HOMOTOPY THEORY OF STOKES STRUCTURES AND DERIVED MODULI

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Abstract. The goal of this paper is to build a theory of Stokes structures in an abstract ∞-categorical context and to show that Stokes structures coming from flat bundles form a locally geometric derived stack of finite presentation. This generalizes existing geometricity results on Stokes structures in four different directions : our result applies in any dimension, ∞-categorical coefficients are allowed, derived structures on moduli spaces are considered and more general spaces than those arising from flat bundles are permitted.

Contents

1. Introduction

Let (E,∇) be a rank n algebraic flat bundle on a smooth complex algebraic variety X. Then, analytic continuation of the solutions of the differential system $\nabla = 0$ gives rise to a representation $\rho\colon \pi_1(X) \to \mathrm{GL}_n(\mathbb{C})$ called the *monodromy representation.* If favourable conditions are imposed, the data of ρ and (E,∇) are equivalent. In that case (E,∇) is called *regular singular* [13] and this case is characterized by the fact that the formal solutions to $\nabla = 0$ automatically converge. In general, the monodromy representation is not enough to capture all the analytic information contained in (E,∇). As already seen by Stokes on the Airy equation [48], formal solutions to $\nabla = 0$ may not converge any more, but their interplay with analytic solutions is highly structured and gives rise to what is nowadays called a *Stokes structure* or a *Stokes filtered local system* [14, 3, 44]. To picture it, let us suppose that X is the affine line and let S^1_∞ be the circle of directions emanating from ∞. Then, the flat bundle (E,∇) has *good formal structure at* ∞, meaning roughly that when restricted to a formal neighbourhood of ∞, it decomposes as a direct sum of regular flat bundles twisted by rank one bundles. The theory of asymptotic developments [46] then ensures the existence of a finite set $\mathrm{St}(E,\nabla) \subset S^1_\infty$ of *Stokes directions* such that for every $d \notin \mathrm{St}(E,\nabla)$, any formal solution $\hat{f}$ to $\nabla = 0$ at ∞ lifts to an analytic solution f in some small enough sector S containing d. We also say that $\hat{f}$ is the asymptotic development of f. By Cauchy's theorem, f admits an analytic continuation to any sector obtained by rotating S. However, the asymptotic development is not preserved under the analytic continuation procedure and may jump when crossing a Stokes line. This is the *Stokes phenomenon.* In practice, these jumps are measured by matrices (one for each Stokes direction) called *Stokes matrices.* Note that Stokes matrices are subjected to choices of basis. To get a more intrinsic presentation, let L be the local system of solutions to $\nabla = 0$ on S^1_∞. Then Deligne and Malgrange observed in [14] that the Stokes phenomenon is recorded by a filtration of L by constructible subsheaves indexed by $\mathcal{O}_{\mathbb{P}^1,\infty}(*\infty)/\mathcal{O}_{\mathbb{P}^1,\infty}$. Concretely for $a \in \mathcal{O}_{\mathbb{P}^1,\infty}(*\infty)/\mathcal{O}_{\mathbb{P}^1,\infty}$, we put

$$L_{\leq a} = \{f \in L \text{ such that } e^{-a} f \text{ has moderate growth } \infty\} .$$

Although this filtration is indexed by an infinite dimensional parameter space, only a finite number of elements, called *irregular values of* (E,∇) contribute in a non trivial way.

On the other hands, representations of the fundamental group naturally form an algebraic variety, the *character variety.* It is thus a natural question to ask whether Stokes structures also form an algebraic variety. This question was answered in [8, 9, 28] in the curve case via GIT methods. See also [6] for a stacky variant in the curve case. In dimension ≥ 2, several major obstacles arise. The first one is that good formal structures breaks down. Still, Sabbah conjectured [43] that good formal structure can be achieved at the cost of enough blow-up above the divisor at infinity. This problem was solved independently by Kedlaya [29, 30] and Mochizuki [37, 35]. Furthermore, given a smooth compact algebraic variety X and a simple normal crossing divisor D, Mochizuki attached to every flat bundle (E,∇) on $U \coloneqq X \setminus D$ with good formal structure along D a Stokes filtered local system $(L, L_{\leq})$ on the real blow-up $\pi\colon \widetilde{X} \to X$ along the components of D, and showed that the data of (E,∇) and $(L, L_{\leq})$ are equivalent. Once strapped in this setting, a second major obstacle in dimension ≥ 2 pertains to the *stratified* nature of good formal structure. To explain it, suppose that $X = \mathbb{C}^2$, let D_1, D_2 be the coordinate axis and let D be their union. Then, very roughly, good formal structure holds separately on the formal neighbourhoods of 0, $D_1 \setminus \{0\}$ and $D_2 \setminus \{0\}$. In dimension 1, the points at infinity are isolated so their contributions to the moduli of Stokes structures don't interact and can thus be analysed separately. In higher dimension, the contributions of $D_1 \setminus \{0\}$, $D_2 \setminus \{0\}$ and 0 are necessarily intricated. This in particular makes it unclear how to use the moduli of Stokes torsors from [49] as smooth atlases in global situations. In this paper, we generalize all known construction of the moduli of (possibly *ramified*) Stokes filtered local systems in four different ways:

(1) our result applies in any dimension;

(2) ∞-categorical coefficients are allowed;

(3) derived structures on moduli spaces are considered;

(4) more general spaces than those arising from flat bundles are permitted.

For representability results along these lines in the De Rham side see [38]. Before presenting our main theorem, let us explain how stratified homotopy theory enters the game.

Let X be a complex manifold and let D be a normal crossing divisor. The starting point is a transfer of Mochizuki's notion of higher dimensional Stokes filtered local system in a purely categorical setup. The first step of this transfer is channeled by the *topological exodromy equivalence* from [32, 41, 54, 25] which converts a hyperconstructible hypersheaf with respect to a stratification P of X into a functor from the ∞-category of Exit Paths $\Pi_\infty(X,P)$ attached to (X,P). By design, the objects of $\Pi_\infty(X,P)$ are the points of X and the morphisms between two points x and y can be thought of as continuous paths $\gamma\colon [0,1]\to X$ such that $\gamma((0,1])$ lies in the same stratum as y. Let $\mathscr{I}\subset \mathcal{O}_X(*D)/\mathcal{O}_X$ be a good sheaf of irregular values (see Recollection 16.3.4). Let $\pi\colon \widetilde{X}\to X$ be the real blow-up of X along D (see Construction 16.1.4). A point $x\in\widetilde{X}$ with $\pi(x)\in D$ can be thought of as a line passing through $\pi(x)$ and a section of $\pi^*\mathscr{I}$ near x as a meromorphic function defined on some small multi-sector emanating from $\pi(x)$. For two such sections a and b, the relation

$$a \leq_x b \text{ if and only if } e^{a-b} \text{ has moderate growth at } x$$

defines an order on the germs of $\pi^*\mathscr{I}$ at x. This collection of orders upgrades $\pi^*\mathscr{I}$ into a sheaf of posets that turns out to be constructible for a suitable choice of finite subanalytic stratification P of $\widetilde{X}$. Through the exodromy equivalence, $\pi^*\mathscr{I}$ thus corresponds to a functor $\Pi_\infty(\widetilde{X},P)\to \mathbf{Poset}$ which in turn corresponds to a cocartesian fibration in posets $\mathcal{I}\to\Pi_\infty(\widetilde{X},P)$ via the Grothendieck construction. In this language, Stokes filtered local systems are special functors $F\colon\mathcal{I}\to\mathcal{E}$ that we call *Stokes functors*, where $\mathcal{E}$ is the category of $\mathbb{C}$-vector spaces. A substantial part of the paper is devoted to the analysis of the two conditions that make these functors special.

Splitting condition. This condition is punctual. For $x\in\widetilde{X}$, let $\mathcal{I}_x\in\mathbf{Poset}$ be the fibre of $\mathcal{I}\to\Pi_\infty(\widetilde{X},P)$ above x and consider the restricted functor $F_x\colon\mathcal{I}_x\to\mathcal{E}$. Let $i_{\mathcal{I}_x}\colon\mathcal{I}_x^{\mathrm{set}}\to\mathcal{I}_x$ be the underlying set of $\mathcal{I}_x$. Let $i_{\mathcal{I}_x,!}\colon \mathrm{Fun}(\mathcal{I}_x^{\mathrm{set}},\mathcal{E})\to\mathrm{Fun}(\mathcal{I}_x,\mathcal{E})$ be the left Kan extension of $i^*_{\mathcal{I}_x}\colon\mathrm{Fun}(\mathcal{I}_x,\mathcal{E})\to\mathrm{Fun}(\mathcal{I}_x^{\mathrm{set}},\mathcal{E})$. Then F_x is requested to lie in the essential image of $i_{\mathcal{I}_x,!}$. Unravelling the definition, the means that there is $V\colon\mathcal{I}_x\to\mathcal{E}$ such that for every $a\in\mathcal{I}_x$, we have

$$F_x(a)\simeq\bigoplus_{b\leq a \text{ in } \mathcal{I}_x} V(b)\ .$$

Induction condition. If $\gamma\colon x\to y$ is an exit path for $(\widetilde{X},P)$, it pertains to a prescription of F_y by F_x via γ referred as *induction* in [37]. If $\gamma\colon\mathcal{I}_x\to\mathcal{I}_y$ is the morphism of posets induced by $\gamma\colon x\to y$ and if $\gamma_!\colon\mathrm{Fun}(\mathcal{I}_x,\mathcal{E})\to\mathrm{Fun}(\mathcal{I}_y,\mathcal{E})$ is the left Kan extension of the pull-back $\gamma^*\colon\mathrm{Fun}(\mathcal{I}_y,\mathcal{E})\to\mathrm{Fun}(\mathcal{I}_x,\mathcal{E})$, Mochizuki's condition translates purely categorically into the requirement that the natural map $\gamma_!(F_x)\to F_y$ is an equivalence.

Remark 1.1. When the sheaf $\mathscr{I}$ is trivial, the splitting condition is trivial and the induction condition is an instance of parallel transport from x to y. So in this case, Stokes functors are nothing but local systems on $\widetilde{X}$ (see Construction 16.1.4).

The splitting and induction conditions have purely categorical formulations. This motivates the following

Definition 1.2. Let $\mathcal{X}$ be an ∞-category. Let $\mathcal{I} \to \mathcal{X}$ be a cocartesian fibration in posets. Let $\mathcal{E}$ be a presentable ∞-category. A *Stokes functor* is a functor $F\colon \mathcal{I} \to \mathcal{E}$ satisfying the splitting and induction conditions. We denote by $\mathrm{St}_{\mathcal{I},\mathcal{E}} \subset \mathrm{Fun}(\mathcal{I},\mathcal{E})$ the full subcategory spanned by Stokes functors and by $\mathrm{St}_{\mathcal{I},\mathcal{E},\omega} \subset \mathrm{St}_{\mathcal{I},\mathcal{E}}$ the subcategory spanned by Stokes functors with value in compact objects of $\mathcal{E}$.

One major obstacle to work ∞-categorically is to make sense of the induction condition in a sufficiently synthetic way to minimize the amount of ∞-categorical data required for its check. This is achieved through the *specialization* formalism developed in a first part of the paper. The gains with this approach are streamlined proofs of crucial properties for Stokes functors: preservation under cartesian pull-back and induction over a fixed base (Corollary 8.3.4), invariance under localisation of the base (Proposition 8.3.5), preservation under graduation (Proposition 9.4.9), explicit description when $\mathcal{X}$ has an initial object (Proposition 8.2.5), spreading out (Theorem 11.4.1), compatibility with tensor product in $\mathbf{Pr}^{\mathrm{L}}$ (Proposition 8.6.5), categorical actions of local systems (Corollary 8.8.5), Van Kampen (Proposition 8.5.1) and existence of t-structures (Proposition 8.7.11). When $\mathcal{X}$ is the ∞-category of Exit Paths of some manifold endowed with a subanalytic stratification, we introduce the following:

Definition 1.3. Let M be a manifold. Let $X \subset M$ be a locally closed subanalytic subset and let $X \to P$ be a subanalytic stratification. A *Stokes fibration* over (X, P) is a cocartesian fibration in posets $\mathcal{I} \to \Pi_\infty(X, P)$. The data of $(X, P, \mathcal{I})$ is referred to as a *Stokes analytic stratified space.*

Similarly to Stokes lines, one can define the Stokes loci:

Definition 1.4. Let $(X, P, \mathcal{I})$ be a Stokes analytic stratified space and let a, b be cocartesian sections of $\mathcal{I} \to \Pi_\infty(X, P)$. Then, the *Stokes locus of* $\{a, b\}$ is the set of points $x \in X$ such that a_x and b_x cannot be compared in $\mathcal{I}_x$.

The simplest Stokes analytic stratified spaces are those for which the order plays no role.

Example 1.5. We say that a Stokes analytic stratified space $(X, P, \mathcal{I})$ is *elementary* if for every presentable stable ∞-category $\mathcal{E}$, the left Kan extension $i_{\mathcal{I}!}\colon \mathrm{Fun}(\mathcal{I}^{\mathrm{set}}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{I}, \mathcal{E})$ induces an equivalence between $\mathrm{St}_{\mathcal{I}^{\mathrm{set}},\mathcal{E}}$ and $\mathrm{St}_{\mathcal{I},\mathcal{E}}$.

The following is the main result of this paper :

Theorem 1.6 (Theorem 16.6.15)**.** *Let X be a complex manifold admitting a smooth compactification. Let D be a normal crossing divisor in X and put $U := X \setminus D$. Let $\pi\colon \widetilde{X} \to X$ be the real-blow up along D and $j\colon U \to \widetilde{X}$ the inclusion. Let $\mathscr{I} \subset (j_*\mathcal{O}_U)/(j_*\mathcal{O}_U)^{\mathrm{lb}}$ be a good sheaf of irregular values and let $\widetilde{X} \to P$ be a finite subanalytic stratification such that $\mathscr{I}$ is P-hyperconstructible. Let $(\widetilde{X}, P, \mathcal{I})$ be the associated Stokes analytic stratified space. Let k be an animated commutative ring. Then the prestack*

$$\mathbf{St}_{\mathcal{I}}\colon \mathrm{dAff}_k^{\mathrm{op}} \to \mathbf{Spc}$$

defined by the rule

$$\mathbf{St}_{\mathcal{I}}(\mathrm{Spec}(A)) := (\mathrm{St}_{\mathcal{I},\mathrm{Mod}_A,\omega})^{\simeq}$$

is locally geometric of finite presentation. Moreover, for every animated commutative k-algebra A and every morphism

$$x\colon \mathrm{Spec}(A) \to \mathbf{St}_{\mathcal{I}}$$

classifying a Stokes functor $F\colon \mathcal{I} \to \mathrm{Perf}_A$, there is a canonical equivalence

$$x^*\mathbb{T}_{\mathbf{St}_{\mathcal{I}}} \simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{I},\mathrm{Mod}_A)}(F, F)[1] \ ,$$

where $\mathbb{T}_{\mathbf{St}_{\mathcal{I}}}$ *denotes the tangent complex of* $\mathbf{St}_{\mathcal{I}}$ *and the right hand side denotes the* Mod_A*-enriched* Hom *of* $\mathrm{Fun}(\mathcal{I}, \mathrm{Mod}_A)$*.*

There are at least three reasons justifying the use of derived algebraic geometry. First, it is sensitive to the full stratified homotopy type $\Pi_\infty(\widetilde{X}, P)$ and not only its underlying 1-category. By analogy with character varieties [21, 50] and the curve case [8, 9, 45], we expect $\mathbf{St}_{\mathcal{I}}$ to carry a shifted symplectic structure in the sense of [39], which is typically invisible from the viewpoint of classical algebraic geometry in dimension ≥ 2. Finally, derived algebraic geometry provides an interpretation of the cohomology of Stokes functors as cotangent complexes for $\mathbf{St}_{\mathcal{I}}$, leading to a better control of its infinitesimal theory than in the classical context. These aspects will be the topics of future works.

The proof of Theorem 1.6 goes through the identification of the prestack $\mathbf{St}_{\mathcal{I}}$ with Toën-Vaquié moduli of objects attached to $\mathrm{St}_{\mathcal{I},\mathrm{Mod}_k}$, which is known to be locally geometric of finite presentation by the main result [51]. At the core of this identification is the following

Theorem 1.7 (Theorem 12.1.3)**.** *In the setting of Theorem 1.6, let* $\mathcal{E}$ *be a presentable stable* ∞*-category. Then, the subcategory* $\mathrm{St}_{\mathcal{I},\mathcal{E}} \subset \mathrm{Fun}(\mathcal{I}, \mathcal{E})$ *is stable under limits and colimits.*

Let us explain why Theorem 1.7 is striking. Let $F_\bullet \colon I \to \mathrm{St}_{\mathcal{I},\mathcal{E}}$ be a diagram of Stokes functors and let $F \coloneqq \lim F_i$ be its limit computed in $\mathrm{Fun}(\mathcal{I}, \mathcal{E})$. Then, for every $i \in I$ and every $x \in \widetilde{X}$, the splitting condition for F_i at x yields an equivalence $F_{i,x} \simeq i_{\mathcal{I}_x,!}(V_i)$ where $V_i \colon \mathcal{I}_x^{\mathrm{set}} \to \mathcal{E}$ is a functor. Observe that these equivalences are non canonical, so they typically cannot be rearranged into a diagram $V_\bullet \colon I \to \mathrm{Fun}(\mathcal{I}_x^{\mathrm{set}}, \mathcal{E})$ realizing the splitting of F at x. What Theorem 1.7 says is that for Stokes stratified spaces coming from the theory of flat bundles, such a rearrangement exists. As immediate corollary of Theorem 1.7, we deduce the following

Theorem 1.8 (Theorem 12.1.1 and Corollary 12.1.6)**.** *In the setting of Theorem 1.7, the following hold;*

(1) For every presentable stable ∞*-category* $\mathcal{E}$*, the* ∞*-category* $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ *is presentable stable.*

(2) For every Grothendieck abelian category $\mathcal{A}$*, the category* $\mathrm{St}_{\mathcal{I},\mathcal{A}}$ *is Grothendieck abelian.*

When $\mathcal{A}$ is the category of vector spaces over a field, (2) reproduces a theorem of Sabbah [44, Corollary 9.20]. This is again striking since over a point, Stokes functors neither form a presentable stable ∞-category nor an abelian category. In the same vein, we show the following:

Theorem 1.9 (Theorem 12.3.5)**.** *In the setting of Theorem 1.6, let* $\mathcal{E}$ *be a* k*-linear presentable stable* ∞*-category of finite type (see Definition 17.3.1). Then,* $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ *is* k*-linear of finite type. In particular,* $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ *is a smooth non-commutative space.*

One could package the above results in the following

Slogan 1.10. *For Stokes stratified spaces coming from flat bundles, the* ∞*-category of Stokes functors inherits the properties of its coefficients.*

We refer to Theorem 14.2.2 for a possible way of transforming this slogan into a precise statement.

In a nutshell, the moduli functor from Theorem 1.6 parametrizes "Stokes filtered perfect complexes". From this perspective, actual Stokes filtered local systems correspond to objects concentrated in degree 0. It turns out that these can also be organized into a sub pre-stack $\mathbf{St}_{\mathcal{I},k}^{\mathrm{flat}} \subset \mathbf{St}_{\mathcal{I},k}$ satisfying the following

Theorem 1.11 (Theorem 13.3.4)**.** *In the setting of Theorem 1.6, the pre-stack* $\mathbf{St}_{\mathcal{I},k}^{\mathrm{flat}}$ *is a derived* 1*-Artin stack locally of finite type.*

In particular, the truncation of $\mathbf{St}^{\mathrm{flat}}_{\mathcal{I},k}$, namely its restriction to discrete k-algebra is an Artin stack locally of finite type in the classical sense.

Although stated in the context coming from flat bundles, the above theorems hold more generally for what we call *families of Stokes analytic stratified spaces locally admitting a piecewise elementary level structure*. To explain this, let us introduce the following

Definition 1.12. Let $(X, P, \mathcal{I})$ be a Stokes analytic stratified space and let $p\colon \mathcal{I} \to \mathcal{J}$ be a morphism of Stokes fibrations over (X, P). We say that $p\colon \mathcal{I} \to \mathcal{J}$ is a *level morphism* if for every $x \in X$ and every $a, b \in \mathcal{I}_x$, we have

$$p(a) < p(b) \text{ in } \mathcal{J}_x \Rightarrow a < b \text{ in } \mathcal{I}_x \ .$$

Definition 1.12 is an axiomatization of a standard dévissage procedure in the classical theory of Stokes structures. If we consider the fibre product $\mathcal{I}_p := \mathcal{J}^{\mathrm{set}} \times_{\mathcal{J}} \mathcal{I}$, this dévissage is traditionally used to reduce the study of $(X, P, \mathcal{I})$ to that of $(X, P, \mathcal{J})$ and $(X, P, \mathcal{I}_p)$. This is effective since the level morphisms naturally occurring classically are so that $\mathcal{J}$ has less objects than $\mathcal{I}$ while $\mathcal{I}_p$ comes with extra properties. In this paper, we show that the power of this reduction procedure has a purely categorical explanation, which seems to be new already in the classical setting (see Theorem 10.2.1 for a more precise statement in a purely ∞-categorical context):

Theorem 1.13. *Let $(X, P, \mathcal{I})$ be a Stokes analytic stratified space and let $p\colon \mathcal{I} \to \mathcal{J}$ be a level graduation morphism of Stokes fibrations over (X, P). Let $\mathcal{E}$ be a presentable stable ∞-category. Then, there is a pullback square*

$$\begin{tikzcd} \mathrm{St}_{\mathcal{I},\mathcal{E}} \arrow[r] \arrow[d] & \mathrm{St}_{\mathcal{J},\mathcal{E}} \arrow[d] \\ \mathrm{St}_{\mathcal{I}_p,\mathcal{E}} \arrow[r] & \mathrm{St}_{\mathcal{J}^{\mathrm{set}},\mathcal{E}} \end{tikzcd}$$

in $\mathbf{Cat}_\infty$.

The extra property of $(X, P, \mathcal{I}_p)$ alluded to is what we call *piecewise elementary* (see Definition 11.3.19). In a nutshell, it means that every point admits a subanalytic closed neighbourhood Z such that the induced Stokes analytic stratified space $(Z, P, \mathcal{I}_p|_Z)$ is elementary in the sense of Example 1.5. That one can find such cover is typically possible when the differences of irregular values have the same pole order. This follows from the following result, whose statement is inspired from [36, Proposition 3.16]:

Theorem 1.14 (Theorem 15.2.4)**.** *Let $(C, P, \mathcal{I})$ be a Stokes analytic stratified space in finite posets where $C \subset \mathbb{R}^n$ is a polyhedron and $\mathcal{I}^{\mathrm{set}} \to \Pi_\infty(C, P)$ is locally constant. Assume that for every distinct cocartesian sections a, b of $\mathcal{I} \to \Pi_\infty(C, P)$, there exists a non zero affine form $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ such that*

(1) The Stokes locus of $\{a, b\}$ is $C \cap \{\varphi = 0\}$ (see Definition 1.4).

(2) $C \setminus \{\varphi = 0\}$ admits exactly two connected components C_1 and C_2.

(3) $a_x < b_x$ in $\mathcal{I}_x$ for every $x \in C_1$ and $a_x < b_x$ for every $x \in C_2$.

Then $(C, P, \mathcal{I})$ is elementary.

Linear overview. The paper is divided in four parts.

Part 1. The constructible sheaf of Stokes data. After reviewing the exodromy equivalence of [41, 25], we discuss a categorical construction known as the *exponential construction*, that plays a central role in this paper. Then, we introduce the notion of *Stokes stratified space* and the related notion of *Stokes loci*. Via the exponential construction, we introduce the constructible sheaf of Stokes data and work out some basic examples.

Part 2. Categorical aspects. In this part, we develop the ∞-categorical framework needed in the study of Stokes functors. We start by refining our analysis of the exponential construction via the *specialization equivalence*. Later, we separately study the property of being *cocartesian* and *punctually split*. This leads to the basic functorialities of the ∞-categories of Stokes functors (see Corollary 8.3.4) and to their fundamental properties such the invariance by localization (see Proposition 8.3.5), Van Kampen (see Proposition 8.5.1) and the existence of t-structures (see Proposition 8.7.11). In the later §§9–10, we develop the theory of *graduation* and the notion of *level structure*. We study the compatibility of the graduation procedure with Stokes functors and the interaction with their basic functorialities. Theorem 10.2.1 is in many ways the crucial result of this part, establishing the categorical basis of the level induction technique used to prove the main results of the next part.

Part 3. Geometric aspects. In this part, we place ourselves in the geometric setting. In Section 11.3, we introduce the fundamental notion of elementarity and its variants and we later prove a spreading out theorem for elementary subsets in the setting of Stokes analytic stratified spaces (see Theorem 11.4.1). Assuming the existence of a ramified piecewise linear level structure, we prove the main theorems concerning Stokes functors: that they form a presentable stable ∞-category (see Theorem 12.1.1), their non-commutative smoothness (see Theorem 12.3.5) and the representability of the derived stack of Stokes structures (see Theorem 13.1.4). In §15, we develop the elementarity criterion based on the geometry of the Stokes loci (see Theorem 15.2.4) and in §16 we study the Stokes stratified spaces arising from the theory of flat bundles, notably establishing the existence of ramified piecewise linear level structures (see Corollary 16.5.5).

Part 4. Categorical complements. This part is essentially intended as an appendix to the main body of the paper. Nevertheless, it turns out that the language of the specialization equivalence is a powerful categorical tool that allows to prove structural results on cocartesian fibrations. In Theorem 17.1.2, we establish a local-to-global principle for compactness of the total space of a cocartesian fibration. In Theorem 18.2.1, we give a new and model-independent proof of Hinich's theorem [27]. Last but not least, in §§19–20 we introduce the notion of *finite étale (cocartesian) fibration*. This notion plays a crucial role in the proof of the retraction lemma (see Corollary 8.8.6) that allows to treat *ramified* Stokes structures.

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Part 1. The constructible sheaf of Stokes data

In this part, we introduce the main geometrical object of interest of the paper: Stokes stratified spaces, i.e. stratified spaces equipped with a constructible sheaf of posets. This sheaf of posets allows to define *Stokes loci* (see Definition 4.2.2), and to introduce the constructible sheaf of *Stokes functors* (see Definition 5.2.3). Much of the later parts of this paper, will be devoted to understand the properties of the global sections of this sheaf.

2. Stratified spaces and constructible sheaves

We begin giving a brief review of the exodromy correspondence [41, 25].

2.1. **Atomic generation.**

Let $\mathcal{C}$ be a presentable ∞-category. Recall that an object $c \in \mathcal{C}$ is *atomic* if the functor

$$\mathrm{Map}_{\mathcal{C}}(c, -) \colon \mathcal{C} \to \mathbf{Spc}$$

preserves *all* colimits. Write $\mathcal{C}^{\mathrm{at}} \subset \mathcal{C}$ for the full subcategory spanned by the atomic objects. We say that $\mathcal{C}$ is *atomically generated* if the unique colimit-preserving extension

$$\mathrm{PSh}(\mathcal{C}^{\mathrm{at}}) \hookrightarrow \mathcal{C}$$

of $\mathcal{C}^{\mathrm{at}} \subset \mathcal{C}$ along the Yoneda embedding is an equivalence.

2.2. **Stratifications and hyperconstructible hypersheaves.**

Recollection 2.2.1. If P be a poset, we endow P with the topology whose open subsets are the closed upward subsets $Q \subset P$. That is for every $a \in Q$ and $b \in P$ such that $b \geqslant a$, we have $b \in Q$.

Definition 2.2.2. Let X be a topological space. Let P be a poset. A *stratification of X by P* is a continuous morphism $X \to P$.

Remark 2.2.3. We abuse notations by denoting a stratification of X by P as (X, P) instead of $X \to P$ and refer to (X, P) as a stratified space. The collection of stratified spaces organize into a category in an obvious manner.

Example 2.2.4. Let X be a topological space. Let $\mathcal{A}$ be a finite family of closed subsets of X. Then, the map $p_{\mathcal{A}} \colon X \to \mathrm{Fun}(\mathcal{A}, \Delta^1)$ sending $x \in X$ to the function $\chi_x \colon \mathcal{A} \to \Delta^1$ defined by

$$\chi_x(F) \coloneqq \begin{cases} 0 & \text{if } x \in F \\ 1 & \text{otherwise} \end{cases}$$

is a stratification of X. We note by $(X, \mathcal{A})$ the associated stratified space.

Remark 2.2.5. When X is a manifold and $\mathcal{A}$ is the set of irreducible components of a strict normal crossing divisor D, we note (X, D) instead of $(X, \mathcal{A})$.

Definition 2.2.6. Let (X, P) be a stratified space. Let $\mathcal{E}$ be a presentable ∞-category. An hypersheaf $F \colon \mathrm{Open}(X)^{\mathrm{op}} \to \mathcal{E}$ with value in $\mathcal{E}$ is *hyperconstructible* if for every $p \in P$, the hypersheaf $i_p^{*,\mathrm{hyp}}(F)$ is locally hyperconstant on X_p, where $i_p \colon X_p \to X$ denotes the canonical inclusion. We denote by $\mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E})$ the full-subcategory of $\mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E})$ spanned by hyperconstructible hypersheaves on (X, P).

2.3. **Exodromic stratified spaces.** Following [12, 25] we introduce the following definition:

Definition 2.3.1. A stratified space (X, P) is said to be *exodromic* if it satisfies the following conditions:

(1) the ∞-category $\mathrm{Cons}_P^{\mathrm{hyp}}(X)$ is atomically generated;

(2) the full subcategory $\mathrm{Cons}_P^{\mathrm{hyp}}(X) \subset \mathrm{Sh}^{\mathrm{hyp}}(X)$ is closed under limits and colimits;

(3) the functor $p^* \colon \mathrm{Fun}(P, \mathbf{Spc}) \to \mathrm{Cons}_P^{\mathrm{hyp}}(X)$ commutes with limits.

