cocycle for $\mathrm{St}_{\mathcal{M}}$ with respect to the cover $\mathcal{S}$. Taking the associated Stokes torsor defines the map LR_P from (3.4.2). We now have to show that this map is an isomorphism.

Let C be a smooth curve passing through P and not contained in D. Then, we have $(\mathrm{St}_{\mathcal{M}})_{|S^1_C} \simeq \mathrm{St}_{\mathcal{M}_C}$ and $\mathcal{M}_C$ admits only one level. Thus, restriction to S^1_C provides a commutative diagram

$$\begin{tikzcd} \Gamma(\partial P, \mathbb{S}to_{\mathcal{M}}) \arrow[r] \arrow[d, "\mathrm{LR}_P"'] & \Gamma(S^1_C, \mathbb{S}to_{\mathcal{M}_C}) \arrow[d] \\ H^1(\partial P, \mathrm{St}_{\mathcal{M}}) \arrow[r] & H^1(S^1_C, \mathrm{St}_{\mathcal{M}_C}) \end{tikzcd} \tag{3.4.4}$$

From Loday-Richaud's theorem, the right vertical map of (3.4.4) is an isomorphism. Taking the invariants under the action of $\pi_1(\mathbb{T}(\mathbf{m}, \pi(0)), \pi(0))$ on the right part of the diagram (3.4.4) gives a commutative diagram

$$\begin{tikzcd} \Gamma(\partial P, \mathbb{S}to_{\mathcal{M}}) \arrow[r, "\sim"] \arrow[d, "\mathrm{LR}_P"'] & \Gamma(S^1_C, \mathbb{S}to_{\mathcal{M}_C})^{\pi_1} \arrow[d, "\wr"] \\ H^1(\partial P, \mathrm{St}_{\mathcal{M}}) \arrow[r, "\mathbf{res}_C"'] & H^1(S^1_C, \mathrm{St}_{\mathcal{M}_C})^{\pi_1} \end{tikzcd} \tag{3.4.5}$$

From lemma 2.6.1, the map $\mathbf{res}_C$ is an isomorphism. Hence, LR_P is an isomorphism. This concludes the proof of proposition 3.4.1. □

3.5. Proof of Theorem 3. We are going to prove something slightly stronger than Theorem 3, that is the following

THEOREM 5. *Let X be a smooth complex algebraic variety and let D be a normal crossing divisor in X. Let $\mathcal{M}$ be a good meromorphic connection on X with poles along D. Suppose that the Stokes hyperplanes of $\mathcal{M}$ don't meet and that locally along D, the connection $\mathcal{M}$ admits only one level. Then, $H^1(\partial D, \mathrm{St}^{<D}_{\mathcal{M}})$ is an affine space.*

PROOF. Let $\mathcal{U}$ be a cover of D by small enough open subsets of D on which the level of $\mathcal{M}$ is defined. Since $R^1 p_* \mathrm{St}^{<D}_{\mathcal{M}}$ is a sheaf, we have

$$H^1(\partial D, \mathrm{St}^{<D}_{\mathcal{M}}) \simeq \lim_{U\in\mathcal{U}} H^1(\partial U, \mathrm{St}^{<D}_{\mathcal{M}}) \tag{3.5.1}$$

Let $U \in \mathcal{U}$. Since $\mathcal{M}$ has locally only one level, if two distinct irregular values $a_0, b_0 \in \Gamma(U, \mathcal{I})$ are such that $a_0 - b_0$ does not have poles along every irreducible component of U, then so does $a - b$ for every $a, b \in \Gamma(U, \mathcal{I})$ distinct. Thus, $\mathrm{St}_{\mathcal{M}}^{<D}$ is trivial on U. So $H^1(\partial U, \mathrm{St}_{\mathcal{M}}^{<D})$ is trivial. Hence the limit (3.5.1) can be taken over the open subsets $U \in \mathcal{U}$ for which for every $a, b \in \Gamma(U, \mathcal{I})$ distinct, the difference $a-b$ has poles along every irreducible component of U. In particular we can suppose that $\mathcal{M}$ is *very good*, that is for every open set $V \subset D$, for every $a, b \in \Gamma(V, \mathcal{I})$ distinct, the difference $a - b$ has poles along every irreducible component of V. In that case, for every $P \in D$, we have

$$(\mathrm{St}_{\mathcal{M}}^{<D})_{|\partial P} \simeq \mathrm{St}_{\mathcal{M}}^{<P} \text{ and } (\mathbb{S}to_{\mathcal{M}}^{<D})_{|\partial P} \simeq \mathbb{S}to_{\mathcal{M}}^{<P} \tag{3.5.2}$$