We denote by **ExStrat** the category of exodromic stratified spaces with stratified morphisms between them.

Example 2.3.2 ([41, Theorem 5.18])**.** Every conically stratified space with locally weakly contractible strata is exodromic.

Definition 2.3.3. Let (X, P) be an exodromic stratified space. We define the ∞-category of exit paths $\Pi_\infty(X, P)$ as the opposite of the full subcategory of $\mathrm{Cons}_P^{\mathrm{hyp}}(X)$ spanned by atomic objects.

Recollection 2.3.4. Let $f \colon (X, P) \to (Y, Q)$ be a morphism between exodromic stratified spaces. By [25, Theorem 3.2.3] the functor $f^{*,\mathrm{hyp}} \colon \mathrm{Cons}_Q^{\mathrm{hyp}}(Y) \to \mathrm{Cons}_P^{\mathrm{hyp}}(X)$ admits a left adjoint

$$f_\sharp^{\mathrm{hyp}} \colon \mathrm{Cons}_P^{\mathrm{hyp}}(X) \to \mathrm{Cons}_Q^{\mathrm{hyp}}(Y)$$

which preserves atomic objects. It therefore induces a well defined functor

$$\Pi_\infty(f) \colon \Pi_\infty(X, P) \to \Pi_\infty(Y, Q) \ .$$

Using the equivalence $\mathbf{Pr}^{\mathrm{L,at}} \simeq \mathbf{Cat}_\infty^{\mathrm{idem}}$, we can promote this construction to a functor

$$\Pi_\infty \colon \mathbf{ExStrat} \to \mathbf{Cat}_\infty \ .$$

Recollection 2.3.5. For an exodromic stratified space (X, P), there is a canonical equivalence

$$\mathrm{Fun}(\Pi_\infty(X, P), \mathbf{Cat}_\infty) \simeq \mathrm{Cons}_P^{\mathrm{hyp}}(X, \mathbf{Cat}_\infty) \tag{2.3.6}$$

referred to as the *exodromy equivalence*. By [25, Theorem 0.3.1], the exodromy equivalence (2.3.6) is especially well-behaved with respect to stratified morphisms. Namely for every morphism $f \colon (X, P) \to (Y, Q)$ between exodromic stratified spaces, the following square

$$\begin{tikzcd} \mathrm{Fun}(\Pi_\infty(Y, Q), \mathbf{Cat}_\infty) \arrow[r, "\sim"] \arrow[d, "\Pi_\infty(f)^*"] & \mathrm{Cons}_Q^{\mathrm{hyp}}(Y, \mathbf{Cat}_\infty) \arrow[d, "f^{\mathrm{hyp},*}"] \\ \mathrm{Fun}(\Pi_\infty(X, P), \mathbf{Cat}_\infty) \arrow[r, "\sim"] & \mathrm{Cons}_P^{\mathrm{hyp}}(X, \mathbf{Cat}_\infty) \end{tikzcd}$$

commutes. In particular, if $\mathcal{F} \in \mathrm{Cons}_P^{\mathrm{hyp}}(X, \mathbf{Cat}_\infty)$ corresponds to $F \colon \Pi_\infty(X, P) \to \mathbf{Cat}_\infty$ trough the exodromy equivalence, then are canonical equivalences $\mathcal{F}(X) \simeq \lim_{\Pi_\infty(X,P)} F$ and $\mathcal{F}_x \simeq F(x)$ for every $x \in X$.

Remark 2.3.7. The exodromy equivalence and its functorialities also hold with coefficients in $\mathbf{Pr}^{\mathrm{L}}$ (see [25, Proposition 4.2.5]).

Proposition 2.3.8 ([25, Theorem 3.3.6])**.** *Let (X, P) be a stratified space and let $R \to P$ be a refinement such that (X, R) is exodromic. Then, (X, P) is exodromic and the induced functor*

$$\Pi_\infty(X, R) \to \Pi_\infty(X, P)$$

exhibits $\Pi_\infty(X, P)$ as the localization of $\Pi_\infty(X, R)$ at the set of arrows sent to equivalences by $\Pi_\infty(X, R) \to R \to P$. In particular, the above functor is final and cofinal.

Remark 2.3.9 ([25, Proposition A.3.16]). In the setting of Proposition 2.3.8, the ∞-category $\Pi_\infty(X, P)$ is compact (resp. finite) if $\Pi_\infty(Y, Q)$ is compact (resp. finite).

Definition 2.3.10 ([25, Definition 5.2.4]). Let (X, P) be a stratified space. We say that (X, P) is *conically refineable* if there exists a refinement $R \to P$ such that (X, R) is conically stratified with locally weakly contractible strata.

Remark 2.3.11. A conically refineable stratified space is exodromic in virtue of Example 2.3.2 and Proposition 2.3.8.

Definition 2.3.12. Let (X, P) be an exodromic stratified space. Let $Z \subset X$ be a locally closed subset such that (Z, P) is exodromic. Let $U \subset X$ be an open neighbourhood of Z. We say that U *is final at* Z if (U, P) is exodromic and if the functor

$$\Pi_\infty(Z, P) \to \Pi_\infty(U, P)$$

is final.

Proposition 2.3.13 ([25, Corollary 3.4.5]). *Let (X, P) be an exodromic stratified space. Let $U_\bullet$ be an étale hypercover of X such that (U_n, P) is exodromic for every $[n] \in \mathbf{\Delta}_s$. Then, the canonical functor*

$$\operatorname{colim} \Pi_\infty(U_\bullet, P) \to \Pi_\infty(X, P)$$

is an equivalence.

Remark 2.3.14. In the setting of Definition 2.3.12, if U is final at Z, then the inclusion $Z \hookrightarrow U$ is a homotopy equivalence. Indeed, [31, Proposition 4.1.1.3-(3)] guarantees that $\Pi_\infty(Z, P) \to \Pi_\infty(U, P)$ is an equivalence after passing to the enveloping ∞-groupoid. But Proposition 2.3.8 implies that

$$\operatorname{Env}(\Pi_\infty(Z, P)) \simeq \Pi_\infty(Z) \qquad \text{and} \qquad \operatorname{Env}(\Pi_\infty(U, P)) \simeq \Pi_\infty(U) \ ,$$

so the conclusion follows.

Definition 2.3.15 ([41, Definition 2.3.2]). Let (X, P) be an exodromic stratified space. Let $Z \subset X$ be a locally closed subset such that (Z, P) is exodromic. We say that (X, P) is *final* Z if the collection of final at Z open neighbourhoods of Z forms a fundamental system of neighbourhoods of Z.

Definition 2.3.16. Let (X, P) be an exodromic stratified space. Let $Z \subset X$ be a locally closed subset such that (Z, P) is exodromic. We say that (X, P) is *hereditary final at* Z if for every open subset $U \subseteq X$, the stratified space (U, P) is final at $U \cap Z$.

2.4. **Triangulations and hereditary finality.** The goal of this subsection is to prove some hereditary final property (Definition 2.3.16) for stratified spaces admitting a locally finite triangulation. Before doing this, we need intermediate notations and lemmas.

Let $K = (V, F)$ be a simplicial complex. We denote by $|K|$ the *geometric realization of* K. By construction, a point in $|K|$ is a function $x \colon V \to [0, 1]$ supported on a face of K and such that $\sum_{v \in V} x(v) = 1$. Let us endow the set of faces F of K with the inclusion. Let $\operatorname{Supp}_K \colon |K| \to F$ be the support function.

Theorem 2.4.1 ([32, Theorem A.6.10]). *Let $K = (V, F)$ be a locally finite simplicial complex. The stratified space $(|K|, F)$ is conically stratified with contractible strata and the structural morphism*

$$\Pi_\infty(|K|, F) \to F$$

is an equivalence of ∞-categories.

Definition 2.4.2. Let K be a simplicial complex and let S be a simplicial subcomplex of K. We say that S is *full* if for every face σ of K, the subset $\sigma \cap S$ is empty or is a face of S.

Lemma 2.4.3. *Let $K = (V, F)$ be a locally finite simplicial complex. Let $S = (V(S), F(S))$ be a full subcomplex of K. Put*

$$U(S, K) \coloneqq \{x \in |K| \text{ such that } V(S) \cap \mathrm{Supp}_K(x) \neq \emptyset\} .$$

Then, $U(S, K)$ is final at $|S|$.

Proof. By Theorem 2.4.1, the category $\Pi_\infty(U(S, K), F)$ identifies with the subposet $P(S)$ of F of faces containing at least one vertex in S. We have to show that the inclusion $F(S) \to P(S)$ is final. Let $\sigma \in P(S)$. Then, $F(S) \times_{P(S)} P(S)_{/\sigma}$ identifies with the poset of faces of K contained in S and σ. Since σ contains at least one vertex of S, the poset $F(S) \times_{P(S)} P(S)_{/\sigma}$ is not empty. Since S is full in K, we deduce that $F(S) \times_{P(S)} P(S)_{/\sigma}$ admits a maximal element, and is thus weakly contractible. This finishes the proof of Lemma 2.4.3. □

Definition 2.4.4. Let (X, P) be a stratified space. A *triangulation of* (X, P) is the data of (K, r) where $K = (V, F)$ is a simplicial complex and $r: (|K|, F) \to (X, P)$ is a refinement. We say that (K, r) is locally finite if K is locally finite.

The existence of a (locally finite) triangulation is compatible with restriction to an open subset.

Lemma 2.4.5. *Let (X, P) be a stratified space admitting a (locally finite) triangulation. Let U be an open subset in X. Then (U, P) admits a (locally finite) triangulation.*

Proof. Let $K = (V, F)$ be a simplicial complex and let $r: (|K|, F) \to (X, P)$ be a refinement. Then, $(r^{-1}(U), F) \to (U, P)$ is a refinement. From [17, Theorem 1], there exists a simplicial complex $L = (V(L), F(L))$ and a refinement $s: (|L|, F(L)) \to (r^{-1}(U), F)$. Hence, $r \circ s: (|L|, F(L)) \to (U, P)$ is a refinement. That L is locally finite if K is locally finite follows from [47, Theorem 8 (p. 119)]. □

Lemma 2.4.6. *Let $r: (X, P) \to (Y, Q)$ be a refinement between exodromic stratified spaces. Let $Z \subset Y$ be a locally closed subset and put $T \coloneqq r^{-1}(Z)$. Let $U \subset X$ be an open subset final at T in the sense of Definition 2.3.15. Then $r(U)$ is final at Z. In particular, if (X, P) is final at T, then (Y, Q) is final at Z.*

Proof. There is a commutative diagram of ∞-categories

$$\begin{tikzcd} \Pi_\infty(T, P) \arrow[r] \arrow[d] & \Pi_\infty(Z, Q) \arrow[d] \\ \Pi_\infty(U, P) \arrow[r] & \Pi_\infty(r(U), Q) \end{tikzcd}$$

where the left vertical functor is final. From Proposition 2.3.8, the horizontal functors are localizations. They are thus final functors from [11, 7.1.10]. From [31, 4.1.1.3], we deduce that $\Pi_\infty(Z, Q) \to \Pi_\infty(r(U), Q)$ is final. Lemma 2.4.6 is thus proved. □

Proposition 2.4.7. *Let (X, P) be an exodromic stratified space admitting a locally finite triangulation. Then, for every locally closed subposet $Q \subset P$, (X, P) is hereditary final at X_Q in the sense of Definition 2.3.16.*

Proof. Let $U \subset X$ be an open subset. We have to show that (U, P) is final at $U \cap X_Q$. By Lemma 2.4.5, (U, P) admits a locally finite triangulation. At the cost of replacing X by U, we are left to show that (X, P) is final at X_Q. Write $Q = F \cap O$ where $F \subset P$ is closed and where $O \subset P$ is open. To show that (X, P) is final at X_Q amounts to show that (X_O, O) is final at X_Q. From Lemma 2.4.5 again, we are left to show that (X, P) is final at X_Q where $Q \subset P$

is closed. Applying Lemma 2.4.5 one last time, we are left to show that there exists an open subset $U \subset X$ final at X_Q where $Q \subset P$ is closed. Let $K = (V, F)$ be a locally finite simplicial complex and let $r\colon (|K|, F) \to (X, P)$ be a refinement. Since $Q \subset P$ is closed, $r^{-1}(Q) \subset F$ is closed. Hence, $r^{-1}(Q)$ is the set of faces of a simplicial subcomplex $S = (V(S), F(S))$ of K. At the cost of replacing K by its barycentric subdivision, we can suppose that S is full in the sense of Definition 2.4.2. By Lemma 2.4.6, it is enough to show that there exists an open subset $U \subset |K|$ containing $|S|$ such that U is final at $|S|$. The existence of such open subset is provided by Lemma 2.4.3. □

2.5. **Subanalytic stratified space.** In this subsection, we introduce the class of exodromic stratified spaces relevant for the study of Stokes structures coming from flat bundles.

Definition 2.5.1. A *subanalytic stratified space* is the data of (M, X, P) where M is a smooth real analytic space, $X \subset M$ a locally closed subanalytic subset and where $X \to P$ is a locally finite stratification by subanalytic subsets.

A morphism $f\colon (M, X, P) \to (N, Y, Q)$ of subanalytic stratified spaces is an analytic morphism $f\colon M \to N$ inducing a stratified morphism $f\colon (X, P) \to (Y, Q)$ such that the graph of $f\colon X \to Y$ is subanalytic.

Notation 2.5.2. We denote by **AnStrat** the category of subanalytic stratified spaces and subanalytic stratified morphisms between them.

Remark 2.5.3. If the stratification $X \to P$ satisfies Whitney's conditions, a theorem of Mather [34] implies that (X, P) is conically stratified with locally weakly contractible strata. In that case we say that (M, X, P) is a *Whitney stratified space*. Note that every subanalytic stratified space admits a Whitney refinement.

Remark 2.5.4 ([25, Theorem 5.3.9])**.** For every subanalytic stratified space (X, P) and every open subset $U \subset X$, the stratified space (U, P) is conically refineable in virtue of Remark 2.5.3. Hence it is exodromic by Remark 2.3.11.

Remark 2.5.5. For a subanalytic stratified space (M, X, P), we will often drop the reference to M and denote it by (X, P).

Proposition 2.5.6 ([25, Proposition 5.2.9])**.** *Let (M, X, P) be a subanalytic stratified space. Then, every point $x \in X$ admits a fundamental system of open neighbourhoods U such that x is an initial object in $\Pi_\infty(U, P)$.*

Proposition 2.5.7 ([25, Theorem 5.3.9])**.** *Let (M, X, P) be a subanalytic stratified space. Assume that X is relatively compact in M. Then, (X, P) is categorically finite, that is $\Pi_\infty(X, P)$ is a finite ∞-category.*

Lemma 2.5.8. *Let (M, X, P) be a subanalytic stratified space. Let $\mathcal{F} \in \mathrm{Sh}^{\mathrm{hyp}}(X, \mathbf{Cat}_\infty)$. Let $\mathcal{A}$ be a finite family of locally closed subanalytic subsets of X such that $\mathcal{F}|_Z \in \mathrm{Loc}^{\mathrm{hyp}}(Z, \mathbf{Cat}_\infty)$ for every $Z \in \mathcal{A}$. Then there is a subanalytic refinement $Q \to P$ such that $\mathcal{F} \in \mathrm{Cons}_Q^{\mathrm{hyp}}(X, \mathbf{Cat}_\infty)$. If furthermore P is finite, Q can be chosen finite as well.*

Proof. For $Z \in \mathcal{A}$, write $Z = F \cap U$ where F is closed and U is open. Then $Z = \overline{Z} \cap U$, so that Z is open in $\overline{Z}$. Hence, the map $M \to \Delta^2$ defined by

$$\overline{Z} \setminus Z \to 0,\ Z \to 1,\ M \setminus \overline{Z} \to 2$$

is a stratification of M. The stratification $M \to P \times \mathrm{Fun}(\mathcal{A}, \Delta^2)$ does the job. □

Lemma 2.5.9. *Let (M, X, P) be a subanalytic stratified space. Then for every locally closed subset $Q \subset P$, (X, P) is hereditary final at X_Q in the sense of Definition 2.3.16.*

Proof. By [22], the stratified space (X, P) admits a locally finite triangulation. Then Lemma 2.5.9 follows from Proposition 2.4.7. $\square$

Proposition 2.5.10. *Let $f\colon (M, X, P) \to (N, Y, Q)$ be a proper morphism between subanalytic stratified spaces. Then the following hold*

(1) There is a subanalytic refinement $S \to Q$ such that for every $\mathcal{F} \in \mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathbf{Cat}_\infty)$, we have $f_*(\mathcal{F}) \in \mathrm{Cons}_S^{\mathrm{hyp}}(Y; \mathbf{Cat}_\infty)$.

(2) For every $\mathcal{F} \in \mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathbf{Cat}_\infty)$, the formation of $f_*(\mathcal{F})$ commutes with base change.

Proof. By [23, 1.7], there is a refinement

$$\begin{tikzcd} (M, X, R) \arrow[r] \arrow[d] & (M, X, P) \arrow[d] \\ (N, Y, S) \arrow[r] & (N, Y, Q) \end{tikzcd}$$

by a morphism of Whitney stratified spaces submersive on each strata. By Thom first isotopy lemma [34], we deduce that $(X, R) \to (Y, S)$ is a stratified bundle above each stratum of (Y, S) By [53, 3.7], the fibres of f are Whitney stratified spaces. They are thus conically stratified spaces with locally weakly contractible strata by Remark 2.5.3. By Lemma 2.5.9, for every locally closed subset $T \subset R$, (X, R) is hereditary final at X_T. Hence, [41, Proposition 6.10.7-(a)] shows that $S \to Q$ satisfies (1). To prove (2), it is enough to prove base change along the inclusion of a point. Then, one further reduces to the case where $f\colon (M, X, P) \to (N, Y, Q)$ is a morphism of Whitney stratified spaces submersive on each strata. In this case, (2) follows from [41, Proposition 6.10.7-(b)]. $\square$

3. Cocartesian fibrations and the exponential construction

We now review some ∞-category theory that has been developed in the companion paper [40]. We need this technology for two reasons: (i) to provide a streamlined definition of the category of *Stokes stratified spaces*, and (ii) to show that we can functorially attach to every Stokes stratified space a constructible sheaf of ∞-categories, whose global sections is exactly the associated ∞-category of Stokes structures.

3.1. **Dual fibrations.**

Following the companion paper [40] we introduce the ∞-category **CoCart**. We start from the cartesian fibration

$$\mathrm{t}\colon \mathbf{Cat}_\infty^{[1]} \coloneqq \mathrm{Fun}(\Delta^1, \mathbf{Cat}_\infty) \to \mathbf{Cat}_\infty$$

sending a functor $\mathcal{A} \to \mathcal{X}$ to its target ∞-category. We then pass to the dual cocartesian fibration, in the following sense:

Definition 3.1.1. Let $p\colon \mathcal{A} \to \mathcal{X}$ be a cartesian fibration and let $\Upsilon_{\mathcal{A}}\colon \mathcal{X}^{\mathrm{op}} \to \mathbf{Cat}_\infty$ be its straightening. The *dual cocartesian fibration* $p^\star\colon \mathcal{A}^\star \to \mathcal{X}^{\mathrm{op}}$ is the cocartesian fibration classified by $\Upsilon_{\mathcal{A}}$.

Recollection 3.1.2. In the setting of the above definition, recall from [5] that objects of $\mathcal{A}^\star$ coincide with the objects of $\mathcal{A}$, while 1-morphisms $a \to b$ in $\mathcal{A}^\star$ are given by spans

$$a \xleftarrow{u} c \xrightarrow{v} b$$

where u is p-cocartesian and $p(v)$ is equivalent to the identity of $p(b)$.

We let

$$\mathbb{B}\colon \mathbf{Cat}_\infty^{[1]\star} \to \mathbf{Cat}_\infty^{\mathrm{op}}$$

be the cocartesian fibration dual to t. Specializing Recollection 3.1.2 to this setting, we see that objects of $\mathbf{Cat}_\infty^{[1]\star}$ are functors $\mathcal{A} \to \mathcal{X}$, and morphisms $\mathbf{f} = (f, u, v)$ from $\mathcal{B} \to \mathcal{Y}$ to $\mathcal{A} \to \mathcal{X}$ are commutative diagrams in $\mathbf{Cat}_\infty$ of the form

$$\begin{tikzcd} \mathcal{B} \arrow[d] & \mathcal{B}_\mathcal{X} \arrow[l, "u"'] \arrow[r, "v"] \arrow[d] & \mathcal{A} \arrow[dl] \\ \mathcal{Y} & \mathcal{X} \arrow[l, "f"'] & \end{tikzcd} \tag{3.1.3}$$

where the square is a pullback. With respect to this description, $\mathbb{B}$ sends $\mathcal{A} \to \mathcal{X}$ to its target (or base) $\mathcal{X}$, and a diagram as above defines a $\mathbb{B}$-cocartesian morphism if and only if v is an equivalence.

We define **CoCart** to be the (non-full) subcategory of $\mathbf{Cat}_\infty^{[1]\star}$ whose objects are cocartesian fibrations, and whose 1-morphisms are commutative diagrams as above where v is required to preserve cocartesian edges. In this way, **CoCart** becomes a cocartesian fibration over $\mathbf{Cat}_\infty^{\mathrm{op}}$ such that $\mathbf{CoCart} \to \mathbf{Cat}_\infty^{[1]\star}$ preserves cocartesian edges. Notice that the fiber at $\mathcal{X} \in \mathbf{Cat}_\infty^{\mathrm{op}}$ coincides with the ∞-category $\mathbf{CoCart}_{/\mathcal{X}}$. We will also need a couple of variants of this construction:

Variant 3.1.4. We let $\mathbf{PosFib} \subset \mathbf{CoCart}$ be the full subcategory spanned by those cocartesian fibrations $\mathcal{A} \to \mathcal{X}$ whose fibers are posets.

Variant 3.1.5. Let $\mathbf{CAT}_\infty$ be the ∞-category of large ∞-categories and consider the following fiber product:

$$\mathcal{C} := \mathrm{Fun}(\Delta^1, \mathbf{CAT}_\infty) \times_{\mathbf{CAT}_\infty} \mathbf{Cat}_\infty ,$$

where we used the target morphism $\mathrm{t}\colon \mathrm{Fun}(\Delta^1, \mathbf{CAT}_\infty) \to \mathbf{CAT}_\infty$. In other words, objects in $\mathcal{C}$ are morphisms $p\colon \mathcal{A} \to \mathcal{X}$ where $\mathcal{X}$ is a small ∞-category and the fibers of p are not necessarily small ∞-categories. The induced morphism $\mathrm{t}\colon \mathcal{C} \to \mathbf{Cat}_\infty$ is a cartesian fibration. Inside the dual cocartesian fibration $\mathcal{C}^\star$, we define $\mathbf{CoCART}$ as the subcategory spanned by cocartesian fibrations and whose 1-morphisms are diagrams (3.1.3) where v preserves cocartesian edges.

Variant 3.1.6. We let $\mathbf{PrFib}^{\mathrm{L}} \subset \mathbf{CoCART}$ be the subcategory spanned by cocartesian fibrations with presentable fibres and whose 1-morphisms are diagrams (3.1.3) that are morphisms in $\mathbf{CoCART}$ such that for every $x \in \mathcal{X}$, the induced functor $v_x\colon \mathcal{B}_{f(x)} \to \mathcal{A}_x$ is a morphism in $\mathbf{Pr}^{\mathrm{L}}$, i.e. is cocontinuous. $\mathbf{PrFib}^{\mathrm{L}}$ is the ∞-category of *presentable cocartesian fibrations* [40, §3.4].

Recollection 3.1.7. Both **CoCart** and $\mathbf{PrFib}^{\mathrm{L}}$ can be promoted to $\mathbf{Cat}_\infty^{\mathrm{op}}$-families of symmetric monoidal ∞-categories $\mathbf{CoCart}^\otimes$ and $\mathbf{PrFib}^{\mathrm{L},\otimes}$, in the sense of [40, Definition A.1]. Concretely, this provides for every $\mathcal{X} \in \mathbf{Cat}_\infty^{\mathrm{op}}$ a symmetric monoidal structure on the fiber $\mathbf{CoCart}_\mathcal{X}$ and $\mathbf{PrFib}^{\mathrm{L}}_\mathcal{X}$ of $\mathbb{B}\colon \mathbf{CoCart} \to \mathbf{Cat}_\infty^{\mathrm{op}}$ and $\mathbb{B}\colon \mathbf{PrFib}^{\mathrm{L}} \to \mathbf{Cat}_\infty^{\mathrm{op}}$. Invoking the straightening equivalence [31, Theorem 3.2.0.1], we find canonical identifications

$$\mathbf{CoCart}_\mathcal{X} \simeq \mathrm{Fun}(\mathcal{X}, \mathbf{Cat}_\infty) \qquad \text{and} \qquad \mathbf{PrFib}^{\mathrm{L}}_\mathcal{X} \simeq \mathrm{Fun}(\mathcal{X}, \mathbf{Pr}^{\mathrm{L}}) . \tag{3.1.8}$$

Under these equivalences, the above symmetric monoidal structures correspond to those induced respectively by the cartesian product on $\mathbf{Cat}_\infty$ and the tensor product on $\mathbf{Pr}^{\mathrm{L}}$ as defined in [32, §4.8.1].

3.2. **Exponential construction.** Fix a presentable ∞-category $\mathcal{E}$.

Construction 3.2.1. Let $p\colon \mathcal{A} \to \mathcal{X}$ be a cocartesian fibration. Let $\Upsilon_{\mathcal{A}}\colon \mathcal{X} \to \mathbf{Cat}_\infty$ be its straightening and consider the functor

$$\mathrm{Fun}_!(\Upsilon_{\mathcal{A}}(-), \mathcal{E})\colon \mathcal{X} \to \mathbf{Pr}^{\mathrm{L}} \ ,$$

where $\mathrm{Fun}_!$ denotes the functoriality given by left Kan extensions. We write

$$\exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X}) \to \mathcal{X}$$

for the presentable cocartesian fibration classifying $\mathrm{Fun}_!(\Upsilon_{\mathcal{A}}(-), \mathcal{E})$. We refer to $\exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X})$ as the *exponential fibration with coefficients in* $\mathcal{E}$ *associated to* $p\colon \mathcal{A} \to \mathcal{X}$.

Example 3.2.2. Assume that $\mathcal{X} = *$ is the category with one object and one (identity) morphism. Then $\mathbf{CoCart}_{\mathcal{X}} \simeq \mathbf{Cat}_\infty$ and $\mathbf{PrFib}^{\mathrm{L}}_{\mathcal{X}} \simeq \mathbf{Pr}^{\mathrm{L}}$. In this case, $\exp_{\mathcal{E}}(\mathcal{A}) \simeq \mathrm{Fun}(\mathcal{A}, \mathcal{E})$.

Example 3.2.3. Assume that $\mathcal{X} = \Delta^1$, so that we can represent $\Phi_{\mathcal{A}}$ as a single functor $f\colon \mathcal{A}_0 \to \mathcal{A}_1$. In this case, the functor $\mathrm{Fun}_!(\Upsilon_{\mathcal{A}}(-), \mathcal{E})\colon \Delta^1 \to \mathbf{Pr}^{\mathrm{L}}$ is identified with the functor

$$f_!\colon \mathrm{Fun}(\mathcal{A}_0, \mathcal{E}) \to \mathrm{Fun}(\mathcal{A}_1, \mathcal{E}) \ ,$$

where $f_!$ denotes the left Kan extension along f. Therefore we can understand $\exp_{\mathcal{E}}(\mathcal{A}/\Delta^1)$ as the presentable cocartesian fibration over Δ^1 whose objects are pairs (F, i) where $i \in \Delta^1$ and $F\colon \mathcal{A}_i \to \mathcal{E}$ is a functor. Besides, using [31, Proposition 2.4.4.2], we deduce that

$$\mathrm{Map}_{\exp_{\mathcal{E}}(\mathcal{A}/\Delta^1)}((F,i),(G,j)) = \begin{cases} \mathrm{Map}_{\mathrm{Fun}(\mathcal{A}_0,\mathcal{E})}(F,G) & \text{if } i = j = 0 \ , \\ \mathrm{Map}_{\mathrm{Fun}(\mathcal{A}_1,\mathcal{E})}(f_!(F),G) & \text{if } i = 0 \text{ and } j = 1 \ , \\ \mathrm{Map}_{\mathrm{Fun}(\mathcal{A}_1,\mathcal{E})}(F,G) & \text{if } i = j = 1 \ , \\ \emptyset & \text{if } i = 1 \text{ and } j = 0 \ . \end{cases}$$

Finally, a morphism $(F, 0) \to (G, 1)$ in $\exp_{\mathcal{E}}(\mathcal{A}/\Delta^1)$ is cocartesian if and only if the induced morphism $f_!(F) \to G$ is an equivalence.

Example 3.2.4. Combining the previous two points with the general properties of the straightening equivalence, we deduce that for any morphism $\gamma\colon x \to y$ in $\mathcal{X}$ the fibers of $\exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X})$ at x and y are canonically identified with $\mathrm{Fun}(\mathcal{A}_x, \mathcal{E})$ and $\mathrm{Fun}(\mathcal{A}_y, \mathcal{E})$, and a morphism $\alpha\colon F \to G$ in $\exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X})$ lying over γ is cocartesian if and only if for any choice of a cocartesian straightening $f_\gamma\colon \mathcal{A}_x \to \mathcal{A}_y$ of γ, α exhibits G as left Kan extension of F along f_γ.

It follows from [40, Variant 3.20 & Remark 3.21] that Construction 3.2.1 can be canonically promoted to an ∞-functor

$$\exp_{\mathcal{E}}\colon \mathbf{CoCart} \to \mathbf{PrFib}^{\mathrm{L}} \ .$$

Let us spell out the functoriality of $\exp_{\mathcal{E}}$ in more concrete terms. With respect to morphisms in $\mathbf{CoCart}$ as in (3.1.3), we will use the following notation:

$$\begin{tikzcd} \mathcal{B} \arrow[d] & \mathcal{B}_{\mathcal{X}} \arrow[l, "u"'] \arrow[r, "v"] \arrow[d] & \mathcal{A} \arrow[dl] & & \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y}) \arrow[d] & \exp_{\mathcal{E}}(\mathcal{B}_{\mathcal{X}}/\mathcal{X}) \arrow[l, "\mathcal{E}^u"'] \arrow[r, "\mathcal{E}^v_!"] \arrow[d] & \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X}) \arrow[dl] \\ \mathcal{Y} & \mathcal{X} \arrow[l, "f"'] & & \overset{\exp_{\mathcal{E}}}{\longmapsto} & \mathcal{Y} & \mathcal{X} \arrow[l, "f"'] & \end{tikzcd} \tag{3.2.5}$$

We refer to the functor $\mathcal{E}^v_!$ as the *exponential induction functor*.

Proposition 3.2.6. *With respect to* (3.2.5), *we have:*

1. *the functor* $\mathcal{E}^u\colon \exp_{\mathcal{E}}(\mathcal{B}_{\mathcal{X}}/\mathcal{X}) \to \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y})$ *makes the the right square a pullback;*
2. *the functor* $\mathcal{E}^v_!$ *preserves cocartesian edges.*

In particular, $\exp_{\mathcal{E}}$ *takes* $\mathbb{B}$*-cocartesian edges in* $\mathbf{CoCart}$ *to* $\mathbb{B}$*-cocartesian edges in* $\mathbf{PrFib}^{\mathrm{L}}$*.*

Proof. Statement (1) simply follows unraveling the definitions, as in [40, Lemma 3.8]. Statement (2) is automatic from the definition of morphisms in $\mathbf{PrFib}^{\mathrm{L}}$, but the reader should observe that for fixed $\mathcal{X} \in \mathbf{Cat}_\infty^{\mathrm{op}}$, the induced functor $\exp_{\mathcal{E},\mathcal{X}} \colon \mathbf{CoCart}_\mathcal{X} \to \mathbf{PrFib}^{\mathrm{L}}_\mathcal{X}$ is precisely given by Construction 3.2.1. In other words, $\mathcal{E}^v_!$ is the unstraightening of the natural transformation

$$\mathrm{Fun}_!(\Upsilon_{\mathcal{B}_\mathcal{X}}(-), \mathcal{E}) \to \mathrm{Fun}_!(\Upsilon_{\mathcal{A}}(-), \mathcal{E})$$

induced by left Kan extension along the natural transformation $\Upsilon_v \colon \Upsilon_{\mathcal{B}_\mathcal{X}} \to \Upsilon_\mathcal{A}$. Therefore, $\mathcal{E}^v_!$ preserves cocartesian edges by construction. □

Corollary 3.2.7. *Consider a commutative diagram in* $\mathbf{CoCart}$

$$\begin{tikzcd}
\mathcal{B}_\mathcal{Y} \arrow[rr, "v_\mathcal{Y}"] \arrow[dr] \arrow[drr, "u_\mathcal{B}"] & & \mathcal{A}_\mathcal{Y} \arrow[dl] \arrow[drr, "u_\mathcal{A}"] & & \\
& \mathcal{Y} \arrow[drr] & \mathcal{B} \arrow[rr, "v"] \arrow[dr] & & \mathcal{A} \arrow[dl] \\
& & & \mathcal{X} &
\end{tikzcd}$$

whose diagonal squares are pullback. Then, the squares of the commutative diagram

$$\begin{tikzcd}
\exp_\mathcal{E}(\mathcal{B}_\mathcal{Y}/\mathcal{Y}) \arrow[rr, "\mathcal{E}^{v_\mathcal{Y}}_!"] \arrow[dr] \arrow[drr, "\mathcal{E}^{u_\mathcal{B}}"] & & \exp_\mathcal{E}(\mathcal{A}_\mathcal{Y}/\mathcal{Y}) \arrow[dl] \arrow[drr, "\mathcal{E}^{u_\mathcal{A}}"] & & \\
& \mathcal{Y} \arrow[drr] & \exp_\mathcal{E}(\mathcal{B}/\mathcal{X}) \arrow[rr, "\mathcal{E}^v_!"] \arrow[dr] & & \exp_\mathcal{E}(\mathcal{A}/\mathcal{X}) \arrow[dl] \\
& & & \mathcal{X} &
\end{tikzcd}$$

are pullback.

Via the identifications of the fibres of the exponential fibration supplied by Example 3.2.2, Corollary 3.2.7 specializes to

Corollary 3.2.8. *In the situation from Corollary 3.2.7 where* $\mathcal{Y}$ *is an object* $x \in \mathcal{X}$*, the squares of the commutative diagram*

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{B}_x, \mathcal{E}) \arrow[rr, "v_{x,!}"] \arrow[dr] \arrow[drr] & & \mathrm{Fun}(\mathcal{A}_x, \mathcal{E}) \arrow[dl] \arrow[drr] & & \\
& * \arrow[drr, "x"'] & \exp_\mathcal{E}(\mathcal{B}/\mathcal{X}) \arrow[rr, "\mathcal{E}^v_!"] \arrow[dr] & & \exp_\mathcal{E}(\mathcal{A}/\mathcal{X}) \arrow[dl] \\
& & & \mathcal{X} &
\end{tikzcd}$$

are pullback, where $v_{x,!}$ *is the left Kan extension along* $v_x \colon \mathcal{A}_x \to \mathcal{B}_x$*.*

Recollection 3.2.9. Assume that $\mathcal{E}$ has an additional symmetric monoidal structure $\mathcal{E}^\otimes$. Then [40, Example 3.22] shows that $\exp_\mathcal{E}$ admits a natural extension

$$\exp_\mathcal{E} \colon \mathbf{CoCart}^\otimes \to \mathbf{PrFib}^{\mathrm{L},\otimes}$$

to a $\mathbf{Cat}_\infty^{\mathrm{op}}$-lax symmetric monoidal functors, in the sense of Definition A.3 in *loc. cit.*

3.3. **Section functors.** Given a cocartesian fibration $\mathcal{A} \to \mathcal{X}$ we can associate to it two different ∞-categories:

$$\Sigma_{\mathcal{X}}(\mathcal{A}/\mathcal{X}) \coloneqq \mathrm{Fun}_{/\mathcal{X}}(\mathcal{X}, \mathcal{A}) \qquad \text{and} \qquad \Sigma^{\mathrm{cocart}}_{\mathcal{X}}(\mathcal{A}/\mathcal{X}) \coloneqq \mathrm{Fun}^{\mathrm{cocart}}_{/\mathcal{X}}(\mathcal{X}, \mathcal{A}) \ .$$

These are respectively the ∞-categories of sections and of cocartesian sections. It follows from [40, Corollary 3.23 & Variant 3.24] that these constructions promote to global functors

$$\Sigma,\ \Sigma^{\mathrm{cocart}} \colon \mathbf{CoCart} \to \mathbf{Cat}_\infty \times \mathbf{Cat}_\infty^{\mathrm{op}} \qquad \text{and} \qquad \Sigma,\ \Sigma^{\mathrm{cocart}} \colon \mathbf{PrFib}^{\mathrm{L}} \to \mathbf{Pr}^{\mathrm{L}} \times \mathbf{Cat}_\infty^{\mathrm{op}} \ .$$

The same considerations of *loc. cit.* shows that the same holds for $\mathbf{C\small{O}C\small{ART}}$ in place of $\mathbf{CoCart}$.

Remark 3.3.1. The functor $\Sigma^{\mathrm{cocart}}_{\mathcal{X}} \colon \mathbf{PrFib}^{\mathrm{L}} \to \mathbf{Pr}^{\mathrm{L}}$ admits a monoidal left adjoint $\mathrm{Triv}_{\mathcal{X}} \colon \mathbf{Pr}^{\mathrm{L}} \to \mathbf{PrFib}^{\mathrm{L}}$ informally given by $\mathcal{E} \to (\mathcal{E} \times \mathcal{X})/\mathcal{X}$. In particular, given an object $\mathcal{A} \to \mathcal{X}$ of $\mathbf{PrFib}^{\mathrm{L}}$ and $\mathcal{E}, \mathcal{E}' \in \mathbf{Pr}^{\mathrm{L}}$, we have

$$\exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X}) \otimes_{\mathcal{X}} \mathrm{Triv}_{\mathcal{X}}(\mathcal{E}') \simeq \exp_{\mathcal{E} \otimes \mathcal{E}'}(\mathcal{A}/\mathcal{X}) \ .$$

Notation 3.3.2. Often we will also write Σ and Σ^{cocart} for the induced functors $\mathbf{PrFib}^{\mathrm{L}} \to \mathbf{Pr}^{\mathrm{L}}$ and its variants obtained composing the above functors with the canonical projection $\mathbf{Pr}^{\mathrm{L}} \times \mathbf{Cat}_\infty^{\mathrm{op}} \to \mathbf{Pr}^{\mathrm{L}}$.

The subtlety here is in the great amount of functoriality encoded in Σ and Σ^{cocart}. To fix ideas, let us discuss the case of $\mathbf{PrFib}^{\mathrm{L}}$ and the functor Σ^{cocart}, although similar considerations will apply to both $\mathbf{CoCart}$ and $\mathbf{C\small{O}C\small{ART}}$ in place of $\mathbf{PrFib}^{\mathrm{L}}$ and Σ in place of Σ^{cocart}. Morphisms in $\mathbf{Pr}^{\mathrm{L}}$ are commutative diagrams of the form

$$\begin{tikzcd} \mathcal{B} \arrow[d] & \mathcal{B}_{\mathcal{X}} \arrow[l, "u"'] \arrow[r, "v"] \arrow[d] & \mathcal{A} \arrow[dl] \\ \mathcal{Y} & \mathcal{X} \arrow[l, "f"'] & \end{tikzcd}$$

where the square is a pullback and v preserves cocartesian edges. Applying Σ^{cocart}, this diagram is sent to the composition

$$\mathrm{Fun}^{\mathrm{cocart}}_{/\mathcal{Y}}(\mathcal{Y}, \mathcal{B}) \xrightarrow{u^*} \mathrm{Fun}^{\mathrm{cocart}}_{/\mathcal{X}}(\mathcal{X}, \mathcal{B}_{\mathcal{X}}) \xrightarrow{v \circ -} \mathrm{Fun}^{\mathrm{cocart}}_{/\mathcal{X}}(\mathcal{X}, \mathcal{A}) \ .$$

Concretely, u^* takes a cocartesian section $s \colon \mathcal{Y} \to \mathcal{B}$, considers the composition $s \circ f$ and applies the universal property of pullbacks to produce a section $u^*(s) \colon \mathcal{X} \to \mathcal{B}_{\mathcal{X}}$ of $\mathcal{B}_{\mathcal{X}} \to \mathcal{X}$. An immediate check reveals that this is again a cocartesian section, so that u^* is in fact well defined. On the other hand, $v \circ -$ takes a cocartesian section $t \colon \mathcal{X} \to \mathcal{B}_{\mathcal{X}}$ to the composite cocartesian section $v \circ t \colon \mathcal{X} \to \mathcal{A}$. That these operations can be performed ∞-functorially in $\mathbf{PrFib}^{\mathrm{L}}$ is precisely the content of [40, Corollary 3.23].

We will often be interested in taking sections of exponential constructions. The following result is essentially a consequence of the theory of lax limits developed in [19]:

Proposition 3.3.3 (See [40, Proposition 4.1])**.** *Let $\mathcal{E}$ be a presentable ∞-category and let $\mathcal{A} \to \mathcal{X}$ be a cocartesian fibration. There are canonical equivalences*

$$\mathrm{Fun}(\mathcal{A}, \mathcal{E}) \simeq \Sigma_{\mathcal{X}}(\exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X})) \simeq \mathrm{Fun}_{/\mathcal{X}}(\mathcal{X}, \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X})) \ .$$

Warning 3.3.4. If instead of applying $\Sigma_{\mathcal{X}}$ we use $\Sigma^{\mathrm{cocart}}_{\mathcal{X}}$, we obtain a full subcategory $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$ of $\mathrm{Fun}(\mathcal{A}, \mathcal{E})$. We refer to objects in $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$ as *cocartesian functors*. We will provide a in Proposition 7.2.3 a characterization intrinsic to $\mathrm{Fun}(\mathcal{A}, \mathcal{E})$ of what it means for a functor $F \colon \mathcal{A} \to \mathcal{E}$ to be cocartesian.

4. Stokes stratified spaces

4.1. **The notion of Stokes stratified space.** We are now ready to introduce the main geometric object of interest of this paper:

Definition 4.1.1. The *category of Stokes stratified spaces* **StStrat** is the fiber product

$$\begin{tikzcd} \mathbf{StStrat} \arrow[r] \arrow[d] & \mathbf{PosFib}^{\mathrm{op}} \arrow[d, "\mathbb{B}^{\mathrm{op}}"] \\ \mathbf{ExStrat} \arrow[r, "\Pi_\infty"] & \mathbf{Cat}_\infty \end{tikzcd} .$$

Remark 4.1.2. It immediately follows from [31, Proposition 2.4.4.2] that mapping spaces in **StStrat** are discrete. Therefore [31, Proposition 2.3.4.18] guarantees that **StStrat** is (categorically equivalent to) a 1-category.

We have two extreme classes of examples:

Example 4.1.3. Let $(X, P) \in \mathbf{ExStrat}$ be an exodromic stratified space. The identity of $\Pi_\infty(X, P)$ is a cocartesian fibration in poset (whose fibers are all the trivial poset). This provides a canonical fully faithful functor $\mathbf{ExStrat} \to \mathbf{StStrat}$, which is left adjoint to the forgetful functor $\mathbf{StStrat} \to \mathbf{ExStrat}$. Therefore, **StStrat** can be seen as an extension of the category of exodromic stratified spaces. The forgetful functor $\mathbf{StStrat} \to \mathbf{ExStrat}$ also has a right adjoint, that sends (X, P) to $(X, P, \emptyset)$, where $\emptyset$ is the empty cocartesian fibration in posets.

Example 4.1.4. Let $\mathcal{I}$ be a poset. Then $\mathcal{I} \to *$ is a cocartesian fibration, so $(*, *, \mathcal{I})$ defines a Stokes stratified space. In other words, Stokes stratified spaces can also be seen as an extension of the category of posets. Nevertheless, this class of examples is badly behaved, and the main theorems of this paper rarely apply to these situations.

Remark 4.1.5. Objects of **StStrat** can be explicitly described as triples $(X, P, \mathcal{I})$, where (X, P) is an exodromic stratified space and $\mathcal{I} \to \Pi_\infty(X, P)$ is a cocartesian fibration in posets. Combining the straightening equivalence [31, Theorem 3.2.0.1]

$$\mathbf{CoCart}_{/\Pi_\infty(X,P)} \simeq \mathrm{Fun}(\Pi_\infty(X, P), \mathbf{Cat}_\infty)$$

with the exodromy equivalence (2.3.6)

$$\mathrm{Fun}(\Pi_\infty(X, P), \mathbf{Cat}_\infty) \simeq \mathrm{Cons}_P^{\mathrm{hyp}}(X, \mathbf{Cat}_\infty) ,$$

we can equivalently describe the datum $\mathcal{I} \to \Pi_\infty(X, P)$ as the datum of a hypersheaf of posets $\mathscr{I}$ on X on (X, P). With respect to this translation, the stalk of $\mathscr{I}$ at a point $x \in X$ coincides with the fiber of $\mathcal{I}$ at x seen as an object in $\Pi_\infty(X, P)$. We occasionally refer to the datum of a cocartesian fibration in posets $\mathcal{I}$ over $\Pi_\infty(X, P)$ as the datum of a *Stokes fibration on* (X, P).

Remark 4.1.6. The forgetful map $\mathbf{StStrat} \to \mathbf{ExStrat}$ is a cartesian fibration, and a morphism $f : (Y, Q, \mathcal{J}) \to (X, P, \mathcal{I})$ is cartesian if and only if the induced square

$$\begin{tikzcd} \mathcal{J} \arrow[r] \arrow[d] & \mathcal{I} \arrow[d] \\ \Pi_\infty(Y, Q) \arrow[r] & \Pi_\infty(X, P) \end{tikzcd}$$

is a pullback.

In practice, we will be interested in a more restricted class of Stokes stratified spaces:

Definition 4.1.7. The ∞-category of *Stokes analytic stratified spaces* **StAnStrat** is the fiber product

$$\begin{tikzcd} \mathbf{StAnStrat} \arrow[r] \arrow[d] & \mathbf{StStrat} \arrow[d] \\ \mathbf{AnStrat} \arrow[r] & \mathbf{ExStrat} \end{tikzcd}$$

where **AnStrat** is the category of subanalytic stratified spaces from Definition 2.5.1 and the bottom horizontal functor is supplied by Remark 2.5.4.

The following example is the prototype of the situations we are interested in. The reader will notice that, in comparison to Examples 4.1.3 and 4.1.4, it is a more intermediate kind of example:

Example 4.1.8. Let $p\colon X \to \mathbb{C}$ be the real blow-up of $\mathbb{C}$ at 0. Concretely, there is an identification $X \simeq [0, +\infty) \times S^1$ through which p reads as $(r, \theta) \to re^{i\theta}$. We think of the boundary $\partial X \simeq S^1$ of X as the circle of directions emanating from 0 in $\mathbb{C}$. Let $I \subset \mathcal{O}_{\mathbb{C},0}(*0)/\mathcal{O}_{\mathbb{C},0}$ be a finite set. For $\theta \in S^1$ we define an order $\leq_\theta$ on I by requiring that $a \leq_\theta b$ if and only if $a = b$ or $a \neq b$ and e^{a-b} has rapid decay in the direction θ. The latter condition means that if we write

$$a - b = f_{a,b} z^{-m_{a,b}}$$

where $m_{a,b} > 0$ is the pole of $a - b$ at 0 and $f_{a,b} \in \mathcal{O}_{\mathbb{C},0}$ is non zero, then

$$F_{a,b}(\theta) := \Re(f_{a,b}(0)e^{-i\theta m_{a,b}}) < 0$$

For $a \neq b$, the locus of directions θ such that $a \leq_\theta b$ is thus a disjoint union of $m_{a,b}$ open intervals of S^1 of length $\pi/m_{a,b}$. Let $S^1 \to P$ be a finite stratification whose closed strata are points and whose open strata are open intervals over which $F_{a,b} < 0$ or $F_{a,b} > 0$ for every $a, b \in I$ distinct. Observe that $\Pi_\infty(S^1, P)$ is equivalent to a poset. Furthermore, for every $\gamma\colon \theta_1 \to \theta_2$ morphism of $\Pi_\infty(S^1, P)$ and for every $a, b \in I$ distinct such that $a \leq_{\theta_1} b$, we have $a \leq_{\theta_2} b$ by design of $S^1 \to P$. Hence, γ induces a morphism of posets $(I, \leq_{\theta_1}) \to (I, \leq_{\theta_2})$. Thus, the orders $(I, \leq_\theta)_{\theta \in S^1}$ organize as a functor $\Pi_\infty(X, P) \to \mathbf{Poset}$ or equivalently as a cocartesian fibration in posets $\mathcal{I} \to \Pi_\infty(X, P)$.

4.2. **Stokes loci.** An important feature of the classical theory of Stokes data is the existence of Stokes lines. Remarkably, it is possible to define Stokes loci for any Stokes stratified space $(X, P, \mathcal{I}) \in \mathbf{StStrat}$, as we are going to discuss now.

Definition 4.2.1. For $(X, P, \mathcal{I}) \in \mathbf{StStrat}$, we denote by $\mathscr{I}$ the hyperconstructible hypersheaf on (X, P) corresponding to the cocartesian fibration $\mathcal{I} \to \Pi_\infty(X, P)$ as in Remark 4.1.5. The objects of

$$\mathscr{I}(X) \simeq \mathrm{Fun}^{\mathrm{cocart}}_{/\Pi_\infty(X,P)}(\Pi_\infty(X, P), \mathcal{I})$$

are the *cocartesian sections of* $\mathcal{I}$ *over* $\Pi_\infty(X, P)$.

Definition 4.2.2. Let $(X, P, \mathcal{I})$ be a Stokes analytic stratified space. Let $\sigma, \tau \in \mathscr{I}(X)$ be cocartesian sections. The *Stokes locus* $X_{\sigma,\tau}$ *of* σ, τ is the set of points $x \in X$ such that $\sigma(x), \tau(x) \in \mathcal{I}_x$ cannot be compared.

Observation 4.2.3. Let $f\colon (Y, Q, \mathcal{J}) \to (X, P, \mathcal{I})$ be a cartesian morphism between Stokes analytic stratified spaces (see Remark 4.1.6). Let $\sigma, \tau \in \mathscr{I}(X)$ be cocartesian sections. Then, we have

$$Y_{f^*\sigma, f^*\tau} = f^{-1}(X_{\sigma,\tau}) \ .$$

Lemma 4.2.4. *Let $(X, P, \mathcal{I})$ be a Stokes analtyic stratified space. Let $\sigma, \tau \in \mathscr{I}(X)$ be cocartesian sections. Then,*

(1) $X_{\sigma,\tau}$ is closed in X.

(2) For every $p \in P$, the set $X_{\sigma,\tau} \cap X_p$ is open and closed in X_p. In particular, $X_{\sigma,\tau}$ is a union of connected components of strata of (X, P).

Proof. At the cost of refining (X, P) by a Whitney stratified space, Observation 4.2.3 implies that we can suppose (X, P) to be conically stratified with locally weakly contractible strata. Let $x \in X - X_{\sigma,\tau}$. We can suppose that $\sigma(x) \leq \tau(x)$ in $\mathcal{I}_x$. Since the strata of (X, P) are locally weakly contractible, Proposition 2.5.6 yields the existence of an open subset $U \subset X$ containing x such that x is an initial object of $\mathrm{Exit}(U, P)$. Hence, for every $y \in U$, there is an exit path $\gamma \colon x \to y$ giving rise to a morphism of posets $\mathcal{I}_x \to \mathcal{I}_y$ sending $\sigma(x)$ to $\sigma(y)$ and $\tau(x)$ to $\tau(y)$. Thus $\sigma(y) \leq \tau(y)$. Hence $U \subset X - X_{\sigma,\tau}$. This proves (1). We now prove (2). From Observation 4.2.3, we can suppose that X is trivially stratified and show that $X_{\sigma,\tau}$ is open and closed in X. From (1), it is enough to show that $X_{\sigma,\tau}$ is open in X. Let $x \in X_{\sigma,\tau}$ and let $U \subset X$ be an open subset containing x such that x is an initial object of $\mathrm{Exit}(U, P)$. Let $y \in U$. Let $\gamma \colon x \to y$ be a path. Since the stratification is trivial, γ is an isomorphism. Thus γ gives rise to an *isomorphism* of posets $\mathcal{I}_x \to \mathcal{I}_y$ sending $\sigma(x)$ to $\sigma(y)$ and $\tau(x)$ to $\tau(y)$. Since $\sigma(x), \tau(x) \in \mathcal{I}_x$ cannot be compared, nor do $\sigma(y), \tau(y) \in \mathcal{I}_y$. Hence, $U \subset X_{\sigma,\tau}$. The proof of Lemma 4.2.4 is thus complete. □

5. The filtered and the Stokes hyperconstructible hypersheaves

Given a Stokes stratified space $(X, P, \mathcal{I})$, we now attach to it two hyperconstructible hypersheaves of categories on (X, P).

5.1. The hyperconstructible hypersheaves of filtered functors.

Observation 5.1.1. By Remark 2.3.7 and Recollection 3.1.7, we have identifications

$$\mathrm{Cons}_P(X; \mathbf{Pr}^{\mathrm{L}}) \simeq \mathrm{Fun}(\Pi_\infty(X, P), \mathbf{Pr}^{\mathrm{L}}) \simeq \mathbf{PrFib}^{\mathrm{L}}_{\Pi_\infty(X,P)} \ .$$

These equivalences give rise to the following canonically commutative diagram:

$$\begin{tikzcd} \mathrm{Cons}^{\mathrm{hyp}}_P(X; \mathbf{Pr}^{\mathrm{L}}) \arrow[r, "\sim"] \arrow[dr, "\Gamma_{X,*}"'] & \mathrm{Fun}(\Pi_\infty(X, P), \mathbf{Pr}^{\mathrm{L}}) \arrow[d, "\lim"] & \mathbf{PrFib}^{\mathrm{L}}_{\Pi_\infty(X,P)} \arrow[l, "\sim"'] \arrow[dl, "\Sigma^{\mathrm{cocart}}"] \\ & \mathbf{Pr}^{\mathrm{L}} & \end{tikzcd}$$

Similar considerations hold if we replace $\mathbf{Pr}^{\mathrm{L}}$ by $\mathbf{Cat}_\infty$ or by $\mathbf{CAT}_\infty$.

Definition 5.1.2. Let $(X, P, \mathcal{I})$ be a Stokes stratified space and let $\mathcal{E}$ be a presentable ∞-category. The *categorical hypersheaf of $\mathcal{I}$-filtered functors on (X, P) with coefficients in $\mathcal{E}$* is the object $\mathfrak{Fil}_{\mathcal{I},\mathcal{E}}$ in $\mathrm{Cons}^{\mathrm{hyp}}_P(X; \mathbf{Pr}^{\mathrm{L}})$ corresponding to $\exp_{\mathcal{E}}(\mathcal{I}/\Pi_\infty(X, P))$ via the equivalences of Observation 5.1.1. The ∞-category of *cocartesian $\mathcal{I}$-filtered functors on* (X, P) is the presentable ∞-category

$$\mathrm{Fil}^{\mathrm{co}}_{\mathcal{I},\mathcal{E}} \coloneqq \mathfrak{Fil}_{\mathcal{I},\mathcal{E}}(X)$$

of global sections of $\mathfrak{Fil}_{\mathcal{I},\mathcal{E}}$.

Remark 5.1.3. Let $(X, P, \mathcal{I})$ be a Stokes stratified space. We can give an explicit description of the hypersheaf $\mathfrak{Fil}_{\mathcal{I},\mathcal{E}}$ as follows. For every open subset $U \subset X$, write

$$j_U \colon \Pi_\infty(U, P) \to \Pi_\infty(X, P)$$

for the canonical map. Let $\Upsilon_{\mathcal{I}} \colon \Pi_\infty(X, P) \to \mathbf{Poset}$ be the straightening of $\mathcal{I}$. Unraveling the equivalences of Observation 5.1.1, we can identify $\mathfrak{Fil}_{\mathcal{I},\mathcal{E}}$ with the presheaf $\mathrm{Open}(X)^{\mathrm{op}} \to \mathbf{Pr}^{\mathrm{L}}$ informally defined by sending an open subset $U \subset X$ to

$$\lim_{\Pi_\infty(U,P)} \mathrm{Fun}_!(\Upsilon_{\mathcal{I}} \circ j_U(-), \mathcal{E}) \ .$$

It is not obvious from this description that $\mathfrak{Fil}_{\mathcal{I},\mathcal{E}}$ satisfies hyperdescent nor that it is P-hyperconstructible: it is rather a consequence of the exodromy equivalence.

Example 5.1.4. Let $(X, P, \mathcal{I})$ be a Stokes stratified space. Let $U \subset X$ be an open subset such that $\Pi_\infty(U, P)$ admits an initial object x. Then the description of $\mathfrak{Fil}_{\mathcal{I},\mathcal{E}}$ given in Remark 5.1.3 yields a canonical equivalence $\mathfrak{Fil}_{\mathcal{I},\mathcal{E}}(U) \simeq \mathrm{Fun}(\mathcal{I}_x, \mathcal{E})$.

Remark 5.1.5. Since all the stratified spaces we care about in this paper admit a fundamental system of open neighbourhood U satisfying the conditions of Example 5.1.4, the description from Remark 5.1.3 is adapted to understand the local behavior of $\mathfrak{Fil}_{\mathcal{I},\mathcal{E}}$. To understand its ∞-category of global sections $\mathrm{Fil}^{\mathrm{co}}_{\mathcal{I},\mathcal{E}}$, it is best to use the perspective provided by Proposition 3.3.3. Indeed, it follows from Warning 3.3.4 that

$$\mathrm{Fil}^{\mathrm{co}}_{\mathcal{I},\mathcal{E}} \simeq \Sigma^{\mathrm{cocart}}(\exp_{\mathcal{E}}(\mathcal{I}/\Pi_\infty(X,P)))$$

is a full subcategory of

$$\Sigma(\exp_{\mathcal{E}}(\mathcal{I}/\Pi_\infty(X,P))) \simeq \mathrm{Fun}(\mathcal{I}, \mathcal{E}) \ .$$

We refer to $\mathrm{Fun}(\mathcal{I}, \mathcal{E})$ as the *∞-category of $\mathcal{I}$-filtered functors on* (X, P), and we will see later in Proposition 7.2.3 a characterization intrinsic to $\mathrm{Fun}(\mathcal{I}, \mathcal{E})$ of what it means for a functor $F \colon \mathcal{I} \to \mathcal{E}$ to belong to the full subcategory $\mathrm{Fil}^{\mathrm{co}}_{\mathcal{I},\mathcal{E}}$.