In that case, we show the following more precise result

Proposition 3.5.3. *Let X be a smooth complex algebraic variety and let D be a normal crossing divisor in X. Let $\mathcal{M}$ be a very good meromorphic connection on X with poles along D. Suppose that the Stokes hyperplanes of $\mathcal{M}$ don't meet and that locally along D, the connection $\mathcal{M}$ admits only one level. Then, there is an isomorphism of schemes*

$$\Gamma(\partial D, \mathbb{S}to_{\mathcal{M}}^{<D}) \longrightarrow H^1(\partial D, \mathrm{St}_{\mathcal{M}}^{<D}) \tag{3.5.4}$$

In particular, $H^1(\partial D, \mathrm{St}_{\mathcal{M}}^{<D})$ is an affine scheme.

Note that $H^1(\partial D, \mathrm{St}_{\mathcal{M}}^{<D})$ is the global section space of the sheaf $R^1p_* \mathrm{St}_{\mathcal{M}}^{<D}$ and that $\Gamma(\partial D, \mathbb{S}to_{\mathcal{M}}^{<D})$ is the global section space of $p_*\mathbb{S}to_{\mathcal{M}}^{<D}$. Hence, it is enough to prove that the sheaves $R^1p_* \mathrm{St}_{\mathcal{M}}^{<D}$ and $p_*\mathbb{S}to_{\mathcal{M}}^{<D}$ are isomorphic. We are going to construct local isomorphisms between these sheaves and then observe that they glue into a global isomorphism.

As recalled in paragraph 1.4, the sheaf $R^1p_* \mathrm{St}_{\mathcal{M}}^{<D}$ is constructible. Hence, there exists a cover $\mathcal{U}$ of D by open subsets of D such that for every $U \in \mathcal{U}$, there exists a point $P \in U$ such that the restriction of torsors

$$H^1(\partial U, \mathrm{St}_{\mathcal{M}}^{<D}) \longrightarrow H^1(\partial P, \mathrm{St}_{\mathcal{M}}^{<D}) \tag{3.5.5}$$

is an isomorphism of schemes. Let $U \in \mathcal{U}$ and let $P \in U$ as above. Let $(x_1, \dots, x_n)$ be local coordinates centred at P such that D is defined by $x_1 \cdots x_m = 0$ in U. At the cost of shrinking $\mathcal{U}$, we can suppose that $\mathcal{M}$ has a unique level $\mathbf{m} \in \mathbb{Z}_{<0}^m$ on U. For $a, b \in \Gamma(U, \mathcal{I})$ distinct, we have

$$a - b = f_{ab} x^{\mathbf{m}}$$

with $f_{ab}(0) \neq 0$. Put $-g_{ab}^{-1} = f_{ab}/|f_{ab}|$. Then, the map $\partial G_{a,b}$ reads

$$((r_k, z_k)_{1\leqslant k\leqslant m}, (x_k)_{m+1\leqslant k\leqslant n}) \longrightarrow -g_{ab}(r_k z_k, x_k)^{-1} z^{\mathbf{m}} \tag{3.5.6}$$

If $\phi_{\mathbf{m}} : \partial U \longrightarrow S^1$ is the map $((r_k, z_k)_{1\leqslant k\leqslant m}, (x_k)_{m+1\leqslant k\leqslant n}) \longrightarrow z^{\mathbf{m}}$, then

$$H_{ab}^{<} \cap \partial U = \{z \in \partial U \text{ with } \phi_{\mathbf{m}}(z) = g_{ab}(z)\}$$

Let us re-index the g_{ab} with $\mathbb{Z}/N\mathbb{Z}$ in such a way that the $(g_i(0))_{i\in\mathbb{Z}/N\mathbb{Z}}$ are in an ascending order with respect to the clockwise orientation on S^1. Note that the g_{ab} are continuous. Hence, at the cost of shrinking $\mathcal{U}$, we can suppose that there exists $\epsilon > 0$ such that g_i maps ∂U into $]g_i(0) - \epsilon, g_i(0) + \epsilon[$ for every $i \in \mathbb{Z}/N\mathbb{Z}$. Since the

Stokes hyperplanes of $\mathcal{M}$ don't meet, the $(g_i(0))_{i\in\mathbb{Z}/N\mathbb{Z}}$ are distinct. Hence, at the cost of shrinking ϵ, the connected components of the

$$\phi_{\mathbf{m}}^{-1}(]g_i(0)-\epsilon, g_{i+1}(0)+\epsilon[), i \in \mathbb{Z}/N\mathbb{Z}$$

form a cover $\mathcal{S}(U) = (S(U)_i)_{i\in\mathbb{Z}/N'\mathbb{Z}}$ of ∂U such that

(1) The open set $S(U)_i$ contains exactly two anti-Stokes hyperplanes.

(2) The open set $S(U)_i \cap S(U)_{i+1}$ contains exactly one anti-Stokes hyperplane for every $i \in \mathbb{Z}/N'\mathbb{Z}$.

(3) We have $S(U)_i \cap S(U)_j \cap S(U)_k = \varnothing$ for every $i,j,k \in \mathbb{Z}/N'\mathbb{Z}$ distinct.

At the cost of shrinking ϵ, any section $g \in \Gamma(\partial U, \mathbb{S}to_{\mathcal{M}}^{<D})$ extends uniquely into a 1-cocycle for $\mathrm{St}_{\mathcal{M}}^{<D}$ with respect to the cover $\mathcal{S}(U)$. Taking the associated Stokes torsor defines a map

$$\mathrm{LR}_U : \Gamma(\partial U, \mathbb{S}to_{\mathcal{M}}^{<D}) \longrightarrow H^1(\partial U, \mathrm{St}_{\mathcal{M}}^{<D})$$

The restriction to ∂P induces a commutative diagram

$$\begin{tikzcd} \Gamma(\partial U, \mathbb{S}to_{\mathcal{M}}^{<D}) \arrow[r] \arrow[d, "\mathrm{LR}_U"'] & \Gamma(\partial P, \mathbb{S}to_{\mathcal{M}}^{<D}) \arrow[d, "\mathrm{LR}_P"] \\ H^1(\partial U, \mathrm{St}_{\mathcal{M}}^{<D}) \arrow[r] & H^1(\partial P, \mathrm{St}_{\mathcal{M}}^{<D}) \end{tikzcd} \tag{3.5.7}$$

From our choice of cover, the bottom arrow in (3.5.7) is an isomorphism. At the cost of refining the cover $\mathcal{U}$, we can suppose that the top arrow in (3.5.7) is an isomorphism. Since the relation (3.5.2) holds, proposition 3.4.1 implies that LR_U is an isomorphism.

If $U, U' \in \mathcal{U}$, observe that the trace of the covers $\mathcal{S}(U)$ and $\mathcal{S}(U')$ on $\partial(U\cap U')$ have a common refinement $\mathcal{V}$ such that the following diagram commutes

$$\begin{tikzcd} & Z^1(\mathcal{S}(U)\cap \partial U', \mathrm{St}_{\mathcal{M}}^{<D}) \arrow[dr] & \\ \Gamma(\partial(U\cap U'), \mathbb{S}to_{\mathcal{M}}^{<D}) \arrow[ur] \arrow[dr] & & Z^1(\mathcal{V}, \mathrm{St}_{\mathcal{M}}^{<D}) \\ & Z^1(\mathcal{S}(U')\cap \partial U, \mathrm{St}_{\mathcal{M}}^{<D}) \arrow[ur] & \end{tikzcd}$$