In the trivial stratification situation, $\mathfrak{Fil}_{\Pi_\infty(X),\mathcal{E}}$ gives back locally constant hypersheaves. Before seeing this, let us introduce the following

Definition 5.1.6. Let X be a topological space. Let $\mathcal{E}$ be a presentable ∞-category. We denote by $\mathfrak{Loc}_{X,\mathcal{E}} \colon \mathrm{Open}(X)^{\mathrm{op}} \to \mathcal{E}$ the presheaf defined by $\mathfrak{Loc}_{X,\mathcal{E}}(U) = \mathrm{Loc}^{\mathrm{hyp}}(U, \mathcal{E})$ for every $U \in \mathrm{Open}(X)$.

Proposition 5.1.7. *Consider a Stokes stratified space of the form $(X, *, \Pi_\infty(X))$. Let $\mathcal{E}$ be a presentable ∞-category. Then, $\mathfrak{Fil}_{\Pi_\infty(X),\mathcal{E}}$ is canonically equivalent to $\mathfrak{Loc}_{X,\mathcal{E}}$ (see Definition 5.1.6).*

Proof. In that case, $\exp_{\mathcal{E}}(\Pi_\infty(X)/\Pi_\infty(X))$ is the constant fibration $\Pi_\infty(X) \times \mathcal{E} \to \Pi_\infty(X)$. Since $\Pi_\infty(X)$ is an ∞-groupoïd, every section of $\exp_{\mathcal{E}}(\Pi_\infty(X)/\Pi_\infty(X))$ is cocartesian. Thus Remark 5.1.3 yields a canonical equivalence

$$\mathfrak{Fil}_{\Pi_\infty(X),\mathcal{E}}(U) \simeq \mathrm{Fun}(\Pi_\infty(U), \mathcal{E})$$

for every $U \in \mathrm{Open}(X)$. Since X is exodromic, the conclusion follows from the monodromy equivalence. □

5.2. **The hyperconstructible hypersheaves of Stokes functors.** The categorical hyperconstructible hypersheaf $\mathfrak{Fil}_{\mathcal{I},\mathcal{E}}$ is not yet the main object of interest of this paper. In order to obtain the subsheaf of $\mathcal{E}$-valued Stokes structures, we need the following easy observation:

Lemma 5.2.1. *Let $\mathcal{X} \in \mathbf{Cat}_\infty$ and let*

$$\begin{tikzcd} \mathcal{B} \arrow[rr, "f"] \arrow[dr, "q"'] & & \mathcal{A} \arrow[dl, "p"] \\ & \mathcal{X} & \end{tikzcd}$$

be a commutative diagram where p and q are cocartesian fibrations and f preserves cocartesian edges. Letting $\mathrm{EssIm}(f)$ *be the essential image of f, the composition*

$$\mathrm{EssIm}(f) \subseteq \mathcal{A} \xrightarrow{p} \mathcal{X}$$

is again a cocartesian fibration. Furthermore, the formation of $\mathrm{EssIm}(f)$ *commutes with pullback along any morphism $\mathcal{Y} \to \mathcal{X}$ in $\mathbf{Cat}_\infty$. In particular, the fibers of* $\mathrm{EssIm}(f)$ *at $x \in \mathcal{X}$ canonically coincide with the essential image of $f_x \colon \mathcal{B}_x \to \mathcal{A}_x$.*

Notation 5.2.2. We let
$$(-)^{\mathrm{set}} \colon \mathbf{Poset} \to \mathbf{Poset}$$
be the functor sending a poset $(I, \leqslant)$ to the underlying set I, seen as a poset with trivial order. By extension, if $\mathcal{X} \in \mathbf{Cat}_\infty$ and $\mathcal{I} \to \mathcal{X}$ is a fibration in posets, we let $\mathcal{I}^{\mathrm{set}}$ be the cocartesian fibration on $\mathcal{X}$ obtained by applying $(-)^{\mathrm{set}}$ fiberwise. In a more verbose way, if $\mathscr{I} \colon \mathcal{X} \to \mathbf{Poset}$ is the unstraightening of $\mathcal{I}$, then $\mathcal{I}^{\mathrm{set}}$ is the cocartesian fibration classifying the composition $(-)^{\mathrm{set}} \circ \mathscr{I} \colon \mathcal{X} \to \mathbf{Poset}$. Notice that $\mathcal{I}^{\mathrm{set}}$ is in fact a left fibration over $\mathcal{X}$ and that it comes equipped with a canonical morphism
$$i_{\mathcal{I}} \colon \mathcal{I}^{\mathrm{set}} \to \mathcal{I}$$
that preserves cocartesian edges over $\mathcal{X}$. It is immediate that this construction promotes to a global functor
$$(-)^{\mathrm{set}} \colon \mathbf{PosFib} \to \mathbf{PosFib} ,$$
equipped with a natural transformation $i \colon (-)^{\mathrm{set}} \to \mathrm{id}_{\mathbf{PosFib}}$.

Let $(X, P, \mathcal{I})$ be Stokes stratified space. Let $\mathcal{E}$ be a presentable ∞-category. The functoriality of the exponential construction induces a well defined exponential induction functor
$$\mathcal{E}^{i_{\mathcal{I}}}_! \colon \exp_{\mathcal{E}}(\mathcal{I}^{\mathrm{set}}/\Pi_\infty(X,P)) \to \exp_{\mathcal{E}}(\mathcal{I}/\Pi_\infty(X,P))$$
in $\mathbf{PrFib}^{\mathrm{L}}_{\Pi_\infty(X,P)}$.

Definition 5.2.3. Let $(X, P, \mathcal{I})$ be Stokes stratified space. Let $\mathcal{E}$ be a presentable ∞-category. The *categorical sheaf of $\mathcal{E}$-valued $\mathcal{I}$-Stokes functors on* (X, P) is the object $\mathfrak{St}_{\mathcal{I},\mathcal{E}}$ in $\mathrm{Cons}_P(X; \mathbf{CAT}_\infty)$ corresponding to $\mathrm{EssIm}(\mathcal{E}^{i_{\mathcal{I}}}_!)$ via the equivalences of Observation 5.1.1. The *∞-category of $\mathcal{E}$-valued $\mathcal{I}$-Stokes functors* is the (large) ∞-category
$$\mathrm{St}_{\mathcal{I},\mathcal{E}} \coloneqq \mathfrak{St}_{\mathcal{I},\mathcal{E}}(X) \in \mathbf{CAT}_\infty$$
of global sections of $\mathfrak{St}_{\mathcal{I},\mathcal{E}}$.

Remark 5.2.4. By Remark 5.1.5, the global sections $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ of $\mathfrak{St}_{\mathcal{I},\mathcal{E}}$ embed fully faithful inside $\mathrm{Fil}^{\mathrm{co}}_{\mathcal{I},\mathcal{E}} \subset \mathrm{Fun}(\mathcal{I}, \mathcal{E})$. One can characterize $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ as the full subcategory of $\mathrm{Fil}^{\mathrm{co}}_{\mathcal{I},\mathcal{E}}$ spanned by punctually split cocartesian functors. See Definition 8.1.1.

Example 5.2.5. Let (X, P) be an exodromic stratified space. Review it as a Stokes stratified space $(X, P, \Pi_\infty(X, P))$, with the trivial cocartesian fibration given by the identity of $\Pi_\infty(X, P)$. Then there is a canonical equivalence
$$\mathfrak{St}_{\Pi_\infty(X,P),\mathcal{E}} \simeq \mathfrak{Loc}_{X,\mathcal{E}} ,$$
where $\mathfrak{Loc}_{X,\mathcal{E}}$ is the categorical sheaf of locally hyperconstant hypersheaves on X (see Definition 5.1.6). The proof relies on some more advanced material that will be developed in the main body of the paper. See Corollary 11.1.13. In other words, Stokes functors provide an extension of the theory of locally hyperconstant hypersheaves.

At the other extreme, we have:

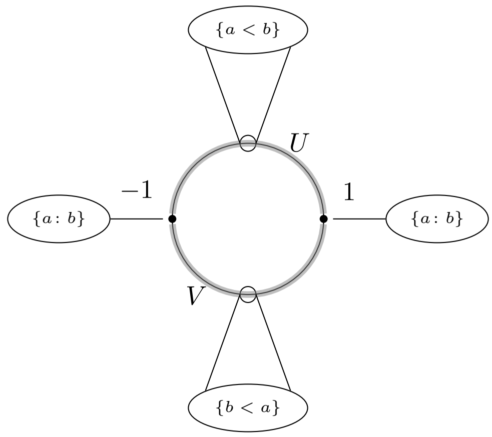
Example 5.2.6. Let $(X, P, \mathcal{I})$ be a Stokes stratified space. Assume that $\Pi_\infty(X, P)$ admits an initial object x. Then, in virtue of Remark 5.1.3, the ∞-category $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ coincides with the essential image of
$$i_{\mathcal{I}_x,!} \colon \mathrm{Fun}(\mathcal{I}^{\mathrm{set}}_x, \mathcal{E}) \to \mathrm{Fun}(\mathcal{I}_x, \mathcal{E}) .$$
Notice that the essential image of this functor is typically *not* presentable. Similarly, if $\mathcal{E}$ is assumed to be stable (or abelian), the (∞-)category $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ is not necessarily stable (nor abelian).

The following example is a particularly simple situation in dimension 1, but covers a large part of the ideas covered in this paper. For this reason, we encourage the reader to keep it in mind while reading the later parts:

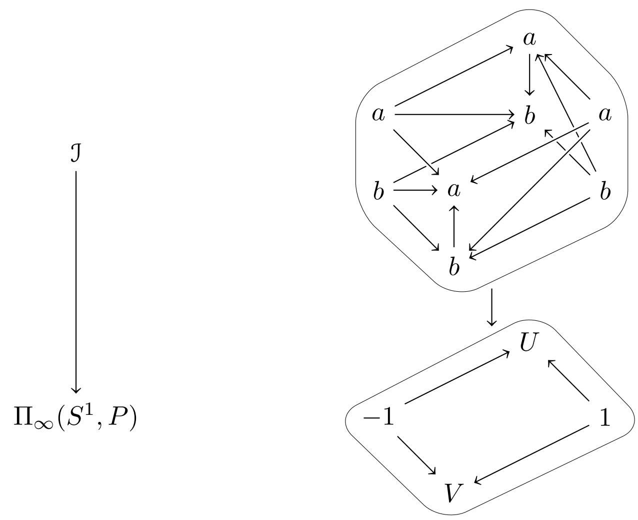
Example 5.2.7. On the circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ consider the stratification over the poset $P = \{0 < 1\}$ whose closed stratum is $\{1, -1\}$. Write

$$U \coloneqq \{z \in S^1 \mid \Im(z) > 0\} \qquad \text{and} \qquad V \coloneqq \{z \in S^1 \mid \Im(z) < 0\} \ .$$

Consider the P-constructible sheaf of posets $\mathscr{I}$ whose underlying sheaf of sets $\mathscr{I}^{\mathrm{set}}$ is the constant sheaf associated to $\{a, b\}$, and whose order is determined by the requirement that $a < b$ over U and $b < a$ over V, while a and b are not comparable at 1 and -1. The situation can be visualized as follows:



After applying the exodromy and the straightening equivalence, we are left with the following cocartesian fibration in posets over $\Pi_\infty(S^1, P)$:



Beware that different copies of a and b represent different objects in $\mathcal{I}$, lying over different objects of $\Pi_\infty(X, P)$. Arrows between identical letters correspond to cocartesian edges in $\mathcal{I}$. Take $\mathcal{E} \coloneqq \mathbf{Mod}_k$, where k is some field. Then both $\mathrm{Fil}^{\mathrm{co}}_{\mathcal{I},\mathcal{E}}$ and $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ can be realized as full subcategories of $\mathrm{Fun}(\mathcal{I}, \mathbf{Mod}_k)$. Although practical for many purposes, this is not the best way to handle these categories. Let us explain in this example how to exploit the sheaf theoretic nature of both $\mathrm{Fil}^{\mathrm{co}}_{\mathcal{I},\mathcal{E}}$

and $\mathrm{St}_{\mathcal{I},\mathcal{E}}$. Define the two opens

$$W_1 \coloneqq \{z \in S^1 \mid \Re(z) > -1\} \qquad \text{and} \qquad W_{-1} \coloneqq \{z \in S^1 \mid \Re(z) < 1\} \ ,$$

and let $W \coloneqq W_1 \cap W_{-1}$ be their intersection. For $i \in \{1, -1\}$, put

$$\mathcal{I}_{W_i} \coloneqq \Pi_\infty(W_i) \times_{\Pi_\infty(S^1,P)} \mathcal{I} \ .$$

Since $\mathfrak{Fil}_{\mathcal{I},\mathcal{E}}$ and $\mathfrak{St}_{\mathcal{I},\mathcal{E}}$ are sheaves, we deduce that the squares

$$\begin{tikzcd} \mathrm{Fil}^{\mathrm{co}}_{\mathcal{I},\mathcal{E}} \arrow[r] \arrow[d] & \mathrm{Fil}^{\mathrm{co}}_{\mathcal{I}_{W_1},\mathcal{E}} \arrow[d] \\ \mathrm{Fil}^{\mathrm{co}}_{\mathcal{I}_{W_{-1}},\mathcal{E}} \arrow[r] & \mathrm{Fil}^{\mathrm{co}}_{\mathcal{I}_W,\mathcal{E}} \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \mathrm{St}_{\mathcal{I},\mathcal{E}} \arrow[r] \arrow[d] & \mathrm{St}_{\mathcal{I}_{W_1},\mathcal{E}} \arrow[d] \\ \mathrm{St}_{\mathcal{I}_{W_{-1}},\mathcal{E}} \arrow[r] & \mathrm{St}_{\mathcal{I}_W,\mathcal{E}} \end{tikzcd}$$

are pullbacks. Now, observe that:

(i) since 1 is initial in $\Pi_\infty(W_1, P)$, we have $\mathrm{Fil}^{\mathrm{co}}_{\mathcal{I}_{W_1},\mathcal{E}} \simeq \mathrm{Fun}(\mathcal{I}_1, \mathcal{E}) \simeq \mathcal{E} \times \mathcal{E}$ and $\mathrm{St}_{\mathcal{I}_{W_1},\mathcal{E}} \simeq \mathrm{St}_{\mathcal{I}_1,\mathcal{E}}$;

(ii) since the order on $\mathcal{I}_1 = \{a \colon b\}$ is trivial, we have $\mathcal{I}_1^{\mathrm{set}} = \mathcal{I}_1$, and therefore $\mathrm{St}_{\mathcal{I}_{W_1},\mathcal{E}} = \mathrm{Fil}^{\mathrm{co}}_{\mathcal{I}_{W_1},\mathcal{E}}$.

A symmetrical reasoning applies with -1 in place of 1. Full faithfulness of $\mathrm{St}_{\mathcal{I}_W,\mathcal{E}} \hookrightarrow \mathrm{Fil}^{\mathrm{co}}_{\mathcal{I}_W,\mathcal{E}}$ ensures that the induced map

$$\mathrm{St}_{\mathcal{I}_{W_1},\mathcal{E}} \times_{\mathrm{St}_{\mathcal{I}_W,\mathcal{E}}} \mathrm{St}_{\mathcal{I}_{W_{-1}},\mathcal{E}} \to \mathrm{St}_{\mathcal{I}_{W_1},\mathcal{E}} \times_{\mathrm{Fil}^{\mathrm{co}}_{\mathcal{I}_W,\mathcal{E}}} \mathrm{St}_{\mathcal{I}_{W_{-1}},\mathcal{E}}$$

is an equivalence. Hence, the canonical map

$$\mathrm{St}_{\mathcal{I},\mathcal{E}} \to \mathrm{Fil}^{\mathrm{co}}_{\mathcal{I},\mathcal{E}}$$

is an equivalence. In particular, $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ is stable.

Part 2. Categorical aspects

In this part, we develop the categorical framework needed to deal with Stokes functors. Stokes functors are by definition filtered functors that are in addition *cocartesian* (see Definition 7.2.1) and *punctually split* (see Definition 8.1.1). These two properties are examined in detail through the lenses of the *specialization equivalence*. Finally, we introduce two fundamental concepts: that of *graduation* (see Definition 9.1.7) and of *level morphism* (see Definition 10.1.1). Theorem 10.2.1 can be seen in many ways as the crucial result of this part: it provides the categorical basis of the level induction technique that will be used in the next part.

6. The specialization equivalence

6.1. Global functoriality statements. Fix a cocartesian fibration $p\colon \mathcal{A} \to \mathcal{X}$ as well as a presentable ∞-category $\mathcal{E}$. Write

$$p_{\mathcal{E}} \colon \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X}) \to \mathcal{X}$$

for the structural map of the exponential construction of p. Recall from Proposition 3.3.3 that there is a canonical equivalence

$$\mathrm{sp}^{\mathcal{E}}_{\mathcal{X},p} \colon \mathrm{Fun}(\mathcal{A},\mathcal{E}) \simeq \mathrm{Fun}_{/\mathcal{X}}(\mathcal{X}, \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X})) \ , \tag{6.1.1}$$

which we refer to as the *specialization equivalence*. When $\mathcal{X}$ and $\mathcal{E}$ are clear out of the context, we will use the notation $\mathrm{sp}_{\mathcal{A}}$ (or even just sp) instead of $\mathrm{sp}^{\mathcal{E}}_{\mathcal{X},p}$.

The right hand side of (6.1.1) is functorial in $p\colon \mathcal{A} \to \mathcal{X}$ with respect to the morphisms in **CoCart**. Explicitly, this means that to every morphism (3.1.3)

$$\begin{tikzcd} \mathcal{B} \arrow[d] & \mathcal{B}_{\mathcal{X}} \arrow[l, "u"'] \arrow[r, "v"] \arrow[d] & \mathcal{A} \arrow[dl] \\ \mathcal{Y} & \mathcal{X} \arrow[l, "f"] & \end{tikzcd}$$

one can first apply $\exp_{\mathcal{E}} \colon \mathbf{CoCart} \to \mathbf{PrFib}^{\mathrm{L}}$ to obtain the morphism

$$\begin{tikzcd} \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y}) \arrow[d] & \exp_{\mathcal{E}}(\mathcal{B}_{\mathcal{X}}/\mathcal{X}) \arrow[l, "\mathcal{E}^u"'] \arrow[r, "\mathcal{E}^v_!"] \arrow[d] & \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X}) \arrow[dl] \\ \mathcal{Y} & \mathcal{X} \arrow[l, "f"] & \end{tikzcd}$$

and then apply the section functor $\Sigma\colon \mathbf{PrFib}^{\mathrm{L}} \to \mathbf{Pr}^{\mathrm{L}}$ to obtain the composition

$$\mathrm{Fun}_{/\mathcal{Y}}(\mathcal{Y}, \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y})) \xrightarrow{\Sigma(\mathcal{E}^u)} \mathrm{Fun}_{/\mathcal{X}}(\mathcal{X}, \exp_{\mathcal{E}}(\mathcal{B}_{\mathcal{X}}/\mathcal{X})) \xrightarrow{\Sigma(\mathcal{E}^v_!)} \mathrm{Fun}_{/\mathcal{X}}(\mathcal{X}, \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X})) \ .$$

We defer to [40] for the justification that these operations can be performed in an ∞-functorial way. The goal of this section is to explain how this functoriality interacts with the specialization equivalence. More precisely, observe that applying (6.1.1) to every term in the above composition, we obtain respectively $\mathrm{Fun}(\mathcal{B},\mathcal{E})$, $\mathrm{Fun}(\mathcal{B}_{\mathcal{X}},\mathcal{E})$ and $\mathrm{Fun}(\mathcal{A},\mathcal{E})$. The following is the main result of this section:

Proposition 6.1.2.

(1) There exists a canonically commutative square

$$
\begin{tikzcd}
\mathrm{Fun}_{/\mathcal{Y}}(\mathcal{Y}, \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y})) \arrow[r, "\Sigma(\mathcal{E}^u)"] \arrow[d, "\mathrm{sp}_{\mathcal{B}}"] & \mathrm{Fun}_{/\mathcal{X}}(\mathcal{X}, \exp_{\mathcal{E}}(\mathcal{B}_{\mathcal{X}}/\mathcal{X})) \arrow[d, "\mathrm{sp}_{\mathcal{B}_{\mathcal{X}}}"] \\
\mathrm{Fun}(\mathcal{B}, \mathcal{E}) \arrow[r, "u^*"] & \mathrm{Fun}(\mathcal{B}_{\mathcal{X}}, \mathcal{E}) \ ,
\end{tikzcd}
\tag{6.1.3}
$$

providing a canonical identification $\Sigma(\mathcal{E}^u) \simeq u^*$.

(2) There exists a canonically commutative square

$$
\begin{tikzcd}
\mathrm{Fun}_{/\mathcal{X}}(\mathcal{X}, \exp_{\mathcal{E}}(\mathcal{B}_{\mathcal{X}}/\mathcal{X})) \arrow[r, "\Sigma(\mathcal{E}^v_!)"] \arrow[d, "\mathrm{sp}_{\mathcal{B}_{\mathcal{X}}}"] & \mathrm{Fun}_{/\mathcal{X}}(\mathcal{X}, \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X})) \arrow[d, "\mathrm{sp}_{\mathcal{A}}"] \\
\mathrm{Fun}(\mathcal{B}_{\mathcal{X}}, \mathcal{E}) \arrow[r, "v_!"] & \mathrm{Fun}(\mathcal{A}, \mathcal{E}) \ ,
\end{tikzcd}
\tag{6.1.4}
$$

providing a canonical identification $\Sigma(\mathcal{E}^v_!) \simeq v_!$.

Before starting the proof, let us record a couple of handy consequences. First, recall from Corollary 3.2.8 that the fiber of $\exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X})$ at $x \in \mathcal{X}$ is canonically identified with $\mathrm{Fun}(\mathcal{A}_x, \mathcal{E})$. In particular, this means that for a functor $F \colon \mathcal{A} \to \mathcal{E}$, the value of its specialization $\mathrm{sp}\, F$ at an object $x \in \mathcal{X}$ is a functor

$$(\mathrm{sp}\, F)_x \colon \mathcal{A}_x \to \mathcal{E} \ .$$

We have:

Corollary 6.1.5. *Let* $j_x \colon \mathcal{A}_x \to \mathcal{A}$ *be the natural inclusion. Then there is a canonical identification*

$$(\mathrm{sp}\, F)_x \simeq j_x^*(F) \ .$$

In particular, for every $a \in \mathcal{A}$ *we have a canonical identification*

$$(\mathrm{sp}\, F)_{p(a)}(a) \simeq F(a) \in \mathcal{E} \ .$$

Proof. The pullback square

$$
\begin{tikzcd}
\mathcal{A} \arrow[d, "p"'] & \mathcal{A}_x \arrow[l] \arrow[d] \\
\mathcal{X} & * \arrow[l, "x"']
\end{tikzcd}
$$

defines a morphism from $\mathcal{A} \to \mathcal{X}$ to $\mathcal{A}_x \to *$ in **CoCart**. It is then enough to apply Proposition 6.1.2-(1) to this morphism. □

Corollary 6.1.6. *Consider a commutative diagram in* **CoCart**

$$
\begin{tikzcd}
\mathcal{B}_{\mathcal{Y}} \arrow[rr, "v_{\mathcal{Y}}"] \arrow[dr] \arrow[drr, "u_{\mathcal{B}}"] & & \mathcal{A}_{\mathcal{Y}} \arrow[dl] \arrow[drr, "u_{\mathcal{A}}"] & & \\
& \mathcal{Y} \arrow[drr] & \mathcal{B} \arrow[rr, "v"] \arrow[dr] & & \mathcal{A} \arrow[dl] \\
& & & \mathcal{X} &
\end{tikzcd}
$$

whose diagonal squares are pullback. Let $\mathcal{E}$ be a presentable ∞-category. Then, the squares

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{B}_{\mathcal{Y}}, \mathcal{E}) & \mathrm{Fun}(\mathcal{A}_{\mathcal{Y}}, \mathcal{E}) \arrow[l, "v_{\mathcal{Y}}^*"'] \\ \mathrm{Fun}(\mathcal{B}, \mathcal{E}) \arrow[u, "u_{\mathcal{B}}^*"] & \mathrm{Fun}(\mathcal{A}, \mathcal{E}) \arrow[l, "v^*"] \arrow[u, "u_{\mathcal{A}}^*"'] \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} \mathrm{Fun}(\mathcal{B}_{\mathcal{Y}}, \mathcal{E}) & \mathrm{Fun}(\mathcal{B}, \mathcal{E}) \arrow[l, "u_{\mathcal{B}}^*"'] \\ \mathrm{Fun}(\mathcal{A}_{\mathcal{Y}}, \mathcal{E}) \arrow[u, "v_{\mathcal{Y}}^*"] & \mathrm{Fun}(\mathcal{A}, \mathcal{E}) \arrow[l, "u_{\mathcal{A}}^*"] \arrow[u, "v^*"'] \end{tikzcd}$$

are respectively horizontally left and right adjointable.

Proof. It is enough to prove the left adjointability statement, which follows by applying the section functor to the commutative diagram

$$\begin{tikzcd} \exp_{\mathcal{E}}(\mathcal{B}_{\mathcal{Y}}/\mathcal{Y}) \arrow[rr, "\mathcal{E}_!^{v_{\mathcal{Y}}}"] \arrow[dr] \arrow[drr, "\mathcal{E}^{u_{\mathcal{B}}}"] & & \exp_{\mathcal{E}}(\mathcal{A}_{\mathcal{Y}}/\mathcal{Y}) \arrow[dl] \arrow[drr, "\mathcal{E}^{u_{\mathcal{A}}}"] & & \\ & \mathcal{Y} \arrow[drr] & \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{X}) \arrow[rr, "\mathcal{E}_!^{v}"] \arrow[dr] & & \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X}) \arrow[dl] \\ & & & \mathcal{X} & \end{tikzcd}$$

supplied by Corollary 3.2.7 and then invoke Proposition 6.1.2. □

6.2. **Some categorical calculus.** As a preliminary for Proposition 6.1.2, we revisit and extend part of the content of [20].