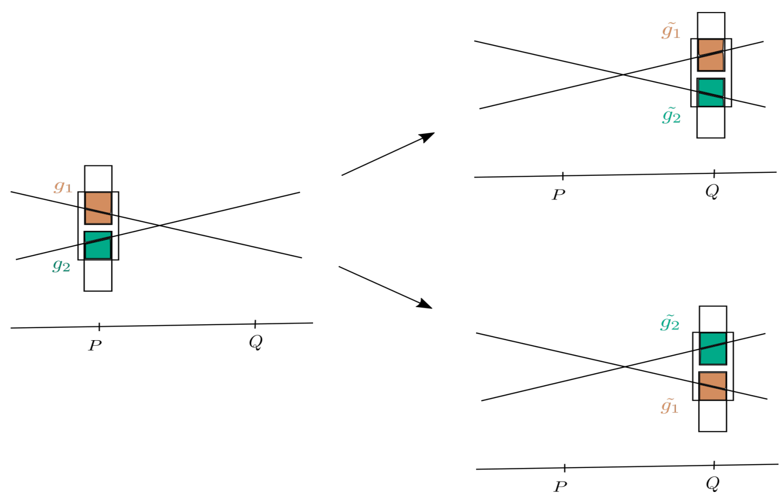
Hence, LR_U and $\mathrm{LR}_{U'}$ coincide on $U \cap U'$ for every $U, U' \in \mathcal{U}$. Thus, the $(\mathrm{LR}_U)_{U\in\mathcal{U}}$ glue into an isomorphism between $p_*\mathbb{S}to_{\mathcal{M}}$ and $R^1p_*\,\mathrm{St}_{\mathcal{M}}^{<D}$. This concludes the proof of proposition 3.5.3.

□

3.6. A conjecture in the multi-level case. In this subsection, we restrict to the case where D is smooth. The question whether $R^1p_*\,\mathrm{St}_{\mathcal{M}}^{<D}$ and $p_*\mathbb{S}to_{\mathcal{M}}^{<D}$ are isomorphic is fruitful since it would imply that *when D is smooth, the moduli of Stokes torsors are affine spaces.* We thus formulate the following

Conjecture. *Let X be a smooth complex algebraic variety. Let D be a smooth divisor in X. Let $\mathcal{M}$ be a good meromorphic connection on X with poles along D. Let $p : \widetilde{X} \longrightarrow X$ be the real blow-up of X along D. Then, the local systems $R^1p_* \operatorname{St}^{<D}_{\mathcal{M}}$ and $p_*\mathbb{S}to^{<D}_{\mathcal{M}}$ are isomorphic.*

In the several level case, the difficulty comes from the fact that the parallel transports for $R^1p_* \operatorname{St}^{<D}_{\mathcal{M}}$ and $p_*\mathbb{S}to^{<D}_{\mathcal{M}}$ produce different cocycles that are not equal on the nose, but might be cohomologous. The picture below illustrates this phenomenon. The picture on the left features part of our initial element of the



Stokes group above P. In this situation, two anti-Stokes hyperplanes L_1 and L_2 intersect once along the path joining P to Q. Let us call x the intersection point. Since the anti-Stokes hyperplanes are parallel to the Stokes hyperplanes, there is a neighbourhood Ω of x in ∂D not meeting any Stokes line coming from the differences of irregular values giving rise to L_1 and L_2. In particular, g_1 and g_2 extend uniquely into sections $\widetilde{g_1}$ and $\widetilde{g_2}$ of $\operatorname{St}^{<D}_{\mathcal{M}}$ over Ω. When applying the parallel transport for $R^1p_* \operatorname{St}^{<D}_{\mathcal{M}}$, we end up with the cocycle in the upper right picture. The bottom right picture represents the effect of the parallel transport for $p_*\mathbb{S}to^{<D}_{\mathcal{M}}$. Finally, one passes from one cocycle to the other by permuting $\widetilde{g_1}$ and $\widetilde{g_2}$. Since the Stokes sheaf is not commutative, it is not a priori clear that these cocycles are cohomologous.