Recollection 6.2.1. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a functor of ∞-categories. The pullback

$$f^* \coloneqq \mathcal{X} \times_{\mathcal{Y}} - \colon \mathbf{Cat}_{\infty/\mathcal{Y}} \to \mathbf{Cat}_{\infty/\mathcal{X}}$$

preserves (co)cartesian fibrations and therefore it gives rise to functors

$$f^*\colon \mathbf{Cart}_{\mathcal{Y}} \to \mathbf{Cart}_{\mathcal{X}} \qquad \text{and} \qquad f^*\colon \mathbf{CoCart}_{\mathcal{Y}} \to \mathbf{CoCart}_{\mathcal{X}} \ .$$

Under the straightening equivalences, we see these functors admit both a left and a right adjoint, denoted respectively

$$f_!^{\mathrm{c}}, f_*^{\mathrm{c}}\colon \mathbf{Cart}_{\mathcal{X}} \to \mathbf{Cart}_{\mathcal{Y}} \qquad \text{and} \qquad f_!^{\mathrm{cc}}, f_*^{\mathrm{cc}}\colon \mathbf{CoCart}_{\mathcal{X}} \to \mathbf{CoCart}_{\mathcal{Y}} \ .$$

Recall the following standard notation in category theory:

Notation 6.2.2. Let $\mathcal{X}$ be an ∞-category. We write $\mathrm{Tw}(\mathcal{X})$ for the associated ∞-category of twisting arrows, see [32, §5.2.1] and $\lambda\colon \mathrm{Tw}(\mathcal{X}) \to \mathcal{X} \times \mathcal{X}^{\mathrm{op}}$ for the right fibration constructed in [32, Proposition 5.2.1.11]. Given a functor

$$F\colon \mathcal{X} \times \mathcal{X}^{\mathrm{op}} \to \mathbf{Cat}_\infty \ ,$$

we respectively write

$$\int^{\mathcal{X}} F \qquad \text{and} \qquad \int_{\mathcal{X}} F$$

for the coend and the end of F, i.e. for the colimit and the limit of the composite

$$\mathrm{Tw}(\mathcal{X}) \xrightarrow{\lambda} \mathcal{X} \times \mathcal{X}^{\mathrm{op}} \xrightarrow{F} \mathbf{Cat}_\infty \ .$$

Notation 6.2.3. Write $\pi_{\mathcal{X}}\colon \mathcal{X} \times \mathcal{X}^{\mathrm{op}} \to \mathcal{X}$ and $\pi_{\mathcal{X}^{\mathrm{op}}}\colon \mathcal{X} \times \mathcal{X}^{\mathrm{op}} \to \mathcal{X}^{\mathrm{op}}$ for the canonical projections. Given two functors

$$F\colon \mathcal{X} \to \mathbf{Cat}_\infty \qquad \text{and} \qquad G\colon \mathcal{X}^{\mathrm{op}} \to \mathbf{Cat}_\infty \ ,$$

we write $F \boxtimes G$ for the functor

$$F \boxtimes G \coloneqq \pi_{\mathcal{X}}^*(F) \times \pi_{\mathcal{X}^{\mathrm{op}}}^*(G) \ .$$

When $\mathcal{A} \to \mathcal{X}$ is a cocartesian fibration and $\mathcal{B} \to \mathcal{X}$ is a cartesian fibration, we write

$$\int^{\mathcal{X}} \mathcal{A} \boxtimes \mathcal{B} \coloneqq \int^{\mathcal{X}} \Upsilon_{\mathcal{A}} \boxtimes \Upsilon_{\mathcal{B}} \qquad \text{and} \qquad \int_{\mathcal{X}} \mathcal{A} \boxtimes \mathcal{B} \coloneqq \int_{\mathcal{X}} \Upsilon_{\mathcal{A}} \boxtimes \Upsilon_{\mathcal{B}} \ .$$

To state the first fundamental result, we need to introduce one final notation:

Notation 6.2.4. Let $\mathcal{X}$ and $\mathcal{E}$ be two ∞-categories. For $\mathcal{A} \to \mathcal{X}$ a cartesian fibration, write $\Upsilon_{\mathcal{A}} \colon \mathcal{X}^{\mathrm{op}} \to \mathbf{Cat}_\infty$ for its straightening and $\mathcal{E}^{\mathcal{A}}_{\mathrm{cc}}$ for the *cocartesian* fibration classifying the functor

$$\mathrm{Fun}(\Upsilon_{\mathcal{A}}, \mathcal{E}) \colon \mathcal{X} \to \mathbf{Cat}_\infty \ .$$

Similarly, for a cocartesian fibration $\mathcal{B} \to \mathcal{A}$, write $\Upsilon_{\mathcal{B}} \colon \mathcal{X} \to \mathbf{Cat}_\infty$ for its straightening and $\mathcal{E}^{\mathcal{B}}_{\mathrm{c}}$ for the *cartesian* fibration classifying the functor

$$\mathrm{Fun}(\Upsilon_{\mathcal{B}}, \mathcal{E}) \colon \mathcal{X}^{\mathrm{op}} \to \mathbf{Cat}_\infty \ .$$

Notice that given a functor $f \colon \mathcal{Y} \to \mathcal{X}$, there are canonical equivalences

$$f^* \mathcal{E}^{\mathcal{A}}_{\mathrm{cc}} \simeq \mathcal{E}^{f^*(\mathcal{A})}_{\mathrm{cc}} \qquad \text{and} \qquad f^* \mathcal{E}^{\mathcal{B}}_{\mathrm{c}} \simeq \mathcal{E}^{f^*(\mathcal{B})}_{\mathrm{c}} \ . \tag{6.2.5}$$

Lemma 6.2.6. *Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a functor of ∞-categories.*

(1) For $\mathcal{B} \to \mathcal{X}$ a cartesian fibration and $\mathcal{A} \to \mathcal{Y}$ a cocartesian fibration, there is a canonical equivalence

$$\int^{\mathcal{Y}} \mathcal{A} \boxtimes f^{\mathrm{c}}_!(\mathcal{B}) \simeq \int^{\mathcal{X}} f^*(\mathcal{A}) \boxtimes \mathcal{B} \ .$$

(2) For $\mathcal{B} \to \mathcal{X}$ a cocartesian fibration and $\mathcal{A} \to \mathcal{Y}$ a cartesian fibration, there is a canonical equivalence

$$\int^{\mathcal{Y}} \mathcal{A} \boxtimes f^{\mathrm{cc}}_!(\mathcal{B}) \simeq \int^{\mathcal{X}} f^*(\mathcal{A}) \boxtimes \mathcal{B} \ .$$

Proof. To prove (1), it suffices to fix $\mathcal{E} \in \mathbf{Cat}_\infty$ and observe that there is the following chain of natural equivalences:

$$\begin{aligned}
\mathrm{Map}_{\mathbf{Cat}_\infty}\Big(\int^{\mathcal{Y}} \mathcal{A} \boxtimes f^{\mathrm{c}}_!(\mathcal{B}), \mathcal{E}\Big) &\simeq \int_{\mathcal{Y}} \mathrm{Map}_{\mathbf{Cat}_\infty}(\Upsilon_{\mathcal{A}} \boxtimes \Upsilon_{f^{\mathrm{c}}_!(\mathcal{B})}, \mathcal{E}) \\
&\simeq \int_{\mathcal{Y}} \mathrm{Map}_{\mathbf{Cat}_\infty}(\Upsilon_{f^{\mathrm{c}}_!(\mathcal{B})}, \mathrm{Fun}(\Upsilon_{\mathcal{A}}, \mathcal{E})) \\
&\simeq \mathrm{Map}_{\mathbf{Cart}_{\mathcal{Y}}}(f^{\mathrm{c}}_!(\mathcal{B}), \mathcal{E}^{\mathcal{A}}_{\mathrm{c}}) && \text{By [20, Prop. 6.9]} \\
&\simeq \mathrm{Map}_{\mathbf{Cart}_{\mathcal{X}}}(\mathcal{B}, f^* \mathcal{E}^{\mathcal{A}}_{\mathrm{c}}) \\
&\simeq \mathrm{Map}_{\mathbf{Cart}_{\mathcal{X}}}(\mathcal{B}, \mathcal{E}^{f^*(\mathcal{A})}_{\mathrm{c}}) && \text{By eq. (6.2.5)} \\
&\simeq \int_{\mathcal{X}} \mathrm{Map}_{\mathbf{Cat}_\infty}(\Upsilon_{\mathcal{B}}, \mathrm{Fun}(\Upsilon_{f^*(\mathcal{A})}, \mathcal{E})) \\
&\simeq \mathrm{Map}_{\mathbf{Cat}_\infty}\Big(\int^{\mathcal{X}} f^*(\mathcal{A}) \boxtimes \mathcal{B}, \mathcal{E}\Big) \ ,
\end{aligned}$$

so the conclusion follows from the Yoneda lemma. As for (2), it follows by the same argument, using $\mathcal{E}^{\mathcal{A}}_{\mathrm{cc}}$ instead of $\mathcal{E}^{\mathcal{A}}_{\mathrm{c}}$ and working in $\mathbf{CoCart}_{\mathcal{Y}}$ instead of in $\mathbf{Cart}_{\mathcal{Y}}$. ☐

Next, recall the following:

Theorem 6.2.7 ([20, Theorem 4.5])**.** *Let $\mathcal{X}$ be an ∞-category. The forgetful functor*

$$\mathrm{U}_{\mathcal{X}} \colon \mathbf{Cart}_{\mathcal{X}} \to \mathbf{Cat}_{\infty/\mathcal{X}}$$

admits a left adjoint $\mathrm{F}_{\mathcal{X}}$.

Remark 6.2.8. Given a functor $f\colon \mathcal{Y} \to \mathcal{X}$, we refer to $\mathrm{F}_{\mathcal{X}}(f)$ as the *free cartesian fibration over $\mathcal{X}$ generated by f*. It follows from the explicit description provided in [20, Definition 4.1 & Remark 4.4], that $\mathrm{F}_{\mathcal{X}}$ satisfies the following two conditions:

(1) when $f = \mathrm{id}_{\mathcal{X}}$, $\mathrm{F}_{\mathcal{X}}(\mathrm{id}_{\mathcal{X}}) = \mathcal{X}^{[1]}$, and the structural map is $\mathrm{ev}_1\colon \mathcal{X}^{[1]} \to \mathcal{X}$. In other words, $\mathrm{F}_{\mathcal{X}}(\mathrm{id}_{\mathcal{X}})$ classifies the functor
$$\mathcal{X}_{-/}\colon \mathcal{X}^{\mathrm{op}} \to \mathbf{Cat}_\infty \ .$$

(2) For a general $f\colon \mathcal{Y} \to \mathcal{X}$, one has the following commutative diagram
$$\begin{tikzcd} \mathrm{F}_{\mathcal{X}}(f) \arrow[r] \arrow[d] & \mathcal{X}^{[1]} \arrow[r, "\mathrm{ev}_1"] \arrow[d, "\mathrm{ev}_0"] & \mathcal{X} \\ \mathcal{Y} \arrow[r, "f"] & \mathcal{X} & \end{tikzcd} ,$$
where the left square is a pullback and where the top horizontal composition is the structural map of the cartesian fibration $\mathrm{F}_{\mathcal{X}}(f)$.

Lemma 6.2.9. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a functor of ∞-categories. Then there is a canonical equivalence*
$$f_!^{\mathrm{c}}(\mathrm{F}_{\mathcal{X}}(\mathrm{id}_{\mathcal{X}})) \simeq \mathrm{F}_{\mathcal{Y}}(f)$$
in $\mathbf{Cart}_{\mathcal{Y}}$.

Proof. Indeed, for every cartesian fibration $\mathcal{B} \to \mathcal{Y}$, we have:
$$\begin{aligned} \mathrm{Map}_{\mathbf{Cart}_{\mathcal{Y}}}(f_!(\mathrm{F}_{\mathcal{X}}(\mathrm{id}_{\mathcal{X}})), \mathcal{B}) &\simeq \mathrm{Map}_{\mathbf{Cart}_{\mathcal{X}}}(\mathrm{F}_{\mathcal{X}}(\mathrm{id}_{\mathcal{X}}), f^*(\mathcal{B})) \\ &\simeq \mathrm{Map}_{/\mathcal{X}}(\mathcal{X}, f^*(\mathcal{B})) \\ &\simeq \mathrm{Map}_{/\mathcal{Y}}(\mathcal{X}, \mathcal{B}) \\ &\simeq \mathrm{Map}_{\mathbf{Cart}_{\mathcal{Y}}}(\mathrm{F}_{\mathcal{Y}}(f), \mathcal{B}) \ , \end{aligned}$$
so the conclusion follows from the Yoneda lemma. ☐

Finally, observe that [20, Proposition 7.1] can be rewritten as follows:

Corollary 6.2.10. *Let $\mathcal{X}$ be an ∞-category and let $\mathcal{A} \to \mathcal{X}$ be a cocartesian fibration. Then there is a canonical equivalence*
$$\mathcal{A} \simeq \int^{\mathcal{X}} \mathcal{A} \boxtimes \mathrm{F}_{\mathcal{X}}(\mathrm{id}_{\mathcal{X}})$$
in $\mathbf{Cat}_\infty$.

6.3. **Exponential pullback vs. global pullback.** Before proving Proposition 6.1.2-(1), let us revisit the proof of the equivalence (6.1.1) in terms of the categorical calculus we just introduced.

Recollection 6.3.1 ([40, Proposition 4.1])**.** Fix a cocartesian fibration $p\colon \mathcal{A} \to \mathcal{X}$ and a presentable ∞-category $\mathcal{E}$. Using the equivalence $\mathbf{Pr}^{\mathrm{L}} \simeq (\mathbf{Pr}^{\mathrm{R}})^{\mathrm{op}}$, we see that the presentable fibration $\exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X}) \to \mathcal{X}$ is at the same time a cocartesian and a cartesian fibration. Seen as a cartesian fibration, it classifies the functor
$$\mathrm{Fun}(\Upsilon_{\mathcal{A}}, \mathcal{E})\colon \mathcal{X}^{\mathrm{op}} \to \mathbf{Pr}^{\mathrm{R}} \ .$$

We use this second description to compute the sections of $\exp_{\mathcal{E},\mathcal{X}}(\mathcal{A})$. Then the specialization equivalence $\mathrm{sp}_{\mathcal{A}}$ is identified with the following composition of equivalences:

$$\begin{aligned}
\mathrm{Fun}_{/\mathcal{X}}(\mathcal{X}, \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X})) &\simeq \mathrm{Fun}^{\mathrm{cart}}_{/\mathcal{X}}(\mathrm{F}_{\mathcal{X}}(\mathrm{id}_{\mathcal{X}}), \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X})) \\
&\simeq \int_{\mathcal{X}} \mathrm{Fun}(\mathcal{X}_{-/}, \mathrm{Fun}(\Upsilon_{\mathcal{A}}, \mathcal{E})) && \text{By [20, Prop. 6.9] \& Rem. 6.2.8} \\
&\simeq \mathrm{Fun}\Big(\int^{\mathcal{X}} \mathcal{A} \boxtimes \mathrm{F}_{\mathcal{X}}(\mathrm{id}_{\mathcal{X}}), \mathcal{E}\Big) \\
&\simeq \mathrm{Fun}(\mathcal{A}, \mathcal{E}) && \text{By Cor. 6.2.10.}
\end{aligned}$$

We are now ready for:

Proof of Proposition 6.1.2-(1). Fix a pullback square

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "u"] \arrow[d] & \mathcal{A} \arrow[d] \\
\mathcal{X} \arrow[r, "f"] & \mathcal{Y}
\end{tikzcd} \tag{6.3.2}$$

where the vertical functors are cocartesian fibrations. Recall from Proposition 3.2.6-(1) the canonical equivalence

$$f^*(\exp_{\mathcal{E}}(\mathcal{A}/\mathcal{Y})) \simeq \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{X})$$

We therefore obtain a canonical equivalence

$$\begin{aligned}
\Sigma_{\mathcal{X}}(\exp_{\mathcal{E}}(\mathcal{B}/\mathcal{X})) &= \mathrm{Fun}_{/\mathcal{X}}(\mathcal{X}, \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{X})) \\
&\simeq \mathrm{Fun}_{/\mathcal{Y}}(\mathcal{X}, \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{Y})) \\
&\simeq \mathrm{Fun}^{\mathrm{cart}}_{/\mathcal{Y}}(\mathrm{F}_{\mathcal{Y}}(f), \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{Y})) \ .
\end{aligned}$$

Similarly,

$$\Sigma_{\mathcal{Y}}(\exp_{\mathcal{E}}(\mathcal{A}/\mathcal{Y})) \simeq \mathrm{Fun}^{\mathrm{cart}}_{/\mathcal{Y}}(\mathrm{F}_{\mathcal{Y}}(\mathrm{id}_{\mathcal{Y}}), \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{Y})) \ .$$

Since $\mathrm{id}_{\mathcal{Y}}$ is the final object in $\mathbf{Cat}_{\infty/\mathcal{Y}}$, we find a canonical map

$$\alpha_f \colon \mathrm{F}_{\mathcal{Y}}(f) \to \mathrm{F}_{\mathcal{Y}}(\mathrm{id}_{\mathcal{Y}})$$

in $\mathbf{Cart}_{\mathcal{Y}}$ between free cartesian fibrations, and unwinding the definitions we find that the sections of the exponential pullback $\Sigma(\mathcal{E}^u)$ are canonically identified with the functor

$$\alpha_f^* \colon \mathrm{Fun}^{\mathrm{cart}}_{/\mathcal{Y}}(\mathrm{F}_{\mathcal{Y}}(\mathrm{id}_{\mathcal{Y}}), \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{Y})) \to \mathrm{Fun}^{\mathrm{cart}}_{/\mathcal{Y}}(\mathrm{F}_{\mathcal{Y}}(f), \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{Y})) \ .$$

Applying the same chain of equivalences of Recollection 6.3.1, we find a canonical identification of α_f^* with the map

$$\mathrm{Fun}\Big(\int^{\mathcal{Y}} \mathcal{A} \boxtimes \mathrm{F}_{\mathcal{Y}}(\mathrm{id}_{\mathcal{Y}}), \mathcal{E}\Big) \to \mathrm{Fun}\Big(\int^{\mathcal{Y}} \mathcal{A} \boxtimes \mathrm{F}_{\mathcal{Y}}(f), \mathcal{E}\Big)$$

induced by pullback along the canonical map

$$\beta_f \colon \mathcal{A} \simeq \int^{\mathcal{Y}} \mathcal{A} \boxtimes \mathrm{F}_{\mathcal{Y}}(\mathrm{id}_{\mathcal{Y}}) \to \int^{\mathcal{Y}} \mathcal{A} \boxtimes \mathrm{F}_{\mathcal{Y}}(f)$$

constructed out of α_f. Recall now from Lemma 6.2.9 that there is a canonical equivalence

$$\mathrm{F}_{\mathcal{Y}}(f) \simeq f^{\mathrm{c}}_!(\mathrm{F}_{\mathcal{X}}(\mathrm{id}_{\mathcal{X}})) \ ,$$

so that Lemma 6.2.6 and Corollary 6.2.10 supply a canonical identification

$$\int^{\mathcal{Y}} \mathcal{A} \boxtimes \mathrm{F}_{\mathcal{Y}}(f) \simeq \int^{\mathcal{Y}} \mathcal{A} \boxtimes f^{\mathrm{c}}_!(\mathrm{F}_{\mathcal{X}}(\mathrm{id}_{\mathcal{X}})) \simeq \int^{\mathcal{X}} f^*(\mathcal{A}) \boxtimes \mathrm{F}_{\mathcal{X}}(\mathrm{id}_{\mathcal{X}}) \simeq \mathcal{B} \ .$$

Unwinding the definitions, we see that β_f is identified with u, whence the conclusion. $\square$

6.4. **Exponential induction vs. global induction.** We now deal with Proposition 6.1.2-(2). Fix an ∞-category $\mathcal{X}$ and consider a morphism

$$\begin{tikzcd} \mathcal{B} \arrow[rr, "v"] \arrow[dr] & & \mathcal{A} \arrow[dl] \\ & \mathcal{X} & \end{tikzcd}$$

in $\mathbf{CoCart}_{\mathcal{X}}$. Applying $\exp_{\mathcal{E}}(-/\mathcal{X})$, we find the morphism

$$\begin{tikzcd} \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{X}) \arrow[rr, "\mathcal{E}_!^v"] \arrow[dr] & & \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X}) \arrow[dl] \\ & \mathcal{X} & \end{tikzcd}$$

in $\mathbf{PrFib}_{\mathcal{X}}^{\mathrm{L}}$.

Lemma 6.4.1. *The functor $\mathcal{E}_!^v$ admits a right adjoint*

$$\mathcal{E}^{v,*} \colon \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X}) \to \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{X})$$

relative to $\mathcal{X}$.

Proof. Since both $\exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X})$ and $\exp_{\mathcal{E}}(\mathcal{B}/\mathcal{X})$ are cocartesian fibrations and $\mathcal{E}_!^v$ preserves cocartesian edges, applying [32, Proposition 7.3.2.6] shows that it is enough to prove that for every $x \in \mathcal{X}$, the induced functor on the fibers at x

$$\mathcal{E}_{!,x}^v \colon \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X})_x \to \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{X})_x$$

admits a right adjoint. However, Corollary 3.2.8 identifies this functor with the left Kan extension

$$v_{x,!} \colon \mathrm{Fun}(\mathcal{A}_x, \mathcal{E}) \to \mathrm{Fun}(\mathcal{B}_x, \mathcal{E}) \ ,$$

which is tautologically left adjoint to the restriction v_x^*. The conclusion follows. $\square$

At this point, Proposition 6.1.2 immediately follows from the following more precise statement:

Proposition 6.4.2. *Keeping the same notations as above, both diagrams*

$$\begin{tikzcd} \mathrm{Fun}_{/\mathcal{X}}(\mathcal{X}, \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X})) \arrow[r, "\Sigma_{\mathcal{X}}(\mathcal{E}_!^v)"] \arrow[d, "\mathrm{sp}_{\mathcal{A}}"] & \mathrm{Fun}_{/\mathcal{X}}(\mathcal{X}, \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{X})) \arrow[d, "\mathrm{sp}_{\mathcal{B}}"] \\ \mathrm{Fun}(\mathcal{A}, \mathcal{E}) \arrow[r, "v_!"] & \mathrm{Fun}(\mathcal{B}, \mathcal{E}) \end{tikzcd}$$

and

$$\begin{tikzcd} \mathrm{Fun}_{/\mathcal{X}}(\mathcal{X}, \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{X})) \arrow[r, "\Sigma_{\mathcal{X}}(\mathcal{E}^{v,*})"] \arrow[d, "\mathrm{sp}_{\mathcal{A}}"] & \mathrm{Fun}_{/\mathcal{X}}(\mathcal{X}, \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X})) \arrow[d, "\mathrm{sp}_{\mathcal{B}}"] \\ \mathrm{Fun}(\mathcal{B}, \mathcal{E}) \arrow[r, "v^*"] & \mathrm{Fun}(\mathcal{A}, \mathcal{E}) \end{tikzcd}$$

are canonically commutative.

Proof. Since $\mathcal{E}_!^v$ is left adjoint to $\mathcal{E}^{v,*}$ by Lemma 6.4.1, it follows that $\Sigma_{\mathcal{X}}(\mathcal{E}_!^v)$ is left adjoint to $\Sigma_{\mathcal{X}}(\mathcal{E}^{v,*})$. Since $\mathrm{sp}_{\mathcal{A}}$ and $\mathrm{sp}_{\mathcal{B}}$ are equivalences, it is then enough to prove the commutativity of the second diagram. Notice that since v preserves cocartesian arrows, it induces a natural transformation

$$\alpha_v \colon \Upsilon_{\mathcal{A}} \times \mathcal{X}_{-/} \to \Upsilon_{\mathcal{B}} \times \mathcal{X}_{-/}$$

of functors $\mathcal{X} \times \mathcal{X}^{\mathrm{op}} \to \mathbf{Cat}_\infty$. Following the construction of the specialization equivalence (see Recollection 6.3.1), we reduce to check that the map

$$\mathcal{A} \simeq \int^{\mathcal{X}} \mathcal{A} \boxtimes \mathrm{F}_{\mathcal{X}}(\mathrm{id}_{\mathcal{X}}) \to \int^{\mathcal{X}} \mathcal{B} \boxtimes \mathrm{F}_{\mathcal{X}}(\mathrm{id}_{\mathcal{X}}) \simeq \mathcal{B}$$

induced by α_v is canonically identified with v. This follows from Corollary 6.2.10 and the Yoneda lemma. $\square$

6.5. **Change of coefficients.** It follows from [40, Variant 3.20 & Remark 3.21-(1)] and the functoriality of the tensor product of presentable ∞-categories that the exponential construction $\exp_{\mathcal{E}}$ depends functorially on $\mathcal{E}$. In other words, we have a bifunctor

$$\exp \colon \mathbf{CoCart} \times \mathbf{Pr}^{\mathrm{L}} \to \mathbf{PrFib}^{\mathrm{L}} ,$$

that sends a pair $(p \colon \mathcal{A} \to \mathcal{X}, \mathcal{E})$ to the presentable fibration $p \colon \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X}) \to \mathcal{X}$.

Let $f \colon \mathcal{E} \to \mathcal{E}'$ be a morphism in $\mathbf{Pr}^{\mathrm{L}}$ and fix a cocartesian fibration $p \colon \mathcal{A} \to \mathcal{X}$. The functor f induces morphisms

$$f^{\mathcal{A}/\mathcal{X}} \colon \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X}) \to \exp_{\mathcal{E}'}(\mathcal{A}/\mathcal{X}) \qquad \text{and} \qquad f \colon \mathrm{Fun}(\mathcal{A}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{A}, \mathcal{E}') ,$$

in $\mathbf{PrFib}^{\mathrm{L}}$ and in $\mathbf{Pr}^{\mathrm{L}}$, respectively. Here we wrote f in place of the more accurate $f \circ (-)$, to keep the notations light. These two operations are related by the following relation:

Proposition 6.5.1. *Keeping the above notations, the diagram*

$$\begin{tikzcd}
\mathrm{Fun}_{/\mathcal{X}}(\mathcal{X}, \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X})) \arrow[r, "\Sigma_{\mathcal{X}}(f^{\mathcal{A}/\mathcal{X}})"] \arrow[d, "\mathrm{sp}^{\mathcal{E}}_{\mathcal{A}}"] & \mathrm{Fun}_{/\mathcal{X}}(\mathcal{X}, \exp_{\mathcal{E}'}(\mathcal{A}/\mathcal{X})) \arrow[d, "\mathrm{sp}^{\mathcal{E}'}_{\mathcal{A}}"] \\
\mathrm{Fun}(\mathcal{A}, \mathcal{E}) \arrow[r, "f"] & \mathrm{Fun}(\mathcal{A}, \mathcal{E}')
\end{tikzcd}$$

commutes.

Proof. This simply follows unraveling the chain of equivalences in Recollection 6.3.1 and observing that they are natural in $\mathcal{E}$. $\square$

Finally, let us observe that $f^{\mathcal{A}/\mathcal{X}}$ is natural in $\mathcal{A} \to \mathcal{X}$:

Proposition 6.5.2. *Let*

$$\begin{tikzcd}
\mathcal{B} \arrow[d] & \mathcal{B}_{\mathcal{X}} \arrow[l, "u"'] \arrow[r, "v"] \arrow[d] & \mathcal{A} \arrow[dl] \\
\mathcal{Y} & \mathcal{X} \arrow[l, "f"'] &
\end{tikzcd}$$

be a morphism in $\mathbf{CoCart}$ *and let* $f \colon \mathcal{E} \to \mathcal{D}$ *be a morphism in* $\mathbf{Pr}^{\mathrm{L}}$*. Then the diagram*

$$\begin{tikzcd}
\exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y}) \arrow[d, "f^{\mathcal{B}/\mathcal{Y}}"] & \exp_{\mathcal{E}}(\mathcal{B}_{\mathcal{X}}/\mathcal{X}) \arrow[l, "\mathcal{E}^u"'] \arrow[r, "\mathcal{E}^v_!"] \arrow[d, "f^{\mathcal{B}_{\mathcal{X}}/\mathcal{X}}"] & \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X}) \arrow[d, "f^{\mathcal{A}/\mathcal{X}}"] \\
\exp_{\mathcal{D}}(\mathcal{B}/\mathcal{Y}) & \exp_{\mathcal{D}}(\mathcal{B}_{\mathcal{X}}/\mathcal{X}) \arrow[l, "\mathcal{D}^u"'] \arrow[r, "\mathcal{D}^v_!"] & \exp_{\mathcal{D}}(\mathcal{A}/\mathcal{X})
\end{tikzcd}$$

commutes and the left square is a pullback. In particular, the diagram

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{B}, \mathcal{E}) \arrow[r, "u^*"] \arrow[d, "f"] & \mathrm{Fun}(\mathcal{B}_{\mathcal{X}}, \mathcal{E}) \arrow[r, "v_!"] \arrow[d, "f"] & \mathrm{Fun}(\mathcal{A}, \mathcal{E}) \arrow[d, "f"] \\
\mathrm{Fun}(\mathcal{B}, \mathcal{D}) \arrow[r, "u^*"] & \mathrm{Fun}(\mathcal{B}_{\mathcal{X}}, \mathcal{D}) \arrow[r, "v_!"] & \mathrm{Fun}(\mathcal{A}, \mathcal{E})
\end{tikzcd}$$

commutes.

Proof. The first half simply follows from the bifunctoriality of $\exp\colon \mathbf{CoCart} \times \mathbf{Pr}^{\mathrm{L}} \to \mathbf{PrFib}^{\mathrm{L}}$. The second half follows applying Σ and combining Propositions 6.1.2 and 6.5.1. Alternatively, the second half can be proven directly observing that, since f commutes with colimits, it also commutes with the formation of arbitrary left Kan extensions. $\square$

7. Cocartesian functors

7.1. **The space of specialization morphisms.** Fix a cocartesian fibration $p\colon \mathcal{A} \to \mathcal{X}$ as well as a presentable ∞-category $\mathcal{E}$. Write

$$p_{\mathcal{E}}\colon \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X}) \to \mathcal{X}$$

for the structural morphism of the exponential construction of p. Recall from Proposition 3.3.3 that there is a canonical equivalence

$$\mathrm{sp}^{\mathcal{E}}_{\mathcal{X},p}\colon \mathrm{Fun}(\mathcal{A},\mathcal{E}) \simeq \mathrm{Fun}_{/\mathcal{X}}(\mathcal{X}, \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X})) \ ,$$

which we refer to as the *specialization equivalence*. When $\mathcal{X}$, p and $\mathcal{E}$ are clear out of the context, we drop the decorations and write sp instead of $\mathrm{sp}^{\mathcal{E}}_{\mathcal{X},p}$.

Remark 7.1.1. Recall from Example 3.2.2-(1) that the fiber of $\exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X})$ at $x \in \mathcal{X}$ is canonically identified with $\mathrm{Fun}(\mathcal{A}_x, \mathcal{E})$. In particular, for $F\colon \mathcal{A} \to \mathcal{E}$, the value $(\mathrm{sp}\, F)_x$ of the section $\mathrm{sp}\, F$ on x is a functor $(\mathrm{sp}\, F)_x\colon \mathcal{A}_x \to \mathcal{E}$. Denoting by $j_x\colon \mathcal{A}_x \to \mathcal{A}$ the natural inclusion, Corollary 6.1.5 supplies a canonical identification $(\mathrm{sp}\, F)_x \simeq j_x^*(F)$.

Definition 7.1.2. Let $F \in \mathrm{Fun}(\mathcal{A},\mathcal{E})$ be a functor and let $\gamma\colon x \to y$ be a morphism in $\mathcal{X}$. The *space of specialization morphisms for F relative to γ* is the space $\mathrm{SP}_\gamma(F)$

$$(\mathrm{sp}\, F)_x \xrightarrow{\beta} G \xrightarrow{\alpha} (\mathrm{sp}\, F)_y$$

where β is a $p_{\mathcal{E}}$-cocartesian lift of γ in $\exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X})$. In this case, we say that α is a *specialization morphism for F relative to γ*.

Remark 7.1.3. Since $\exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X})$ is also a cartesian fibration, there is a dual notion of *cospecialization morphism*, that are obtained choosing $p_{\mathcal{E}}$-cartesian lifts of γ.

We immediately discuss a fundamental example.