4. Application

The role of this paragraph is to provide non trivial examples of meromorphic connections with poles along the fibre of an abelian scheme over a curve. This answers a question asked to the author by Y. André. Trivial examples would be split connections, connections coming from the base curve and their tensor product with regular singular connexions on our abelian scheme. We also want to exclude Stokes structures coming from sub-connections of the previous form. We thus prove the following

Proposition 4.0.1. *Let $\mathcal{C}$ be a complex elliptic curve. Put $\mathcal{A} = \mathcal{C} \times \mathbb{A}^1_{\mathbb{C}}$ and let $p : \mathcal{A} \longrightarrow \mathbb{A}^1_{\mathbb{C}}$ be the second projection. Let $\mathcal{A}_0$ be the fibre of p above 0. Then,* *there exists a rank* 4 *meromorphic connection* $\mathcal{M}$ *defined in a neighbourhood of* $\mathcal{A}_0$ *and with poles along* $\mathcal{A}_0$ *such that*

(1) The connection $\mathcal{M}$ *is not of the form* $\mathcal{M}_1 \otimes p^*\mathcal{M}_2$ *where* $\mathcal{M}_1$ *is a regular connection on* $\mathcal{A}$ *with poles along* $\mathcal{A}_0$ *and where* $\mathcal{M}_2$ *is a meromorphic connection on* S *with poles at* 0*.*

(2) The connection $\mathcal{M}$ *underlies a non trivial Stokes structure.*

(3) The Stokes structure of any strict sub-connection of $\mathcal{M}$ *is trivial.*

PROOF. Let (γ_1, γ_2) be a basis for the fundamental group of $\mathcal{C}(\mathbb{C})$. Let γ_3 be a basis for the fundamental group of $\mathbb{C}^*$. Let $A_1, A_2 \in \mathrm{GL}_2(\mathbb{C})$ be matrices such that A_1 has a unique eigenvalue λ_1 and A_2 has a unique eigenvalue λ_2 with $\lambda_1 \neq \lambda_2$ and such that $\ker({}^tA_2 - A_1)$ is one dimensional. For example, take z_1, z_2, z_3, z_4 non zero complex numbers such that

$$A_1 = \begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix}$$

has a unique non zero eigenvalue distinct from z_1, and put

$$A_2 = \begin{pmatrix} z_1 & z_2 \\ 0 & z_1 \end{pmatrix}$$

For $i = 1, 2$, let $\rho_i : \pi_1(\mathcal{A}\backslash\mathcal{A}_0(\mathbb{C})) \longrightarrow \mathrm{GL}_2(\mathbb{C})$ be the representation associating A_i to γ_1 and γ_2 and Id to γ_3. Let $\mathcal{R}_i$ be the regular meromorphic connection on $\mathcal{A}$ with poles along $\mathcal{A}_0$ induced by ρ_i. Let $a \in \mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}(*0)$ with a pole of order $d > 0$ at 0. Put

$$\mathcal{N} := p^*\mathcal{E}^a \otimes \mathcal{R}_1 \oplus \mathcal{R}_2 \tag{4.0.2}$$

Since $\mathcal{R}_1$ and $\mathcal{R}_2$ are non isomorphic, $\mathcal{N}$ satisfies the condition (1). From Theorem 5, the moduli space $H^1(\partial\mathcal{A}_0, \mathrm{St}_{\mathcal{N}}^{<\mathcal{A}_0})$ is a complex affine space. From [**Tey18**, 2.4.4], we have

$$\dim H^1(\partial\mathcal{A}_0, \mathrm{St}_{\mathcal{N}}^{<\mathcal{A}_0}) = \dim H^1(\mathcal{A}_0, \mathrm{Irr}^*_{\mathcal{A}_0} \mathrm{End}\,\mathcal{N})$$

From our choice for $\mathcal{R}_1$ and $\mathcal{R}_2$, note that $\mathrm{End}\,\mathcal{N}$ contains $p^*\mathcal{E}^a$ and $p^*\mathcal{E}^{-a}$ as direct summands. Hence,

$$\begin{aligned} \dim H^1(\partial\mathcal{A}_0, \mathrm{St}_{\mathcal{N}}^{<\mathcal{A}_0}) &\geqslant 2\dim H^1(\mathcal{A}_0, \mathrm{Irr}^*_{\mathcal{A}_0} p^*\mathcal{E}^a) \\ &\geqslant 2\dim H^1(\mathcal{A}_0, p^{-1}\,\mathrm{Irr}^*_{\mathcal{A}_0} \mathcal{E}^a) \\ &\geqslant 2\dim H^1(\mathcal{A}_0, \mathbb{C}^d[-1]) \\ &\geqslant 2d \end{aligned}$$

Hence, $H^1(\partial\mathcal{A}_0, \mathrm{St}_{\mathcal{N}}^{<\mathcal{A}_0})$ is a non trivial affine space. From [**Tey18**, 2.2.1], a point of $H^1(\partial\mathcal{A}_0, \mathrm{St}_{\mathcal{N}}^{<\mathcal{A}_0})$ distinct form the trivial torsor gives rise to a germ of meromorphic connection $\mathcal{M}$ with poles along $\mathcal{A}_0$ satisfying conditions (1) and (2). Indeed, the connection $\mathcal{M}$ satisfies (1) because $\mathcal{N}$ does, and by construction, $\mathcal{M}$ underlies a non trivial Stokes structure.

We are left to show that $\mathcal{M}$ satisfies condition (3). That is, for every strict sub-connection $\mathcal{M}'$ of $\mathcal{M}$, we have

$$H^1(\partial\mathcal{A}_0, \mathrm{St}_{\mathcal{M}'}^{<\mathcal{A}_0}) \simeq 0$$

Let $\mathcal{N}'$ be the split formal model of $\mathcal{M}'$ along $\mathcal{A}_0$. Then, $\mathcal{N}'$ identifies to a non trivial strict sub-connection of $\mathcal{N}$. For $i = 1, 2$, let χ_i be the rank one connection on $\mathcal{A}$ corresponding to the character of $\pi_1(\mathcal{A}\backslash\mathcal{A}_0(\mathbb{C})) \longrightarrow \mathbb{C}^*$ associating λ_i to γ_1 and γ_2, and 1 to γ_3. Then, the non trivial strict sub-connections of $\mathcal{N}$ are

$$p^*\mathcal{E}^a \otimes \chi_1, \chi_2, \mathcal{R}_2, p^*\mathcal{E}^a \otimes \mathcal{R}_1, p^*\mathcal{E}^a \otimes \chi_1 \oplus \chi_2, p^*\mathcal{E}^a \otimes \chi_1 \oplus \mathcal{R}_2, p^*\mathcal{E}^a \otimes \mathcal{R}_1 \oplus \chi_2$$

If $\mathcal{N}'$ is regular or is the twist of a regular connection by an exponential one, then $\mathcal{M}'$ does not underlie a non trivial Stokes structure. Hence, we can suppose that $\mathcal{N}'$ is one of the three last connections given in the above list. From Theorem 5, the moduli space $H^1(\partial\mathcal{A}_0, \mathrm{St}^{<\mathcal{A}_0}_{\mathcal{M}'})$ is a complex affine space. Let us show that it is trivial. From [**Tey18**, 2.4.4], we have

$$\dim H^1(\partial\mathcal{A}_0, \mathrm{St}^{<\mathcal{A}_0}_{\mathcal{M}'}) = \dim H^1(\mathcal{A}_0, \mathrm{Irr}^*_{\mathcal{A}_0} \operatorname{End}\mathcal{M}')$$

From [**Sab17**, 1.2], there is a non canonical isomorphism

$$\mathrm{Irr}^*_{\mathcal{A}_0} \operatorname{End}\mathcal{M}' \simeq \mathrm{Irr}^*_{\mathcal{A}_0} \operatorname{End}\mathcal{N}'$$

Hence, we are left to show that

$$H^1(\mathcal{A}_0, \mathrm{Irr}^*_{\mathcal{A}_0} \operatorname{End}\mathcal{N}') \simeq 0$$