Notation 7.1.4. Let $p\colon \mathcal{A} \to \mathcal{X}$ be a cocartesian fibration. For $\sigma\colon \Delta^n \to \mathcal{X}$, write $\mathcal{A}_\sigma \coloneqq \Delta^n \times_{\mathcal{X}} \mathcal{A}$ and $p_\sigma\colon \mathcal{A}_\sigma \to \Delta^n$ for the induced cocartesian fibration. Notice that Proposition 3.2.6 provides a canonical and functorial identification

$$\exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X})_\sigma \simeq \exp_{\mathcal{E}}(\mathcal{A}_\sigma/\Delta^n) \ .$$

Example 7.1.5. Let $p\colon \mathcal{A} \to \mathcal{X}$ be a cocartesian fibration and let $\gamma\colon x \to y$ be a morphism in $\mathcal{X}$. *Choose* a straightening

$$f\colon \mathcal{A}_x \to \mathcal{A}_y$$

for $p_\gamma\colon \mathcal{A}_\gamma \to \Delta^1$. The functor f fits in the following triangle

$$\begin{tikzcd} \mathcal{A}_x \arrow[rr, "f_\gamma"] \arrow[dr, "j_x"'] & & \mathcal{A}_y \arrow[dl, "j_y"] \\ & \mathcal{A} & \end{tikzcd}$$

where j_x and j_y denote the canonical inclusions of the fibers of p inside $\mathcal{A}$. This triangle is *not commutative* but we can choose a natural transformation

$$s \colon j_x \to j_y \circ f_\gamma$$

in $\mathrm{Fun}(\mathcal{A}_x, \mathcal{E})$ with the property that for every $a \in \mathcal{A}_x$ the morphism $s_\gamma(a) \colon j_x(a) \to j_y(f_\gamma(a))$ is p-cocartesian in $\mathcal{A}$. Applying the contravariant functor $\mathrm{Fun}(-, \mathcal{E})$ we obtain a natural transformation

$$s^* \colon j_x^* \to f_\gamma^* \circ j_y^*$$

of functors from $\mathrm{Fun}(\mathcal{A}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{A}_x, \mathcal{E})$. There is therefore an induced Beck-Chevalley morphism

$$\alpha_{f,s} \colon f_{\gamma,!} \circ j_x^* \to j_y^* \ . \tag{7.1.6}$$

Unraveling the definition of $\exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X})$ we see that for every $F \colon \mathcal{A} \to \mathcal{E}$, the induced morphism

$$\alpha_{f,s}(F) \colon f_{\gamma,!} j_x^*(F) \to j_y^*(F)$$

is a specialization morphism for F relative to γ.

Remark 7.1.7. Let $p \colon \mathcal{A} \to \mathcal{X}$ be a cocartesian fibration and let $F \colon \mathcal{A} \to \mathcal{E}$ be a fixed functor. Since $p_{\mathcal{E}} \colon \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X}) \to \mathcal{X}$ is a cocartesian fibration, it immediately follows that the space $\mathrm{SP}_\gamma(F)$ is contractible. Observe that, in the setting of Example 7.1.5, neither f nor s are uniquely determined in a strict sense (although the spaces of choices for the pair (f, s) is contractible). Every such choice gives rise to an element $\mathrm{SP}_\gamma(F)$, whose underlying specialization morphism is $\alpha_{f,s}(F)$. The contractibility of $\mathrm{SP}_\gamma(F)$ shows that the actual choices for f and s are immaterial as they give rise to equivalent specialization morphisms, and this in a homotopy unambiguous way.

Example 7.1.8. We maintain the notation introduced in Example 7.1.5. It is worth unpacking the specialization equivalence when $\mathcal{X} = \Delta^1$. Write $\Delta^1 = \{\gamma \colon x \to y\}$ and fix a cocartesian fibration $p \colon \mathcal{A} \to \Delta^1$ together with a straightening $f_\gamma \colon \mathcal{A}_x \to \mathcal{A}_y$ and a natural transformation $s \colon j_x \to j_y \circ f_\gamma$ as in Example 7.1.5. Notice that $\mathrm{Tw}(\Delta^1)$ can be represented as

$$\begin{tikzcd} & & x \arrow[dll, equal] \arrow[d, "\gamma"] \arrow[drr, "\gamma"] & & \\ x \arrow[d, equal] & & y & & y \arrow[d, equal] \\ x \arrow[urr, "\gamma"] & & & & y, \arrow[ull, equal] \end{tikzcd}$$

where the vertical arrows are the objects of $\mathrm{Tw}(\Delta^1)$. In other words, $\mathrm{Tw}(\Delta^1)$ is equivalent to $\mathrm{Span} = \{* \leftarrow * \to *\}$. It follows that the chain of equivalences of Recollection 6.3.1 in this case simply asserts that the square

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{A}, \mathcal{E}) \arrow[r, "j_y^*"] \arrow[d] & \mathrm{Fun}(\mathcal{A}_y, \mathcal{E}) \arrow[d, "f_\gamma^*"] \\ \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\mathcal{A}_x, \mathcal{E})) \arrow[r, "\mathrm{ev}_1"] & \mathrm{Fun}(\mathcal{A}_x, \mathcal{E}). \end{tikzcd}$$

is a pullback. Unraveling the definitions, we see that the left vertical map sends $F \colon \mathcal{A} \to \mathcal{E}$ to $s^* \colon j_x^*(F) \to f_\gamma^*(j_y^*(F))$. Vice-versa, given

$$F_x \colon \mathcal{A}_x \to \mathcal{E} \ , \qquad F_y \colon \mathcal{A}_y \to \mathcal{E}$$

and a natural transformation

$$\alpha \colon F_x \to f_\gamma^*(F_y) \ ,$$

we can produce a functor $F \colon \mathcal{A} \to \mathcal{E}$ together with the following data:

(1) equivalences $\beta_x \colon F_x \simeq j_x^*(F)$ and $\beta_y \colon F_y \simeq j_y^*(F)$;

(2) whenever $\phi \colon a \to f_\gamma(a)$ is a p-cocartesian morphism in $\mathcal{A}$, an equivalence

$$\beta_a \colon F(\phi) \simeq \alpha(a)$$

in $\mathrm{Map}_{\mathcal{E}}(F_x(a), F_y(f_\gamma(a)))$.

The above analysis allows to obtain an improvement on Corollary 6.1.5. To state it, we need to first introduce the following:

Notation 7.1.9. Let $\gamma \colon x \to y$‘be a morphism in $\mathcal{X}$. Let $F \in \mathrm{Fun}(\mathcal{A}_x, \mathcal{E})$ and $G \in \mathrm{Fun}(\mathcal{A}_y, \mathcal{E})$ and let $\alpha \colon F \to G$ be a morphism in $\exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X})$ lying over γ. We can factor α as

$$F \xrightarrow{\alpha_0} G' \xrightarrow{\alpha_1} G \ ,$$

where α_1 is $p_{\mathcal{E}}$-cartesian. Unraveling the definitions, we see that for every p-cocartesian lift $\phi \colon a \to b$ of γ, α_1 induces a canonical equivalence $\alpha_1(\phi) \colon G'(a) \simeq G(b)$, and in particular we obtain a well defined morphism

$$\alpha(\phi) \coloneqq \alpha_1(\phi) \circ \alpha_0(a) \colon F(a) \to G(b)$$

in $\mathcal{E}$.

Corollary 7.1.10. *Let $F \colon \mathcal{A} \to \mathcal{E}$ be a functor and let $\phi \colon a \to b$ be a p-cocartesian morphism in $\mathcal{A}$. Then there is a canonical identification*

$$F(\phi) \simeq (\mathrm{sp}\, F)_{p(\phi)}(\phi)$$

of morphisms in $\mathcal{E}$.

Proof. Using Proposition 6.1.2-(1) we can assume without loss of generality that $\mathcal{X} = \Delta^1$. Choose a straightening $f_\gamma \colon \mathcal{A}_x \to \mathcal{A}_y$ together with a morphism $s \colon j_x \to j_y \circ f_\gamma$ as in Example 7.1.5. Using Corollary 6.1.5 we see that $(\mathrm{sp}\, F)_\gamma$ can be factored as

$$j_x^*(F) \xrightarrow{s^*} f_\gamma^*(j_y^*(F)) \longrightarrow j_y^*(F) \ ,$$

where the second morphism is $p_{\mathcal{E}}$-cartesian. With these choices, the notation introduced in Notation 7.1.9 collapses to $(\mathrm{sp}\, F)_{p(\phi)}(a) \simeq s^*(a)$, so the conclusion follows from the analysis of the specialization equivalence over Δ^1 carried out in Example 7.1.8. □

7.2. **Cocartesian functors.** We fix as usual a cocartesian fibration $p \colon \mathcal{A} \to \mathcal{X}$ and a presentable ∞-category $\mathcal{E}$. We let $p_{\mathcal{E}} \colon \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X}) \to \mathcal{X}$ be the canonical projection.

Definition 7.2.1. Let $F \colon \mathcal{A} \to \mathcal{E}$ be a functor and let $\gamma \colon x \to y$ be a morphism in $\mathcal{X}$. We say that F is *cocartesian at* γ if every specialization morphism for F relative to γ is an equivalence in $\mathrm{Fun}(\mathcal{A}_y, \mathcal{E})$.

We say that F is *cocartesian* if it is cocartesian at every morphism γ of $\mathcal{X}$. We write $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$ for the full subcategory of $\mathrm{Fun}(\mathcal{A}, \mathcal{E})$ spanned by cocartesian functors.

Remark 7.2.2. Recall from Remark 7.1.7 that $\mathrm{SP}_\gamma(F)$ is a contractible space. In particular, in order to check that F is cocartesian at γ, it is enough to check that there exists *one* specialization morphism α that is an equivalence.

We now collect a couple of elementary facts concerning these objects. We keep the cocartesian fibration $p \colon \mathcal{A} \to \mathcal{X}$ and the presentable ∞-category $\mathcal{E}$ fixed in all the following statements:

Proposition 7.2.3. *Let $F \colon \mathcal{A} \to \mathcal{E}$ be a functor and let $\gamma \colon x \to y$ be a morphism in $\mathcal{X}$. The following statements are equivalent:*

(1) F is cocartesian at γ;

(2) the specialization $\operatorname{sp} F \colon \mathcal{X} \to \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X})$ takes γ to a $p_{\mathcal{E}}$-cocartesian edge;

(3) let $f_\gamma \colon \mathcal{A}_x \to \mathcal{A}_y$ be any *straightening for $p_\gamma \colon \mathcal{A}_\gamma \to \Delta^1$. Then the canonical Beck-Chevalley transformation* (7.1.6)

$$f_{\gamma,!} j_x^*(F) \to j_y^*(F)$$

is an equivalence.

Proof. Any element of $\mathrm{SP}_\gamma(F)$ corresponds to a factorization

$$\begin{tikzcd} (\operatorname{sp} F)_x \arrow[rr, "\beta"] \arrow[dr, "(\operatorname{sp} F)_\gamma"'] & & G \arrow[dl, "\alpha"] \\ & (\operatorname{sp} F)_y & \end{tikzcd}$$

inside $\exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X})$, where β is $p_{\mathcal{E}}$-cocartesian and α is the associated specialization morphism. It follows that $(\operatorname{sp} F)_\gamma$ is $p_{\mathcal{E}}$-cocartesian if and only if α is an equivalence. This shows that (1) $\Leftrightarrow$ (2). The equivalence (2) $\Leftrightarrow$ (3) follows now from Example 7.1.5. $\square$

Corollary 7.2.4. *Denoting $\Upsilon_{\mathcal{A}} \colon \mathcal{X} \to \mathbf{Cat}_\infty$ the straightening of the cocartesian fibration $\mathcal{A} \to \mathcal{X}$, there are canonical equivalences*

$$\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E}) \simeq \Sigma_{\mathcal{X}}^{\mathrm{cocart}}(\exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X})) \simeq \lim_{\mathcal{X}} \mathrm{Fun}_!(\Upsilon_{\mathcal{A}}, \mathcal{E})$$

In particular:

(1) $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$ is presentable;

(2) if $\mathcal{E}$ is stable, $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$ is stable.

Proof. Combining the specialization equivalence (6.1.1) and the equivalence (1) $\Leftrightarrow$ (2) of Proposition 7.2.3, we see that $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$ coincides with the full subcategory of $\mathrm{Fun}_{/\mathcal{X}}(\mathcal{X}, \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X}))$ spanned by cocartesian sections This proves the first equivalence, and the second follows directly from [31, Proposition 3.3.3.1]. For point (1) it is now sufficient to observe that the functor $\mathrm{Fun}_!(\Upsilon_{\mathcal{A}}, \mathcal{E}) \colon \mathcal{X} \to \mathbf{Cat}_\infty$ takes values in $\mathbf{Pr}^{\mathrm{L}}$, so the conclusion follows from [31, Propositon 5.5.3.13]. Similarly, point (2) follows from [32, Theorem 1.1.4.4]. $\square$

Warning 7.2.5. There is another natural condition that we can impose on a functor $F \colon \mathcal{A} \to \mathcal{E}$: namely, we can ask that F takes p-cocartesian arrows in $\mathcal{A}$ to equivalences in $\mathcal{E}$. This condition cuts a full subcategory $\mathrm{Fun}'(\mathcal{A}, \mathcal{E})$ of $\mathrm{Fun}(\mathcal{A}, \mathcal{E})$, that however does not coincide with $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$. Indeed, [31, Corollary 3.3.4.3] yields an identification

$$\mathrm{Fun}'(\mathcal{A}, \mathcal{E}) \simeq \mathrm{Fun}\big(\operatorname*{colim}_{\mathcal{X}} \Upsilon_{\mathcal{A}}, \mathcal{E}\big) \simeq \lim_{\mathcal{X}^{\mathrm{op}}} \mathrm{Fun}^*(\Upsilon_{\mathcal{A}}, \mathcal{E}) \simeq \Sigma_{\mathcal{X}}^{\mathrm{cart}}(\exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X})) ,$$

where $\Sigma_{\mathcal{X}}^{\mathrm{cart}}$ denotes the functor of *cartesian* sections.

Corollary 7.2.6. *A functor $F \colon \mathcal{A} \to \mathcal{E}$ is cocartesian at every equivalence of $\mathcal{X}$.*

Proof. Immediate from the equivalence (1) $\Leftrightarrow$ (2) of Proposition 7.2.3 and [31, 2.4.1.5]. $\square$

Corollary 7.2.7. *Let*

$$\begin{tikzcd} & y \arrow[dr, "\gamma_1"] & \\ x \arrow[ur, "\gamma_0"] \arrow[rr, "\gamma_2"] & & z. \end{tikzcd}$$

be a commutative triangle in $\mathcal{X}$. Let $F \colon \mathcal{A} \to \mathcal{E}$ be a functor, and assume that it is cocartesian at γ_0. Then F is cocartesian at γ_1 if and only if it is cocartesian at γ_2.

Proof. Immediate from the equivalence (1) $\Leftrightarrow$ (2) of Proposition 7.2.3 and from [31, 2.4.1.7]. $\square$

Corollary 7.2.8. *Let $p : \mathcal{A} \to \mathcal{X}$ be a cocartesian fibration. Let $\gamma : x \to y$ be a morphism in $\mathcal{X}$. Let $\mathcal{E}$ be a presentable ∞-category. Then, the full subcategory of* $\mathrm{Fun}(\mathcal{A}, \mathcal{E})$ *spanned by functors cocartesian at γ is stable under colimits.*

Proof. This follows from the equivalence (1) $\Leftrightarrow$ (3) in Proposition 7.2.3 and the fact that the functors $f_{\gamma,!}$, j_x^* and j_y^* commute with colimits. $\square$

Proposition 7.2.9. *Let $p\colon \mathcal{A} \to \mathcal{X}$ be a cocartesian fibration and let $\mathcal{E}$ be a presentable ∞-category. Then* $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$ *is stable under colimits* $\mathrm{Fun}(\mathcal{A}, \mathcal{E})$*. In particular,* $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$ *is a coreflective subcategory of* $\mathrm{Fun}(\mathcal{A}, \mathcal{E})$*, that is the inclusion*

$$\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E}) \hookrightarrow \mathrm{Fun}(\mathcal{A}, \mathcal{E}) \tag{7.2.10}$$

admits a right adjoint.

Proof. We know from Corollary 7.2.4 that $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$ is presentable. It is thus enough to check that $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$ is stable under colimits in $\mathrm{Fun}(\mathcal{A}, \mathcal{E})$, which follows from Corollary 7.2.8. $\square$

Definition 7.2.11. Let $p\colon \mathcal{A} \to \mathcal{X}$ be a cocartesian fibration and let $\mathcal{E}$ be a presentable ∞-category. We denote by

$$(-)^{\mathrm{cocart}}\colon \mathrm{Fun}(\mathcal{A}, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$$

the right adjoint of the inclusion (7.2.15), and refer to $(-)^{\mathrm{cocart}}$ as the *cocartesianization functor.*

Remark 7.2.12. The functor $(-)^{\mathrm{cocart}}$ can be explicitly computed in some specific situations. See Corollary 7.6.12.

Under extra stability and fiberwise compactness conditions, Corollary 7.2.8 and Proposition 7.2.9 have the following counterparts for limits :

Lemma 7.2.13. *Let $p : \mathcal{A} \to \mathcal{X}$ be a cocartesian fibration and let $\gamma : x \to y$ be a morphism in $\mathcal{X}$ such that $\mathcal{A}_x$ is compact and $\mathcal{A}_y$ is proper (see Definition 17.1.1). Let $\mathcal{E}$ be a presentable stable ∞-category. Then, the full subcategory of* $\mathrm{Fun}(\mathcal{A}, \mathcal{E})$ *spanned by functors cocartesian at γ is closed under limits.*

Proof. This follows from the equivalence (1) $\Leftrightarrow$ (3) in Proposition 7.2.3 and the fact that the functors $f_{\gamma,!}$, j_x^* and j_y^* commute with limits in virtue of Proposition 17.2.3. $\square$

Proposition 7.2.14. *Let $p : \mathcal{A} \to \mathcal{X}$ be a cocartesian fibration with compact and proper fibers. Let $\mathcal{E}$ be a presentable stable ∞-category. Then* $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$ *is stable under limits* $\mathrm{Fun}(\mathcal{A}, \mathcal{E})$*. In particular* $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$ *is a reflective subcategory of* $\mathrm{Fun}(\mathcal{A}, \mathcal{E})$*, that is the inclusion*

$$\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E}) \hookrightarrow \mathrm{Fun}(\mathcal{A}, \mathcal{E}) \tag{7.2.15}$$

admits a left adjoint.

Proof. We know from Corollary 7.2.4 that $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$ is presentable. It is thus enough to check that $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$ is stable under limits in $\mathrm{Fun}(\mathcal{A}, \mathcal{E})$, which follows from Lemma 7.2.13. $\square$

7.3. **Functoriality of cocartesian functors.** We fix as usual a cocartesian fibration $p\colon \mathcal{A} \to \mathcal{X}$ and a presentable ∞-category $\mathcal{E}$. We saw in Corollary 7.2.4 that there is a canonical equivalence

$$\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E}) \simeq \Sigma^{\mathrm{cocart}}(\exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X})) \ .$$

Therefore, it follows from [40, Corollary 3.23] that this construction depends functorially on the cocartesian fibration $\mathcal{A} \to \mathcal{X}$ seen as an element of **CoCart**. We now make this explicit in terms of the *lax* functoriality of $\mathrm{Fun}(\mathcal{A}, \mathcal{E}) \simeq \Sigma(\exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X}))$ in $\mathcal{A} \to \mathcal{X}$.

Proposition 7.3.1. *Let*

$$\begin{tikzcd} \mathcal{B} \arrow[d] & \mathcal{B}_{\mathcal{X}} \arrow[l, "u"'] \arrow[r, "v"] \arrow[d] & \mathcal{A} \arrow[dl] \\ \mathcal{Y} & \mathcal{X} \arrow[l, "f"] & \end{tikzcd}$$

be a morphism in **CoCart**. *Then:*

(1) *if* $F\colon \mathcal{B} \to \mathcal{E}$ *is a cocartesian functor, the same goes for* $u^*(F)\colon \mathcal{B}_{\mathcal{X}} \to \mathcal{E}$*;*

(2) *if* $G\colon \mathcal{B}_X \to \mathcal{E}$ *is a cocartesian functor, then the same goes for* $v_!(G)\colon \mathcal{A} \to \mathcal{E}$.

In particular the functors

$$u^*\colon \operatorname{Fun}(\mathcal{B}, \mathcal{E}) \to \operatorname{Fun}(\mathcal{B}_{\mathcal{X}}, \mathcal{E}) \qquad \text{and} \qquad v_!\colon \operatorname{Fun}(\mathcal{B}_{\mathcal{X}}, \mathcal{E}) \to \operatorname{Fun}(\mathcal{A}, \mathcal{E})$$

restrict to well-defined functors

$$u^*\colon \operatorname{Fun}^{\mathrm{cocart}}(\mathcal{B}, \mathcal{E}) \to \operatorname{Fun}^{\mathrm{cocart}}(\mathcal{B}_{\mathcal{X}}, \mathcal{E}) \qquad \text{and} \qquad v_!\colon \operatorname{Fun}^{\mathrm{cocart}}(\mathcal{B}_{\mathcal{X}}, \mathcal{E}) \to \operatorname{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E}) \ .$$

This proposition results of the following two more precise lemmas:

Lemma 7.3.2. *Let*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "u"] \arrow[d] & \mathcal{B} \arrow[d] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

be a pullback square in $\mathbf{Cat}_\infty$*, where the vertical morphisms are cocartesian fibrations. Fix a morphism* γ *in* $\mathcal{X}$ *and a functor* $F\colon \mathcal{B} \to \mathcal{E}$. *Then* $u^*(F)$ *is cocartesian at* γ *if and only if* F *is cocartesian at* $f(\gamma)$.

Proof. Under the specialization equivalence (6.1.1) and Proposition 6.1.2-(1), the statement follows from Proposition 7.2.3 and from [31, Proposition 2.4.1.3-(2)] applied to the square

$$\begin{tikzcd} \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X}) \arrow[r, "\mathcal{E}^u"] \arrow[d] & \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y}) \arrow[d] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \ , \end{tikzcd}$$

which is a pullback thanks to Proposition 3.2.6-(1). ☐

Lemma 7.3.3. *Let* $\mathcal{X}$ *be an* ∞*-category and consider a morphism*

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "v"] \arrow[dr] & & \mathcal{B} \arrow[dl] \\ & \mathcal{X} & \end{tikzcd}$$

in $\mathbf{CoCart}_{\mathcal{X}}$. *Let* γ *be a morphism in* $\mathcal{X}$ *and let* $F\colon \mathcal{A} \to \mathcal{E}$ *be a functor. If* F *is cocartesian at* γ*, then the same goes for* $v_!(F)$.

Proof. In virtue of Proposition 7.2.3, we have to prove that the section

$$\operatorname{sp}(v_!(F))\colon \mathcal{X} \to \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{X})$$

takes γ to a cocartesian edge in $\exp_{\mathcal{E}}(\mathcal{B}/\mathcal{X})$. Using Proposition 6.1.2-(2), we find a canonical identification

$$\operatorname{sp}(v_!(F)) \simeq \mathcal{E}^v_! \circ \operatorname{sp}(F) \ ,$$

where $\mathcal{E}^v_!\colon \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X}) \to \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{X})$ is the exponential induction functor. The conclusion now follows from Proposition 3.2.6-(2), that guarantees that $\mathcal{E}^v_!$ preserves cocartesian edges. ☐

We conclude with a handy consequence:

Corollary 7.3.4. *In the setting of Lemma 7.3.2, the composition*

$$u_*^{\mathrm{cocart}} := (-)^{\mathrm{cocart}} \circ u_* \colon \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{B}, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$$

is right adjoint to the pull-back functor $u^* \colon \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{B}, \mathcal{E})$.

Proof. The functors at play are well-defined from Proposition 7.3.1-(1) and Definition 7.2.11. Corollary 7.3.4 is then a routine computation. □

Lemma 7.3.5. *Let*

$$\begin{tikzcd} \mathcal{B} \arrow[d] & \mathcal{B}_{\mathcal{X}} \arrow[l, "u"'] \arrow[r, "v"] \arrow[d] & \mathcal{A} \arrow[dl] \\ \mathcal{Y} & \mathcal{X} \arrow[l, "f"'] & \end{tikzcd}$$

be a morphism in **CoCart**. *Assume that* $\mathcal{A} \to \mathcal{X}$ *and* $\mathcal{B} \to \mathcal{X}$ *have compact and proper fibers, and that* $\mathcal{E}$ *is presentable stable. Then, the functors*

$$u^* \colon \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{B}, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{B}_{\mathcal{X}}, \mathcal{E}) \qquad \text{and} \qquad v_! \colon \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{B}_X, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$$

commute with limits and colimits.

Proof. From Proposition 7.2.9 and Proposition 7.2.14, $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{B}, \mathcal{E})$ and $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{B}_{\mathcal{X}}, \mathcal{E})$ are stable under limits and colimits in $\mathrm{Fun}(\mathcal{B}, \mathcal{E})$ and $\mathrm{Fun}(\mathcal{B}_{\mathcal{X}}, \mathcal{E})$ respectively. Hence, it is enough to show that the functors

$$u^* \colon \mathrm{Fun}(\mathcal{B}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{B}_{\mathcal{X}}, \mathcal{E}) \qquad \text{and} \qquad v_! \colon \mathrm{Fun}(\mathcal{B}_{\mathcal{X}}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{A}, \mathcal{E})$$

commute with limits and colimits. For the former, this is obvious. For the latter, this follows from Proposition 17.2.3. □

7.4. **Van Kampen for cocartesian functors.** Consider the following general fact:

Lemma 7.4.1 (Van Kampen for filtered functors)**.** *Let* $\mathcal{X}_\bullet \colon I \to \mathbf{Cat}_\infty$ *be a diagram with colimit* $\mathcal{X}$. *Let* $p \colon \mathcal{A} \to \mathcal{X}$ *be a cocartesian fibration and set*

$$\mathcal{A}_\bullet := \mathcal{X}_\bullet \times_{\mathcal{X}} \mathcal{A} \colon I \to \mathbf{Cat}_\infty \ .$$

Then the canonical morphism

$$\operatorname*{colim}_{i \in I} \mathcal{A}_i \to \mathcal{A}$$

is an equivalence. In particular, for every presentable ∞*-category* $\mathcal{E}$ *the canonical morphism*

$$\mathrm{Fun}(\mathcal{A}, \mathcal{E}) \to \lim_{i \in I^{\mathrm{op}}} \mathrm{Fun}^*(\mathcal{A}_i, \mathcal{E}) \tag{7.4.2}$$

is an equivalence.

Proof. Since $p \colon \mathcal{A} \to \mathcal{X}$ is a cocartesian fibration, it is in particular an exponentiable fibration thanks to [2, Lemma 2.15]. In particular, the functor

$$p^* \colon (\mathbf{Cat}_\infty)_{/\mathcal{X}} \to (\mathbf{Cat}_\infty)_{/\mathcal{A}}$$

is a left adjoint. It follows in particular that it preserves all colimits. Now the conclusion follows from the fact that for every ∞-category $\mathcal{C}$, the forgetful functor

$$(\mathbf{Cat}_\infty)_{/\mathcal{C}} \to \mathbf{Cat}_\infty$$

is conservative and preserves all colimits. □

To prove a Van Kampen result for cocartesian functors, we need a couple of categorical preliminaries. Recall the following definitions:

Definition 7.4.3. The *maximal spine* of the standard n-simplex Δ^n is the sub-simplicial set formed by the consecutive 1-simplexes $\Delta^n_{\{0,1\}}, \Delta^n_{\{1,2\}}, \ldots, \Delta^n_{\{n-1,n\}}$.

Remark 7.4.4. Notice that the maximal spine of Δ^2 coincides with Λ^2_1. On the other hand, for $n \geqslant 3$ every horn Λ^n_i cointains the maximal spine of Δ^n.

Definition 7.4.5. Let $\mathcal{C}$ be a quasi-category and let $S \subset \mathcal{C}$ be a collection of 1-simplexes. We say that S is *closed under identities* if whenever $f: x \to y$ belongs to S, then id_x and id_y belong to S as well.

Construction 7.4.6. Let $\mathcal{C}$ be a quasicategory and let $S \subset \mathcal{C}$ be a collection of 1-simplexes. Define $\mathcal{C}_S$ as the full sub-simplicial set of $\mathcal{C}$ defined by the following condition: an n-simplex $\sigma: \Delta^n \to \mathcal{C}$ belongs to $\mathcal{C}_S$ if and only if the restriction of σ to the maximal spine of Δ^n factors through S.

Lemma 7.4.7. *Let $\mathcal{C}$ be a quasi-category and let $S \subset \mathcal{C}$ be a collection of 1-simplexes. If S is closed under identities, then $\mathcal{C}_S$ is the smallest full sub-quasicategory of $\mathcal{C}$ containing $\mathcal{C}$.*

Proof. Let $\mathcal{C}'$ be the smallest full sub-quasicategory of $\mathcal{C}$ containing $\mathcal{C}$. It immediately follows from Remark 7.4.4 that $\mathcal{C}_S$ is a quasi-category, and therefore that $\mathcal{C}' \subseteq \mathcal{C}_S$. Vice-versa, iteratively applying the lifting condition against inner horns we deduce that any sub-quasicategory containing S must contain $\mathcal{C}_S$. Thus, $\mathcal{C}' = \mathcal{C}_S$ as full sub-quasicategories of $\mathcal{C}$. □

Notation 7.4.8. Let $f: \mathcal{Y} \to \mathcal{X}$ be a morphism of quasicategories. We denote by S_f the collection of 1-simplexes of $\mathcal{X}$ that lie in the essential image of f.

Lemma 7.4.9. *Let $\mathcal{X}_\bullet: I \to \mathbf{Cat}_\infty$ be a diagram with colimit $\mathcal{X}$. Let $f_i: \mathcal{X}_i \to \mathcal{X}$ be the structural morphisms and define*

$$S \coloneqq \bigcup_{i \in I} S_{f_i} \ .$$

Then S is closed under identities and the inclusion $\mathcal{X}_S \subseteq \mathcal{X}$ is an equivalence in $\mathbf{Cat}_\infty$.

Proof. That S is closed under identities simply follows from the definitions. Notice that $\mathcal{X}_S$ is itself an ∞-category and that the inclusion $i: \mathcal{X}_S \hookrightarrow \mathcal{X}$ is fully faithful. By definition, every f_i factors as

$$\overline{f}_i: \mathcal{X}_i \to \mathcal{X}_S \ .$$

Therefore, the universal property of the colimit provides a canonical map $p: \mathcal{X} \to \mathcal{X}_S$ together with an equivalence $i \circ p \simeq \mathrm{id}_{\mathcal{X}}$. This implies that i is essentially surjective. Being already fully faithful, it follows that it is an equivalence. □

We are now ready for:

Proposition 7.4.10 (Van Kampen for cocartesian functors)**.** *Let $\mathcal{X}_\bullet: I \to \mathbf{Cat}_\infty$ be a diagram with colimit $\mathcal{X}$. Let $p: \mathcal{A} \to \mathcal{X}$ be a cocartesian fibration and set*

$$\mathcal{A}_\bullet \coloneqq \mathcal{X}_\bullet \times_{\mathcal{X}} \mathcal{A}: I \to \mathbf{Cat}_\infty \ .$$

Let $\mathcal{E}$ be a presentable ∞-category. Then the equivalence of Lemma 7.4.1 restricts to an equivalence

$$\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E}) \simeq \lim_{i \in I} \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}_i, \mathcal{E}) \ .$$

Proof. Using Proposition 7.3.1-(1), we see that the canonical map (7.4.2) induces a well defined map between cocartesian functors making the diagram

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{A}, \mathcal{E}) \arrow[r] & \lim\limits_{i\in I} \mathrm{Fun}(\mathcal{A}_i, \mathcal{E}) \\ \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E}) \arrow[r] \arrow[u, hook] & \lim\limits_{i\in I} \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}_i, \mathcal{E}) \arrow[u, hook] \end{tikzcd} .$$

Since the top horizontal arrow is an equivalence and the vertical ones are fully faithful, it follows that the bottom horizontal functor is fully faithful as well. To conclude the proof, it is enough to show that a functor $F\colon \mathcal{A} \to \mathcal{E}$ is cocartesian if and only if for every $i \in I$ its image in $\mathrm{Fun}(\mathcal{A}_i, \mathcal{E})$ is cocartesian. The "only if" follows from Proposition 7.3.1-(1). For the converse, observe first that combining Corollary 7.2.7 and Lemma 7.4.9 we deduce that F is cocartesian if and only if it is cocartesian at every morphism in the essential image of the structural map $f_i\colon \mathcal{X}_i \to \mathcal{X}$. At this point, the conclusion follows from Lemma 7.3.2. □

7.5. **Change of coefficients for cocartesian functors.** Fix a cocartesian fibration $p\colon \mathcal{A} \to \mathcal{X}$ and let $f\colon \mathcal{E} \to \mathcal{E}'$ be a morphism in $\mathbf{Pr}^{\mathrm{L}}$. Recall from Section 6.5 that this induces a transformation

$$f^{\mathcal{A}/\mathcal{X}}\colon\ \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X}) \to \exp_{\mathcal{E}'}(\mathcal{A}/\mathcal{X})$$

in $\mathbf{PrFib}^{\mathrm{L}}$. In particular:

Proposition 7.5.1. *The transformation $f^{\mathcal{A}/\mathcal{X}}$ preserves cocartesian edges. Therefore, the induced functor*

$$f\colon\ \mathrm{Fun}(\mathcal{A}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{A}, \mathcal{E}')$$

preserves cocartesian functors and induces a well defined morphism

$$f\colon\ \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E}') .$$

Proof. Since $f^{\mathcal{A}/\mathcal{X}}$ is a morphism in $\mathbf{PrFib}^{\mathrm{L}}$, it automatically preserves cocartesian edges. The second half follows then from the identification $f \simeq \Sigma_{\mathcal{X}}(f^{\mathcal{A}/\mathcal{X}})$ supplied by Proposition 6.5.1. □

We now study the change of coefficients via the tensor product in $\mathbf{Pr}^{\mathrm{L}}$. Recall that for every pair of presentable ∞-categories $\mathcal{E}$ and $\mathcal{E}'$ and for every ∞-category $\mathcal{A}$, there is a canonical equivalence

$$\mathrm{Fun}(\mathcal{A}, \mathcal{E}) \otimes \mathcal{E}' \simeq \mathrm{Fun}(\mathcal{A}, \mathcal{E} \otimes \mathcal{E}') .$$

Under suitable finiteness assumptions, we are going to see that this equivalence preserves cocartesian functors.

Definition 7.5.2. Define $\mathbf{Pr}^{\mathrm{L,R}}$ as the (non full) subcategory of $\mathbf{Pr}^{\mathrm{L}}$ whose objects are presentable ∞-categories and morphisms are functors that are both left and right adjoints.

Definition 7.5.3. Let $\mathbf{PrFib}^{\mathrm{L,R}}$ be the full subcategory of $\mathbf{PrFib}^{\mathrm{L}}$ corresponding to $\mathrm{Fun}(\mathcal{X}, \mathbf{Pr}^{\mathrm{L,R}})$ under the straightening equivalence (3.1.8).

Example 7.5.4. Let $p\colon \mathcal{A} \to \mathcal{X}$ be a cocartesian fibration with compact and proper fibers (see Definition 17.1.1). Let $\mathcal{E}$ be a stable presentable ∞-category. Then the exponential fibration $p_{\mathcal{E}}\colon \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X}) \to \mathcal{X}$ defines an object in $\mathbf{PrFib}^{\mathrm{L,R}}$: indeed, we have to check that for every morphism $\gamma\colon x \to y$ in $\mathcal{X}$ and any choice of a straightening $f_\gamma\colon \mathcal{A}_x \to \mathcal{A}_y$, the induced functor

$$f_{\gamma,!}\colon\ \mathrm{Fun}(\mathcal{A}_x, \mathcal{E}) \to \mathrm{Fun}(\mathcal{A}_y, \mathcal{E})$$

commutes with limits and colimits, and this follows from Proposition 17.2.3.

Our main use of $\mathbf{Pr}^{\mathrm{L,R}}$ will be through the following lemma from [25, 2.7.9].

Lemma 7.5.5. *Let A be a small ∞-category and let $\mathcal{C}_\bullet \colon A \to \mathbf{Pr}^{\mathrm{L,R}}$ be a diagram of ∞-categories. Then,*

(1) The limits of $\mathcal{C}_\bullet$ when computed in $\mathbf{Pr}^{\mathrm{R}}$, $\mathbf{Pr}^{\mathrm{L}}$, or $\mathbf{Cat}_\infty$ all agree.

(2) For any presentable ∞-category $\mathcal{E}$, the natural morphism
$$\lim_{\alpha \in A} \mathcal{E} \otimes \mathcal{C}_\alpha \to \mathcal{E} \otimes \lim_{\alpha \in A} \mathcal{C}_\alpha$$
in $\mathbf{Pr}^{\mathrm{L}}$ is an equivalence. (Here, both limits are computed in $\mathbf{Pr}^{\mathrm{L}}$).

Lemma 7.5.6. *Let $\mathcal{X}$ be an ∞-category and let $\mathcal{E}$ be a presentable ∞-category. Then commutative diagram*
$$\begin{tikzcd} \mathbf{Pr}^{\mathrm{L}} \arrow[r, "\mathrm{Triv}_{\mathcal{X}}"] \arrow[d, "(-)\otimes\mathcal{E}"] & \mathbf{PrFib}^{\mathrm{L}} \arrow[d, "(-)\otimes_{\mathcal{X}}\mathrm{Triv}_{\mathcal{X}}(\mathcal{E})"] \\ \mathbf{Pr}^{\mathrm{L}} \arrow[r, "\mathrm{Triv}_{\mathcal{X}}"] & \mathbf{PrFib}^{\mathrm{L}} \end{tikzcd}$$
is horizontally right adjointable on objects of $\mathbf{PrFib}^{\mathrm{L,R}}$. That is, for every object $p \colon \mathcal{A} \to \mathcal{X}$ of $\mathbf{PrFib}^{\mathrm{L,R}}$, the Beck-Chevalley transformation
$$\Sigma^{\mathrm{cocart}}_{\mathcal{X}}(\mathcal{A}/\mathcal{X}) \otimes \mathcal{E} \to \Sigma^{\mathrm{cocart}}_{\mathcal{X}}(\mathcal{A} \otimes_{\mathcal{X}} \mathrm{Triv}_{\mathcal{X}}(\mathcal{E}))$$
is an equivalence.

Proof. Let $\Upsilon_{\mathcal{A}} : \mathcal{X} \to \mathbf{Pr}^{\mathrm{L}}$ be the straightening of $\mathcal{A} \to \mathcal{X}$. From [31, 3.3.3.2], the Beck-Chevalley transformation reads as the following natural morphism in $\mathbf{Pr}^{\mathrm{L}}$
$$\big(\lim_{x \in \mathcal{X}} \mathcal{A}_x\big) \otimes \mathcal{E} \to \lim_{x \in \mathcal{X}} (\mathcal{A}_x \otimes \mathcal{E}) \ .$$
Then, Lemma 7.5.6 follows from Lemma 7.5.5. ☐

Corollary 7.5.7. *Let $p \colon \mathcal{A} \to \mathcal{X}$ be an object of $\mathbf{CoCart}$ with proper and compact fibers. Let $\mathcal{E}, \mathcal{E}'$ be presentable ∞-categories such that $\mathcal{E}$ is stable. Then, the canonical transformation*
$$\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E}) \otimes \mathcal{E}' \to \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E} \otimes \mathcal{E}')$$
is an equivalence.

Proof. Recall from Remark 3.3.1 that there is a canonical equivalence
$$\exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X}) \otimes_{\mathcal{X}} \mathrm{Triv}_{\mathcal{X}}(\mathcal{E}) \simeq \exp_{\mathcal{E}\otimes\mathcal{E}'}(\mathcal{A}/\mathcal{X}) \ .$$
By Example 7.5.4 the exponential fibration $\exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X})$ belongs to $\mathbf{PrFib}^{\mathrm{L,R}}$. Thus, the conclusion follows applying $\Sigma^{\mathrm{cocart}}_{\mathcal{X}}$ to the above equivalence and using Lemma 7.5.6. ☐

7.6. **Cocartesian functors in presence of an initial object.** We saw in Proposition 7.2.9 that the inclusion of cocartesian functors inside all functors always admits a right adjoint $(-)^{\mathrm{cocart}}$. The goal of this section is to provide an explicit description of this functor in the special case where the base $\mathcal{X}$ admits an initial object. We start with the following construction:

Construction 7.6.1. Fix a cocartesian fibration $p \colon \mathcal{A} \to \mathcal{X}$ and let $\gamma \colon x \to y$ be a morphism in $\mathcal{X}$. Define
$$\overline{\gamma}_! \coloneqq j_y^* \circ j_{x,!} \colon \mathrm{Fun}(\mathcal{A}_x, \mathcal{E}) \to \mathrm{Fun}(\mathcal{A}_y, \mathcal{E}) \ .$$
Write ε_x for the counit of the adjunction $j_{x,!} \dashv j_x^*$. It induces a natural transformation
$$\alpha_\gamma \coloneqq \varepsilon_x j_x^* \colon \overline{\gamma}_! \circ j_x^* \to j_y^* \ .$$

Fix now a straightening
$$f\colon \mathcal{A}_x \to \mathcal{A}_y$$
for $p_\gamma\colon \mathcal{A}_\gamma \to \Delta^1$ together with a natural transformation $s\colon j_x \to j_y \circ f$ as in Example 7.1.5. Write η_x for the unit of $j_{x,!} \dashv j_x^*$ and consider the transformation
$$\beta_{f,s}\colon f_! \xrightarrow{f_!(\eta_x)} f_! \circ j_x^* \circ j_{x!} \xrightarrow{\alpha_{f,s}(j_{x!})} j_y^* \circ j_{x!} = \overline{\gamma}_!.$$

Proposition 7.6.2. *In the setting of Construction 7.6.1, the diagram*
$$\begin{tikzcd} f_! \circ j_x^* \arrow[r, "\beta_{f,s}(j_x^*)"] \arrow[dr, "\alpha_{f,s}"'] & j_y^* \circ j_{x,!} \circ j_x^* \arrow[d, "j_y^*(\varepsilon_x)"] \\ & j_y^* \end{tikzcd}$$
is canonically commutative. If in addition x is an initial object of $\mathcal{X}$, then the natural transformation $\beta_{f,s}\colon f_! \to \overline{\gamma}_!$ is an equivalence. In this case, for every $F \in \mathrm{Fun}(\mathcal{A}, \mathcal{E})$, the morphism
$$j_y^*(\varepsilon_x)\colon j_y^* j_{x,!} j_x^*(F) \to j_y^*(F)$$
is a specialization morphism for F relative to γ.

Proof. For what concerns the commutativity, a standard diagram chase reduces it to the triangular identities for $j_{0,!} \dashv j_0^*$. We leave the details to the reader. We check that $\beta_{f,s}$ is an equivalence under the assumption that x is an initial object of $\mathcal{X}$. Unraveling the definitions, we reduce ourselves to check that for every $a \in \mathcal{A}_y$, the canonical functor induced by the pair (f, s)
$$\mathcal{A}_x \times_{\mathcal{A}_y} (\mathcal{A}_y)_{/a} \to \mathcal{A}_x \times_{\mathcal{A}} \mathcal{A}_{/j_y(a)} \tag{7.6.3}$$
is cofinal. We are going to show that it is an equivalence. To do this, [31, 2.2.3.3] ensures that it is enough to show that (7.6.3) is a pointwise equivalence over $\mathcal{A}_x$. The restriction of (7.6.3) above $b \in \mathcal{A}_x$ reads as
$$\mathrm{Map}_{\mathcal{A}_y}(f(b), a) \to \mathrm{Map}_{\mathcal{A}}(j_x(b), j_y(a)) \ . \tag{7.6.4}$$
Since x is an initial object in $\mathcal{X}$, $\mathrm{Map}_{\mathcal{X}}(x, y)$ is contractible. Thus, every morphism $j_x(b) \to j_y(a)$ lies over $\gamma\colon x \to y$. Since $s\colon j_x(b) \to j_y(f(b))$ is a p-cocartesian lift of γ, [31, 2.4.4.2] implies that (7.6.4) is an equivalence.

□

For a general morphism $\gamma\colon x \to y$ in $\mathcal{X}$, we can always replace $p\colon \mathcal{A} \to \mathcal{X}$ by $p_\gamma\colon \mathcal{A}_\gamma \to \Delta^1$ in order to ensure that the hypothesis of Proposition 7.6.2 is satisfied. This yields:

Definition 7.6.5. Let $p\colon \mathcal{A} \to \mathcal{X}$ be a cocartesian fibration. Let $F \in \mathrm{Fun}(\mathcal{A}, \mathcal{E})$ be a functor and let $\gamma\colon \Delta^1 \to \mathcal{X}$ be a morphism in $\mathcal{X}$. Let $j_\gamma\colon \mathcal{A}_\gamma \to \mathcal{A}$, $j_{\gamma,x}\colon \mathcal{A}_x \to \mathcal{A}_\gamma$ and $j_{\gamma,y}\colon \mathcal{A}_y \to \mathcal{A}_\gamma$ be the natural functors. Observe that $j_\gamma \circ j_{\gamma,y} \simeq j_y$, and similarly for x. The *strict induction functor relative to* γ is the functor
$$\gamma_{\mathcal{A},!} \coloneqq j_{\gamma,y}^* \circ j_{\gamma,x,!}\colon \mathrm{Fun}(\mathcal{A}_x, \mathcal{E}) \to \mathrm{Fun}(\mathcal{A}_y, \mathcal{E}).$$
The *strict specialization morphism for F relative* γ is the natural transformation
$$\mathrm{sp}_{\mathcal{A},\gamma}(F) \coloneqq j_{\gamma,y}^*(\varepsilon_x j_\gamma^*)\colon \gamma_{\mathcal{A},!}(j_x^* F) \to j_y^*(F),$$
where ε_x denotes the counit of the adjunction $j_{\gamma,x,!} \dashv j_{\gamma,x}^*$.

When $\mathcal{A}$ is clear from the context, we write $\gamma_!$ and $\mathrm{sp}_\gamma(F)$ instead of $\gamma_{\mathcal{A}!}$ and $\mathrm{sp}_{\mathcal{A},\gamma}$.

Remark 7.6.6. The terminology is due to the fact that neither $\gamma_{\mathcal{A},!}$ nor $\mathrm{sp}_{\mathcal{A},\gamma}(F)$ depend on the choice of a straightening of $p\colon \mathcal{A} \to \mathcal{X}$.

Proposition 7.6.2 leads to a complete understanding of $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$ when $\mathcal{X}$ has an initial object. Before stating the main result of this section, let us collect a couple of general facts:

Proposition 7.6.7. *Let $\mathcal{X}$ and $\mathcal{Y}$ be small ∞-categories and let*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "g"] \arrow[d, "q"] & \mathcal{D} \arrow[d, "p"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

be a pullback square in $\mathbf{CAT}_\infty$, with p being a cocartesian fibration. If f is a final functor, then the induced pull-back functor

$$f^* \colon \mathrm{Fun}^{\mathrm{cocart}}_{/\mathcal{Y}}(\mathcal{Y}, \mathcal{D}) \to \mathrm{Fun}^{\mathrm{cocart}}_{/\mathcal{X}}(\mathcal{X}, \mathcal{C})$$

is an equivalence of ∞-categories.

Proof. Let $\Upsilon_{\mathcal{D}}$ and $\Upsilon_{\mathcal{C}}$ be the straightenings of $p \colon \mathcal{D} \to \mathcal{Y}$ and of $q \colon \mathcal{C} \to \mathcal{X}$, respectively. Since the given square is a pullback, there is a natural equivalence $\Upsilon_{\mathcal{C}} \simeq \Upsilon_{\mathcal{D}} \circ f$. We find:

$$\begin{aligned} \mathrm{Fun}^{\mathrm{cocart}}_{/\mathcal{Y}}(\mathcal{Y}, \mathcal{D}) &\simeq \lim_{\mathcal{X}} \Upsilon_{\mathcal{D}} && \text{By [31, Prop. 3.3.3.1]} \\ &\simeq \lim_{\mathcal{Y}} \Upsilon_{\mathcal{D}} \circ f && f \text{ is cofinal} \\ &\simeq \lim_{\mathcal{Y}} \Upsilon_{\mathcal{C}} \\ &\simeq \mathrm{Fun}^{\mathrm{cocart}}_{/\mathcal{X}}(\mathcal{X}, \mathcal{C}) && \text{By [31, Prop. 3.3.3.1],} \end{aligned}$$

and the conclusion follows. □

In the particular case where $\mathcal{C} = \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y})$, we find:

Corollary 7.6.8. *Let*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "u"] \arrow[d, "q"] & \mathcal{B} \arrow[d, "p"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

be a pullback in $\mathbf{Cat}_\infty$, with p being a cocartesian fibration. Let $\mathcal{E}$ be a presentable ∞-category. If f is a final functor, then

$$u^* \colon \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{B}, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E}) \tag{7.6.9}$$

is an equivalence.

Proof. Apply Proposition 7.6.7 to $\exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y})$ and use Proposition 6.1.2-(1). □

Observation 7.6.10. Recall from Corollary 7.3.4 that u^* admits a right adjoint u_*^{cocart}. It follows formally that in the situation of Corollary 7.6.8, the u_*^{cocart} realizes the inverse of u^*.

Corollary 7.6.11. *Let $\mathcal{X}$ be an ∞-category with an initial object x. Let $p \colon \mathcal{A} \to \mathcal{X}$ be a cocartesian fibration and let $\mathcal{E}$ be a presentable ∞-category. Then:*

(1) the functor $j_{x,!} \colon \mathrm{Fun}(\mathcal{A}_x, \mathcal{E}) \to \mathrm{Fun}(\mathcal{A}, \mathcal{E})$ factors through $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$.

(2) The adjunction

$$j_{x,!} \colon \mathrm{Fun}(\mathcal{A}_x, \mathcal{E}) \leftrightarrows \mathrm{Fun}(\mathcal{A}, \mathcal{E}) \colon j_x^*$$

restricts to an equivalence of ∞-categories between $\mathrm{Fun}(\mathcal{A}_x, \mathcal{E})$ and $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$.

Proof. We prove (1). Let $F \in \mathrm{Fun}(\mathcal{A}_x, \mathcal{E})$ and let $\gamma\colon y \to z$ be a morphism in $\mathcal{X}$. We need to show that $j_{x,!}(F)$ is cocartesian at γ. Since x is initial in $\mathcal{X}$, we can find a commutative triangle

$$\begin{tikzcd} & y \arrow[dr, "\gamma"] & \\ x \arrow[ur, "\gamma_0"] \arrow[rr, "\gamma_2"] & & z. \end{tikzcd}$$

in $\mathcal{X}$. From Corollary 7.2.7, it is enough to prove that $j_{x,!}(F)$ is cocartesian at γ_0 and γ_2. Equivalently, we can suppose that $y = x$. Now we apply Proposition 7.6.2 to $j_{x,!}(F)$: notice that since x is initial, the inclusion $\{x\} \hookrightarrow \mathcal{X}$ is fully faithful and therefore that $j_x\colon \mathcal{A}_x \to \mathcal{A}$ is fully faithful as well. Thus, the unit transformation $F \to j_x^* j_{x,!}(F)$ is an equivalence, and therefore the strict specialization morphism provided by Proposition 7.6.2 is an equivalence as well.

We now prove (2). Since $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$ is fully faithful inside $\mathrm{Fun}(\mathcal{A}, \mathcal{E})$ and since $j_{x,!}$ factors through $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$, we see that the adjunction $j_{x,!} \dashv j_x^*$ descends to an adjunction

$$j_{x,!}\colon \mathrm{Fun}(\mathcal{A}_x, \mathcal{E}) \leftrightarrows \mathrm{Fun}(\mathcal{A}, \mathcal{E})\colon j_x^* \ .$$

It is therefore enough to prove that j_x^* is an equivalence. Since the inclusion $\{x\} \hookrightarrow \mathcal{X}$ is final, this follows from the limit-description of cocartesian functors provided in Corollary 7.2.4. See also Corollary 7.6.8 below. ☐

Corollary 7.6.12. *Let $\mathcal{X}$ be an ∞-category with an initial object x. Let $p\colon \mathcal{A} \to \mathcal{X}$ be a cocartesian fibration and let $\mathcal{E}$ be a presentable ∞-category. Then there is a natural equivalence*

$$(-)^{\mathrm{cocart}} \simeq j_{x,!} \circ j_x^*$$

of functors from $\mathrm{Fun}(\mathcal{A}, \mathcal{E})$ *to* $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$.

Proof. Fix $F \in \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$ and let $G \in \mathrm{Fun}(\mathcal{A}, \mathcal{E})$. We have

$$\begin{aligned} \mathrm{Map}_{\mathrm{Fun}(\mathcal{A},\mathcal{E})}(F, G) &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{A},\mathcal{E})}(j_{x,!} j_x^*(F), G) && \text{By Cor. 7.6.11} \\ &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{A}_x,\mathcal{E})}(j_x^*(F), j_x^*(G)) \\ &\simeq \mathrm{Map}_{\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}_x,\mathcal{E})}(j_{x,!} j_x^*(F), j_{x,!} j_x^*(G)) \\ &\simeq \mathrm{Map}_{\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}_x,\mathcal{E})}(F, j_{x,!} j_x^*(G)) \end{aligned}$$

where the last two equivalences are again due to Corollary 7.6.11. Therefore, $j_{x,!} \circ j_x^*$ is right adjoint to the inclusion of $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$ into $\mathrm{Fun}(\mathcal{A}, \mathcal{E})$, whence the conclusion. ☐

For later use, let us extract the formal argument used to prove Corollary 7.6.11-(2):

Lemma 7.6.13. *Let*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "u"] \arrow[d, "q"] & \mathcal{B} \arrow[d, "p"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

be a pullback in $\mathbf{Cat}_\infty$*, with p being a cocartesian fibration. Let $\mathcal{E}$ be a presentable ∞-category. Assume that $u^*\colon$ $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{B}, \mathcal{E})$ is an equivalence of ∞-categories. Then, the following conditions are equivalent:*

(1) The functor $u_!\colon$ $\mathrm{Fun}(\mathcal{B}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{A}, \mathcal{E})$ preserves cocartesian functors;

(2) The adjunction $u_! \dashv u^*$ restricts to an equivalence of ∞-categories between $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$ *and* $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{B}, \mathcal{E})$*;*

(3) For every $F \in \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{B}, \mathcal{E})$, there is a natural equivalence $u_!(F) \simeq u_*^{\mathrm{cocart}}(F)$.

Proof. Notice that both (2) and (3) imply tautologically (1). Since $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$ is a full subcategory of $\mathrm{Fun}(\mathcal{A}, \mathcal{E})$, and similarly for $\mathcal{B}$ in place of $\mathcal{A}$, we see that as soon as (1) is satisfied the induced functor

$$u_! \colon \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{B}, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$$

provides a left adjoint to u^*. So (2) holds, and since u^* is an equivalence, (3) follows from the uniqueness of the inverse. □

7.7. **Invariance of cocartesian functors under localization.** We saw in Corollary 7.6.8 that when f is a final functor,

$$u^* \colon \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{B}, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$$

is an equivalence, with inverse given by u_*^{cocart}. Furthermore, in Corollary 7.6.11, we saw that when f is the inclusion of an initial object, then the inverse can be identified with the much simpler left Kan extension $u_!$. In this section, we analyze a similar situation, where f is assumed to be a localization (recall from [11, Proposition 7.1.10] that all localizations are final), building on the results of the previous section. Our starting point is the following finer analysis of cocartesian functors in this special situation:

Proposition 7.7.1. *Let*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "u"] \arrow[d, "q"] & \mathcal{B} \arrow[d, "p"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} . \end{tikzcd} \tag{7.7.2}$$

be a pullback square in $\mathbf{Cat}_\infty$*, with* p *being a cocartesian fibration. Assume that* f *exhibits* $\mathcal{Y}$ *as a localization of* $\mathcal{X}$ *at a collection of morphisms* W*. Then for every presentable* ∞*-category* $\mathcal{E}$ *and every functor* $G \colon \mathcal{A} \to \mathcal{E}$*, the following conditions are equivalent:*

(1) G *lies in the essential image of* $u^* \colon \mathrm{Fun}(\mathcal{B}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{A}, \mathcal{E})$*;*

(2) G *is cartesian at every mophism in* W*;*

(3) For every $\gamma \in W$*, the morphism* $\mathcal{E}^u((\mathrm{sp}\, G)(\gamma))$ *is an equivalence in* $\exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y})$*;*

(4) G *is cocartesian at every mophism in* W*.*

Proof. Let $W_{\mathcal{A}}$ be the set of cocartesian lifts of morphisms in W. We saw in Theorem 18.2.1 that $u \colon \mathcal{A} \to \mathcal{B}$ exhibits $\mathcal{B}$ as a localization of $\mathcal{A}$ at $W_{\mathcal{A}}$. Thus, (1) is equivalent to ask that G inverts every arrow in $W_{\mathcal{A}}$ and Lemma 18.1.1 shows that this is equivalent to condition (2). Combining the specialization equivalence Proposition 3.3.3 and the global functoriality established in Proposition 6.1.2-(1) and the fact that the front square of (18.2.3) is a pullback, we deduce that (1) is equivalent to ask that $\mathcal{E}^u \circ (\mathrm{sp}\, G) \colon \mathcal{X} \to \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y})$ inverts all arrows in W, i.e. to condition (3). Finally, we prove the equivalence between (3) and (4): let γ be a morphism in W. Combining Proposition 3.2.6-(1) and [31, 2.4.1.12], we see that G is cocartesian at γ if and only if $\mathcal{E}^u \circ (\mathrm{sp}\, G)$ takes γ into a $p_{\mathcal{E}}$-cocartesian morphism in $\exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y})$. Since $\mathcal{E}^u((\mathrm{sp}\, G)(\gamma))$ lies over $f(\gamma)$, which is an equivalence in $\mathcal{Y}$, we see that this happens if and only if $\mathcal{E}^u((\mathrm{sp}\, G)(\gamma))$ is an equivalence in $\exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y})$, whence the conclusion. □

Proposition 7.7.3. *Let*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "u"] \arrow[d, "q"] & \mathcal{B} \arrow[d, "p"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} . \end{tikzcd} \tag{7.7.4}$$

be a pullback square in $\mathbf{Cat}_\infty$*, with* p *being a cocartesian fibration. Assume that* f *is a localization functor and let* $\mathcal{E}$ *be a presentable* ∞*-category. Then:*

(1) *A functor* $F \in \operatorname{Fun}(\mathcal{B}, \mathcal{E})$ *is cocartesian if and only if* $u^*(F)$ *cocartesian.*

(2) *The functor*

$$u_!\colon \operatorname{Fun}(\mathcal{A}, \mathcal{E}) \to \operatorname{Fun}(\mathcal{B}, \mathcal{E})$$

preserves cocartesian functors.

(3) *The adjunction*

$$u_!\colon \operatorname{Fun}(\mathcal{A}, \mathcal{E}) \leftrightarrows \operatorname{Fun}(\mathcal{B}, \mathcal{E})\colon u^*$$

restricts to an equivalence of ∞*-categories between* $\operatorname{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$ *and* $\operatorname{Fun}^{\mathrm{cocart}}(\mathcal{B}, \mathcal{E})$*.*

Proof. Let W be the collection of morphisms that in $\mathcal{X}$ that are inverted by f. We start by proving (1). The "only if" direction is a consequence of Proposition 7.3.1-(1). Suppose on the other hand that $u^*(F)$ is cocartesian. Notice that the homotopy category $\mathrm{h}(\mathcal{Y})$ is the 1-categorical localization of $\mathrm{h}(\mathcal{X})$ at the image of W in $\mathrm{h}(\mathcal{X})$. In particular, every 1-morphism (in $\mathrm{h}(\mathcal{Y})$ and hence) in $\mathcal{Y}$ can be represented as a zig-zag (see [18]):

$$x_0 \to x_1 \leftarrow x_2 \to \cdots \leftarrow x_n$$

in $\mathcal{X}$, where the arrows pointing to the left are in W. Recall from Corollary 7.2.6, that F is cocartesian at every equivalence of $\mathcal{Y}$. Thus, using Corollary 7.2.7 we are left to show that for every morphism γ of $\mathcal{X}$, the functor F is cocartesian at $f(\gamma)$, and this follows from Lemma 7.3.2 and our assumption that $u^*(F)$ is cocartesian.