Since χ_1 and χ_2 are distinct, it is enough to show that for any non trivial character $\chi : \pi_1(\mathcal{A}_0(\mathbb{C})) \longrightarrow \mathbb{C}^*$, we have

$$H^1(\mathcal{A}_0, \mathrm{Irr}^*_{\mathcal{A}_0}(p^*\mathcal{E}^a \otimes \chi)) \simeq 0$$

where we abuse notation by denoting by χ the pull-back to $\mathcal{A}$ of the rank one connection on $\mathcal{A}_0$ induced by χ. Let $p_{\mathcal{A}_0} : \widetilde{\mathcal{A}} \longrightarrow \mathcal{A}$ be the real blow-up of $\mathcal{A}$ along $\mathcal{A}_0$. From [**Sab17**, 3.2], we have

$$\mathrm{Irr}^*_{\mathcal{A}_0}(p^*\mathcal{E}^a \otimes \chi) \simeq Rp_{\mathcal{A}_0 *}\mathcal{H}^0 \mathrm{DR}^{<\mathcal{A}_0}(p^*\mathcal{E}^{-a} \otimes \chi^{-1})$$

Thus, we have to show that for any non trivial character $\chi : \pi_1(\mathcal{A}_0(\mathbb{C})) \longrightarrow \mathbb{C}^*$, we have

$$H^1(\partial\mathcal{A}_0, \mathcal{H}^0 \mathrm{DR}^{<\mathcal{A}_0}(p^*\mathcal{E}^a \otimes \chi)) \simeq 0$$

Let $p_0 : \widetilde{\mathbb{C}} \longrightarrow \mathbb{C}$ be the real blow-up of $\mathbb{C}$ at 0. Observe that $\partial\mathcal{A}_0 \simeq \mathcal{A}_0 \times \partial 0$. Let U be the open subset of $\partial 0$ consisting in the directions along which e^a has rapid decay at 0. Note that $F := \partial 0 \backslash U$ is a union of closed intervals. Let $j : \mathcal{A}_0 \times U \longrightarrow \partial\mathcal{A}_0$ and let $i : \mathcal{A}_0 \times F \longrightarrow \partial\mathcal{A}_0$ be the canonical inclusions. Let L_χ be the local system on $\mathcal{A}_0$ induced by χ. Let $\mathcal{L}_\chi$ be the pull-back of L_χ via $\partial\mathcal{A}_0 \longrightarrow \mathcal{A}_0$. Then, we have

$$\mathcal{H}^0 \mathrm{DR}^{<\mathcal{A}_0}(p^*\mathcal{E}^a \otimes \chi) \simeq j_! j^{-1}\mathcal{L}_\chi$$

Since χ is not trivial, the sheaf $i^{-1}\mathcal{L}_\chi$ does not have non trivial global sections. Hence, the long exact sequence in cohomology induced by

$$0 \longrightarrow j_! j^{-1}\mathcal{L}_\chi \longrightarrow \mathcal{L}_\chi \longrightarrow i_* i^{-1}\mathcal{L}_\chi \longrightarrow 0$$

gives rise to an injective morphism $H^1(\partial\mathcal{A}_0, j_! j^{-1}\mathcal{L}_\chi) \longrightarrow H^1(\partial\mathcal{A}_0, \mathcal{L}_\chi)$. We are thus left to show that $\mathcal{L}_\chi$ is acyclic. Note that

$$R\Gamma(\partial\mathcal{A}_0, \mathcal{L}_\chi) \simeq R\Gamma(\partial 0, R\pi_*\mathcal{L}_\chi)$$

where $\pi : \partial\mathcal{A}_0 \longrightarrow \partial 0$ is the natural projection. Hence, it is enough to prove that $R\pi_*\mathcal{L}_\chi$ is zero. By proper base change theorem, the fibres of $R\pi_*\mathcal{L}_\chi$ identify with $R\Gamma(\mathcal{A}_0, L_\chi)$. We get the desired vanishing from the fact that non trivial rank 1 local systems on elliptic curves are acyclic.

□

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