We now prove the claim (2). Let $G\colon \mathcal{A} \to \mathcal{E}$ be a cocartesian functor. Proposition 7.7.1 ensures the existence of a functor $F\colon \mathcal{B} \to \mathcal{E}$ such that $G \simeq u^*(F)$. Point (1) guarantees that F is cocartesian. At the same time, we know from Theorem 18.2.1 that $u\colon \mathcal{B} \to \mathcal{A}$ is a localization functor. Thus, $u^*\colon \operatorname{Fun}(\mathcal{B}, \mathcal{E}) \to \operatorname{Fun}(\mathcal{A}, \mathcal{E})$ is fully faithul, and therefore the counit transformation $g_! \circ g^* \to \mathrm{id}$ is an equivalence. It follows that

$$F \simeq g_!(g^*(F)) \simeq g_!(G)$$

is cocartesian, and so (2) is proven.

Finally, for (3), recall from [11, Proposition 7.1.10] that localization functors are final. Thus, (3) follows from (2) combined with Corollary 7.6.8 and Lemma 7.6.13. ☐

Corollary 7.7.5. *Let*

$$\begin{tikzcd}
\mathcal{A}' \arrow[rr, "u'"] \arrow[dr, "s'"] \arrow[dd] & & \mathcal{B}' \arrow[dr, "s"] \arrow[dd] & \\
& \mathcal{A} \arrow[rr, "u"] \arrow[dd] & & \mathcal{B} \arrow[dd] \\
\mathcal{X}' \arrow[rr, "f'"] \arrow[dr, "r'"] & & \mathcal{Y}' \arrow[dr, "r"] & \\
& \mathcal{X} \arrow[rr, "f"] & & \mathcal{Y}
\end{tikzcd}$$

be a commutative cube in $\mathbf{Cat}_\infty$*, with the vertical arrows being cocartesian fibrations. Assume that* r *and* r' *are localization functors and that the left and right vertical faces are pullbacks. Let*

$\mathcal{E}$ be a presentable ∞-category. Then, the following diagrams

$$\begin{tikzcd} \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}', \mathcal{E}) \arrow[r, "u'^*"] \arrow[d, "s_!"] & \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{B}', \mathcal{E}) \arrow[d, "s'_!"] \\ \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E}) \arrow[r, "u^*"] & \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{B}, \mathcal{E}), \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{B}, \mathcal{E}) \arrow[r, "g_*^{\mathrm{cocart}}"] \arrow[d, "u'^*"] & \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E}) \arrow[d, "s^*"] \\ \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{B}', \mathcal{E}) \arrow[r, "u'^{\mathrm{cocart}}_*"] & \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}', \mathcal{E}) \end{tikzcd}$$

are canonically commutative.

Proof. Observe that the right square from Corollary 7.7.5 is obtained from the left square by passing to right adjoints. Hence, we are left to prove the commutativity of the left square. Since the top face is commutative, we have

$$s'^* \circ u^* = u'^* \circ s^*$$

From Proposition 7.7.3, the adjunction $s_! \dashv s^*$ induces an equivalence of ∞-categories between $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$ and $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}', \mathcal{E})$ and similarly with $s'_! \dashv s'^*$. The commutativity of the left square thus follows. $\square$

Corollary 7.7.6. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a localization functor between ∞-categories. Let $\mathcal{B}$ be an ∞-category and denote by $u\colon \mathcal{B} \times \mathcal{X} \to \mathcal{B} \times \mathcal{Y}$ the induced functor. Let $\mathcal{E}$ be a presentable ∞-category. The adjunction $u_! \dashv u^*$ induces an equivalence of ∞-categories between* $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A} \times \mathcal{Y}, \mathcal{E})$ *and* $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A} \times \mathcal{X}, \mathcal{E})$.

Proof. Consider the pullback square

$$\begin{tikzcd} \mathcal{A} \times \mathcal{Y} \arrow[r] \arrow[d] & \mathcal{A} \times \mathcal{X} \arrow[d] \\ \mathcal{Y} \arrow[r] & \mathcal{X} \end{tikzcd}$$

and apply Proposition 7.7.3. $\square$

Remark 7.7.7. When applied to the localization $\mathcal{X} \to \mathrm{Env}(\mathcal{X})$, the above corollary says that $u_! \dashv u^*$ induces an equivalence of ∞-categories between $\mathrm{Fun}(\mathcal{A} \times \mathrm{Env}(\mathcal{X}), \mathcal{E})$ and $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A} \times \mathcal{X}, \mathcal{E})$.

7.8. **Exceptional functoriality.** Let

$$\begin{tikzcd} \mathcal{A} \arrow[r, "u"] \arrow[d, "q"] & \mathcal{B} \arrow[d, "p"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd} \tag{7.8.1}$$

be a pullback diagram in $\mathbf{Cat}_\infty$, with p being a cocartesian fibration. We saw in Corollary 7.6.8 that when f is a final functor the pullback

$$u^*\colon \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{B}, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$$

is an equivalence for every presentable ∞-category $\mathcal{E}$. In virtue of Proposition 7.2.9, the inverse to u^* is always given by the functor u_*^{cocart}, which is nevertheless very inexplicit in general. At the same time we saw in two rather different situations (Corollary 7.6.11 and Proposition 7.7.3) that sometimes the inverse can be computed by the left Kan extension $u_!$. In this section, we analyze this phenomenon more in detail, obtaining a sufficient criterion guaranteeing that $u_!$ preserves cocartesian functors, that will be needed later on.

We start with a simple observation:

Proposition 7.8.2. *Let*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "u"] \arrow[d, "q"] & \mathcal{B} \arrow[d, "p"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

be a pullback square in $\mathbf{Cat}_\infty$*, with* p *being a cocartesian fibration. Let* $\mathcal{E}$ *be a presentable* ∞*-category. Assume that* f *is fully faithful, and let* $F \in \mathrm{Fun}(\mathcal{A}, \mathcal{E})$ *be a functor cocartesian at a morphims* $\gamma \colon x \to y$ *in* $\mathcal{X}$*. Then* $u_!(F)$ *is cocartesian at* $f(\gamma)$*.*

Proof. Since f is fully faithful, the same goes for u. Thus, the unit transformation $F \to u^*(u_!(F))$ is an equivalence. Using Lemma 7.3.2, we therefore see that $u_!(F)$ is cocartesian at $f(\gamma)$ if and only if $F \simeq u^*(u_!(F))$ is cocartesian at γ. The conclusion follows. □

We now carry out a finer analysis. Fix $x \in \mathcal{X}$, set $y := f(x)$ and fix as well a morphism $\gamma \colon y \to z$ in $\mathcal{Y}$. Associated to these data, we can form the following commutative cube:

(7.8.3)
$$\begin{tikzcd} \mathcal{B}_y \arrow[rr, "j_{\gamma,x}"] \arrow[dr, "j_x"] \arrow[dd, "p_x"'] & & \mathcal{B}_\gamma \arrow[dr, "j_\gamma"] \arrow[dd, "p_\gamma" near start] & \\ & \mathcal{A} \arrow[rr, crossing over, "u" near start] & & \mathcal{B} \arrow[dd, "p"] \\ * \arrow[rr, "0" near end] \arrow[dr, "y"'] & & \Delta^1 \arrow[dr, "\gamma"] & \\ & \mathcal{X} \arrow[rr, "f"] \arrow[from=uu, crossing over, "q" near start] & & \mathcal{Y}\,, \end{tikzcd}$$

whose vertical faces are pullbacks. Fix a presentable ∞-category $\mathcal{E}$. The commutativity of the top face of the above cube induces a Beck-Chevalley transformation

(7.8.4)
$$j_{\gamma,x,!} \circ j_x^* \to j_\gamma^* \circ u_!$$

of functors from $\mathrm{Fun}(\mathcal{A}, \mathcal{E})$ to $\mathrm{Fun}(\mathcal{B}_\gamma, \mathcal{E})$. We have:

Proposition 7.8.5. *Assume that the Beck-Chevalley transformation* (7.8.4) *is an equivalence. Then for every* $F \in \mathrm{Fun}(\mathcal{B}, \mathcal{E})$*, the functor* $u_!(F) \colon \mathcal{A} \to \mathcal{E}$ *is cocartesian at* γ*.*

Proof. We have to prove that $\mathrm{sp}(u_!(F))$ is cocartesian at γ. By Lemma 7.3.2 applied to $\gamma \colon \Delta^1 \to \mathcal{Y}$, this is equivalent to show that $j_\gamma^*(u_!(F))$ is cocartesian at $0 \to 1$. Since the Beck-Chevalley transformation (7.8.4) is an equivalence, we are reduced to prove that $j_{\gamma,x,!}(j_x^*(F))$ is cocartesian at $0 \to 1$. In other words, we are reduced to prove the statement in the special case where (7.8.1) is the back square of (7.8.3). Since 0 is initial in Δ^1, this follows directly from Corollary 7.6.11. □

We now give a sufficient condition on f and γ ensuring that the Beck-Chevalley transformation (7.8.4) is an equivalence:

Proposition 7.8.6. *In the above setting, assume that:*

(1) for every $(t, \alpha) \in \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}_{/y}$*, the map*
$$\mathrm{Map}_{\mathcal{X}}(t, x) \to \mathrm{Map}_{\mathcal{Y}}(f(t), y)$$
is an equivalence;

(2) for every $(s, \beta) \in \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}_{/z}$*, the composition*
$$\mathrm{Map}_{\mathcal{X}}(s, x) \to \mathrm{Map}_{\mathcal{Y}}(f(s), y) \to \mathrm{Map}_{\mathcal{Y}}(f(s), z)$$
is an equivalence.

Then for every $F \in \mathrm{Fun}(\mathcal{A}, \mathcal{E})$, $u_!(F)$ is cocartesian at γ.

Remark 7.8.7. Notice that condition (1) above is automatically satisfied when f is fully faithful, or when both $\mathrm{Map}_{\mathcal{X}}(t,x)$ and $\mathrm{Map}_{\mathcal{Y}}(f(t),y)$ are both contractible. Similarly, condition (2) is automatically satisfied when both $\mathrm{Map}_{\mathcal{X}}(s,x)$ and $\mathrm{Map}_{\mathcal{Y}}(f(s),z)$ are both contractible.

Proof of Proposition 7.8.6. In virtue of Proposition 7.8.5, it is enough to show that these assumptions guarantee that the Beck-Chevalley transformation (7.8.4) is an equivalence. For this, it is enough to check that for every $b \in \mathcal{B}_\gamma$, the induced functor

$$\mathcal{B}_y \times_{\mathcal{B}_\gamma} (\mathcal{B}_\gamma)_{/b} \longrightarrow \mathcal{A} \times_{\mathcal{B}} \mathcal{B}_{/j_\gamma(b)} \tag{7.8.8}$$

is cofinal. Let $v \coloneqq p_\gamma(b) \in \Delta^1$ and set $w \coloneqq \gamma(v)$ (we have $w = y$ if $v = 0$ and $w = z$ if $v = 1$). Using Lemma 21.2.1, it is sufficient to prove that under our assumptions, the map

$$\{0\} \times_{\Delta^1} \Delta^1_{/v} \longrightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}_{/w} \tag{7.8.9}$$

is cofinal. Observe that the left hand side is contractible (and it coincides with the unique morphism ε from 0 to v in Δ^1). In particular, the map (7.8.9) is cofinal if and only if its image coincides with the final object of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}_{/w}$. Now, unraveling the definitions we see that the above map takes ε to (x, id_y) if $v = 0$ and to (x, γ) if $v = 1$. Thus, we have to prove that (x, id_y) and (x, γ) are final objects in $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}_{/y}$ and in $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}_{/z}$, respectively. Fix $(t, \alpha) \in \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}_{/w}$ and consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}_{/w}}((t,\alpha),(x,\gamma(\varepsilon))) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{X}}(t,x) \arrow[d] \\
\mathrm{Map}_{\mathcal{Y}_{/w}}(\alpha, \gamma(\varepsilon)) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{Y}}(f(t),y) \arrow[d] \\
* \arrow[r, "\alpha"] & \mathrm{Map}_{\mathcal{Y}}(f(t),w) .
\end{tikzcd}$$

The top square is a pullback by definition and the bottom one is a pullback thanks to the dual of [31, Lemma 5.5.5.12]. Our assumptions guarantee that in the two cases under consideration, the right vertical composition is an equivalence. Therefore, it follows that the top left corner is contractible, i.e. that $(x, \gamma(\varepsilon))$ is a final object in $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}_{/w}$, thus completing the proof. $\square$

Corollary 7.8.10. *Let $f \colon \mathrm{J} \to \mathrm{I}$ be a fully faithful functor between posets and consider a pullback square in* $\mathbf{Cat}_\infty$

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "u"] \arrow[d, "q"] & \mathcal{B} \arrow[d, "p"] \\
\mathrm{J} \arrow[r, "f"] & \mathrm{I} ,
\end{tikzcd}$$

where in addition p is a cocartesian fibration. Assume that for every object i in I, the subposet $\mathrm{J}_{/i}$ of J admits a final object. Then, the functor $u_! \colon \mathrm{Fun}(\mathcal{A}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{B}, \mathcal{E})$ preserves cocartesian functors.

Proof. Let $F \colon \mathcal{B} \to \mathcal{E}$ be a cocartesian functor. Let $\gamma \colon i_1 \to i_2$ be a morphism in I. By assumption, there exists a commutative diagram

$$\begin{tikzcd}
 & i_1 \arrow[dr, "\gamma"] & \\
f(j) \arrow[ur] \arrow[rr] & & i_2
\end{tikzcd}$$

in I where j belongs to J. Using Corollary 7.2.7, we see it is enough to show that F is cocartesian at a morphism of the form $f(j) \to i$ where $j \in \mathrm{J}$ and $i \in \mathrm{I}$. Let j_∞ be a final object in $\mathrm{J}_{/i}$. Then, there is a commutative diagram

$$\begin{tikzcd} & f(j_\infty) \arrow[dr] & \\ f(j) \arrow[ur] \arrow[rr] & & i \end{tikzcd}$$

in I. Since $f\colon \mathrm{J} \to \mathrm{I}$ is fully faithful and since $F\colon \mathcal{B} \to \mathcal{E}$ is cocartesian at $j \to j_\infty$, Proposition 7.8.2 ensures that $u_!(F)$ is cocartesian at $f(j) \to f(j_\infty)$. Using again Corollary 7.2.7, we are thus left to show that $u_!(F)$ is cocartesian at $f(j_\infty) \to i$. In that case, the conditions of Proposition 7.8.6 (in the form of Remark 7.8.7) are satisfied and the proof is achieved. ☐

Corollary 7.8.11. *Let*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "u"] \arrow[d, "q"] & \mathcal{B} \arrow[d, "p"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

be a pullback square in $\mathbf{Cat}_\infty$*, with p being a cocartesian fibration. Let $\mathcal{E}$ be a presentable ∞-category. Assume that f is fully faithful and admits a right adjoint g and let $\gamma\colon f \circ g \to \mathrm{id}_{\mathcal{Y}}$ be a counit transformation. Then for every $F \in \mathrm{Fun}(\mathcal{A}, \mathcal{E})$, $u_!(F)$ is cocartesian at γ_y for every $y \in \mathcal{Y}$.*

7.9. **Induced t-structure for cocartesian functors.** Let $p\colon \mathcal{A} \to \mathcal{X}$ be a cocartesian fibration and let $\mathcal{E}$ be a stable presentable ∞-category equipped with an accessible t-structure $\tau = (\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$. Then $\mathrm{Fun}(\mathcal{A}, \mathcal{E})$ has an induced t-structure defined by

$$\mathrm{Fun}(\mathcal{A}, \mathcal{E})_{\geqslant 0} \coloneqq \mathrm{Fun}(\mathcal{A}, \mathcal{E}_{\geqslant 0}) \qquad \text{and} \qquad \mathrm{Fun}(\mathcal{A}, \mathcal{E})_{\leqslant 0} \coloneqq \mathrm{Fun}(\mathcal{A}, \mathcal{E}_{\leqslant 0}) \ .$$

Definition 7.9.1. We say a cocartesian functor $F \in \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$ is *connective (with respect to τ)* if its image in $\mathrm{Fun}(\mathcal{A}, \mathcal{E})$ belongs to $\mathrm{Fun}(\mathcal{A}, \mathcal{E})_{\geqslant 0}$. We let $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})_{\geqslant 0}$ be the full subcategory of $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$ spanned by connective objects.

Proposition 7.9.2. *There exists a unique t-structure on* $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$ *whose connective part coincides with* $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})_{\geqslant 0}$*. In particular, the inclusion* $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E}) \hookrightarrow \mathrm{Fun}(\mathcal{A}, \mathcal{E})$ *is right t-exact.*

Proof. Since $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$ is presentable and stable by Corollary 7.2.4, using [32, Proposition 1.4.4.11] we are reduced to check that $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})_{\geqslant 0}$ is closed under colimits and extensions in $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$. Closure under colimits follows from Corollary 7.2.8, and closure under extensions is automatic. So the conclusion follows. ☐

Lemma 7.9.3. *Assume that $\mathcal{X}$ has an initial object x. Then a cocartesian functor $F \in \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$ is connective if and only if $j_x^*(F) \in \mathrm{Fun}(\mathcal{A}_x, \mathcal{E})$ is connective.*

Proof. The functor $j_x^*\colon \mathrm{Fun}(\mathcal{A}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{A}_x, \mathcal{E})$ is t-exact, so if F is connective then $j_x^*(F)$ is connective as well. For the converse, we have to check that F takes values in $\mathcal{E}_{\geqslant 0}$. It suffices to show that for every $y \in \mathcal{X}$, $j_y^*(F)\colon \mathcal{A}_y \to \mathcal{E}$ takes values in $\mathcal{E}_{\geqslant 0}$. Since x is an initial object, there exists a morphism $\gamma\colon x \to y$ in $\mathcal{X}$. Choose a straightening $f_\gamma\colon \mathcal{A}_x \to \mathcal{A}_y$ for $\mathcal{A}_\gamma$. Then Proposition 7.2.3 provides a canonical identification

$$j_y^*(F) \simeq f_{\gamma,!} j_x^*(F) \ .$$

Since $j_x^*(F)$ takes values in $\mathcal{E}_{\geqslant 0}$ by assumption and since $\mathcal{E}_{\geqslant 0}$ is closed under colimits in $\mathcal{E}$, the conclusion follows from the formula for left Kan extensions. ☐

Corollary 7.9.4. *Assume that $\mathcal{X}$ has an initial object x. Then the adjoint equivalence of Corollary 7.6.11*

$$j_{x,!} \colon \mathrm{Fun}(\mathcal{A}_x, \mathcal{E}) \leftrightarrows \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E}) \colon j_x^*$$

is t-exact.

Proof. Thanks to Lemma 7.9.3, we know that $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})_{\geqslant 0}$ corresponds via the above equivalence to $\mathrm{Fun}(\mathcal{A}_x, \mathcal{E}_{\geqslant 0})$. The conclusion follows from the uniqueness of the t-structure. $\square$

Example 7.9.5. Consider the posets I_0 and I_1 having $I = \{a, b, c, d\}$ as the underlying set and order given by the following Hasse diagrams:

$$I_0 = \left\{ \begin{array}{ccccccc} b & & & & c & & d \\ & \diagdown & & \diagup & & & \\ & & a & & & & \end{array} \right\} , \qquad I_1 = \left\{ \begin{array}{ccccc} & & d & & \\ & \diagup & & \diagdown & \\ b & & & & d \\ & \diagdown & & \diagup & \\ & & a & & \end{array} \right\}$$

The identity of I defines a morphism of posets $f \colon I_0 \to I_1$, which we can reinterpret as a constructible sheaf of posets $\mathcal{I}$ on $([0,1], \{0\})$. Fix a field k and consider the stable derived ∞-category $\mathcal{E} \coloneqq \mathrm{Mod}_k$. Let $F \colon \mathcal{I}_0 \to \mathrm{Mod}_k$ be the functor defined by setting

$$F_a = F_d \coloneqq k \ , \qquad F_b = F_c \coloneqq 0 \ .$$

Then via Corollary 7.9.4, F determines an object in $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{I}, \mathrm{Mod}_k)^{\heartsuit}$. Notice however that

$$f_!(F)_d \simeq F_d \oplus F_a[1] \simeq k \oplus k[1]$$

does not belong to the abelian category $\mathrm{Mod}_k^{\heartsuit}$.

7.10. **Categorical actions on cocartesian functors.** We use the terminology on categorical actions reviewed in Section 20. Fix a presentably symmetric monoidal ∞-category $\mathcal{E}^{\otimes}$. As recalled in Recollection 20.1.2, for every (small) ∞-category $\mathcal{A}$, the functor ∞-category $\mathrm{Fun}(\mathcal{A}, \mathcal{E})$ inherits a symmetric monoidal structure $\mathrm{Fun}(\mathcal{A}, \mathcal{E})^{\otimes}$. When $\mathcal{A}$ is part of a cocartesian fibration $p \colon \mathcal{A} \to \mathcal{X}$, cocartesian functors $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$ form a full subcategory of $\mathrm{Fun}(\mathcal{A}, \mathcal{E})$, but they are not closed under tensor product. Nevertheless, we still see a shadow of the tensor structure of $\mathrm{Fun}(\mathcal{A}, \mathcal{E})$ on cocartesian functors in terms of a categorical action:

Proposition 7.10.1. *Let $p \colon \mathcal{A} \to \mathcal{X}$ be a cocartesian fibration. Then for every $L \in \mathrm{Loc}(\mathcal{X}; \mathcal{E})$ (see Definition 19.1.1) and every $G \in \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$, the functor*

$$p^*(L) \otimes G \colon \mathcal{A} \to \mathcal{E}$$

is again cocartesian. In particular, the standard action of $\mathrm{Loc}(\mathcal{X}; \mathcal{E})$ on $\mathrm{Fun}(\mathcal{A}, \mathcal{E})$ restricts to a categorical action of $\mathrm{Loc}(\mathcal{X}; \mathcal{E})$ on $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$.

Proof. Let $\gamma \colon x \to y$ be a morphism in $\mathcal{X}$ and let $f_\gamma \colon \mathcal{A}_x \to \mathcal{A}_y$ be any straightening for $p_\gamma \colon \mathcal{A}_\gamma \to \Delta^1$. Since j_x^* and j_y^* are symmetric monoidal, we reduce to check that

$$f_{\gamma,!}(j_x^* p^*(L) \otimes j_x^*(G)) \to j_y^*(p^*(L)) \otimes j_y^*(G)$$

is an equivalence. Since $j_x^* \circ p^*(L) \simeq p_x^*(L(x))$, Lemma 20.1.3 supplies a canonical equivalence

$$f_{\gamma,!}(j_x^* p^*(L) \otimes j_x^*(G)) \simeq p_y^*(L(x)) \otimes f_{\gamma,!}(j_x^*(G)) \ .$$

Since G is cocartesian, the canonical comparison map

$$f_{\gamma,!}(j_x^*(G)) \to j_y^*(G)$$

is an equivalence. On the other hand, since L is a local system, the canonical map $L(\gamma) \colon L(x) \to L(y)$ is an equivalence. The conclusion follows. $\square$

Consider now a pullback square

$$\begin{tikzcd} \mathcal{B} \arrow[r, "u"] \arrow[d, "q"] & \mathcal{A} \arrow[d, "p"] \\ \mathcal{Y} \arrow[r, "f"] & \mathcal{X} \end{tikzcd} \tag{7.10.2}$$

in $\mathbf{Cat}_\infty$, where p is a cocartesian fibration. Then Construction 20.2.1 supplies a canonical comparison map

$$\mu \colon \operatorname{Loc}(\mathcal{Y};\mathcal{E}) \otimes_{\operatorname{Loc}(\mathcal{X};\mathcal{E})} \operatorname{Fun}(\mathcal{A},\mathcal{E}) \to \operatorname{Fun}(\mathcal{B},\mathcal{E}) \ .$$

Unraveling the definitions, we see that μ takes $L \otimes G$ to $p^*(L) \otimes G$. In particular, Proposition 7.10.1 shows that μ restricts to a well defined functor

$$\mu^{\mathrm{cocart}} \colon \operatorname{Loc}(\mathcal{Y};\mathcal{E}) \otimes_{\operatorname{Loc}(\mathcal{X};\mathcal{E})} \operatorname{Fun}^{\mathrm{cocart}}(\mathcal{A},\mathcal{E}) \to \operatorname{Fun}^{\mathrm{cocart}}(\mathcal{B},\mathcal{E}) \tag{7.10.3}$$

When f is a finite étale fibration (see Definition 19.2.1), Corollary 20.2.8 shows that μ is an equivalence. The goal of this section is to show that under mild assumptions the same holds in the cocartesian setting.

Observation 7.10.4. In the above setting, assume that f is a finite étale fibration. Then the composition $q' \coloneqq f \circ q \colon \mathcal{B} \to \mathcal{X}$ is a cocartesian fibration. This allows to consider the exponential fibrations

$$\exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y}) \in \mathbf{PrFib}^{\mathrm{L}}_{\mathcal{Y}} \qquad \text{and} \qquad \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{X}) \in \mathbf{PrFib}^{\mathrm{L}}_{\mathcal{X}} \ .$$

We set

$$\operatorname{Fun}^{\mathrm{cocart}}(\mathcal{B}/\mathcal{Y},\mathcal{E}) \coloneqq \Sigma^{\mathrm{cocart}}_{\mathcal{Y}}(\exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y})) \qquad \text{and} \qquad \operatorname{Fun}^{\mathrm{cocart}}(\mathcal{B}/\mathcal{X},\mathcal{E}) \coloneqq \Sigma^{\mathrm{cocart}}_{\mathcal{X}}(\exp_{\mathcal{E}}(\mathcal{B}/\mathcal{X})) \ .$$

Notice that both $\operatorname{Fun}^{\mathrm{cocart}}(\mathcal{B}/\mathcal{Y},\mathcal{E})$ and $\operatorname{Fun}^{\mathrm{cocart}}(\mathcal{B}/\mathcal{X},\mathcal{E})$ are full subcategories of $\operatorname{Fun}(\mathcal{B},\mathcal{E})$.

Construction 7.10.5. Using the notation from Recollection 6.2.1, observe the commutativity of

$$\begin{tikzcd} \mathcal{B} \arrow[r, equal] \arrow[d, "q"] & \mathcal{B} \arrow[d, "f\circ q"] \\ \mathcal{Y} \arrow[r, "f"] & \mathcal{X} \end{tikzcd}$$

provides a canonical transformation

$$\delta \colon \mathcal{B} \to f^*(\mathcal{B})$$

in $\mathbf{CoCart}_{\mathcal{Y}}$. In turn, Proposition 3.2.6-(1) shows that δ induces a morphism

$$\exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y}) \to f^* \exp(\mathcal{B}/\mathcal{X}) \ ,$$

which by adjunction $f^* \dashv f^{\mathrm{cc}}_*$ corresponds to a morphism

$$\alpha \colon \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{X}) \to f^{\mathrm{cc}}_* \exp(\mathcal{B}/\mathcal{Y}) \ . \tag{7.10.6}$$

Proposition 7.10.7. *In the above setting, assume that f is a finite étale fibration. Then the comparison morphism* (7.10.6) *is an equivalence. In particular,*

$$\operatorname{Fun}^{\mathrm{cocart}}(\mathcal{B}/\mathcal{Y},\mathcal{E}) = \operatorname{Fun}^{\mathrm{cocart}}(\mathcal{B}/\mathcal{X},\mathcal{E})$$

as full subcategories of $\operatorname{Fun}(\mathcal{B},\mathcal{E})$.

Proof. The second half follows applying $\Sigma^{\mathrm{cocart}}_{\mathcal{X}}$ to the equivalence (7.10.6). To show that α is an equivalence, it is enough to show that for every $x \in \mathcal{X}$, $j^*_x(\alpha)$ is an equivalence. Unraveling the definitions, we see that

$$\mathcal{B}_x \simeq \coprod_{y \in \mathcal{Y}_x} \mathcal{A}_x \ ,$$

which immediately implies that

$$\exp(\mathcal{B}/\mathcal{X})_x \simeq \prod_{y \in \mathcal{Y}_x} \mathrm{Fun}(\mathcal{A}_x, \mathcal{E}) \ .$$

Similarly, the formula for right Kan extensions paired with [41, Lemma 3.1.1], implies that

$$\big(f_*^{\mathrm{cc}} \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y})\big)_x \simeq \big(f_*^{\mathrm{cc}} f^* \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X})\big)_x \simeq \prod_{y \in \mathcal{Y}_x} \mathrm{Fun}(\mathcal{A}_x, \mathcal{E}) \ .$$

The conclusion follows. □

Corollary 7.10.8. *In the above setting, if f is a finite étale fibration then*

$$u_! \colon \mathrm{Fun}(\mathcal{B}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{A}, \mathcal{E})$$

preserves cocartesian functors and therefore it induces a well defined functor

$$u_! \colon \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{B}, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E}) \ .$$

Proof. Seeing $\mathcal{B}$ fibered over $\mathcal{X}$ via $q' \coloneqq f \circ q$, [31, Proposition 2.4.1.3-(2)] implies that u takes q'-cocartesian edges to p-cocartesian ones. Therefore, Proposition 7.3.1-(2) shows that $u_!$ restricts to a well defined functor

$$u_! \colon \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{B}/\mathcal{X}, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E}) \ .$$

Since $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{B}/\mathcal{X}, \mathcal{E}) = \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{B}/\mathcal{Y}, \mathcal{E})$ by Proposition 7.10.7, the conclusion follows. □

Corollary 7.10.9. *In the above setting, assume that f is a finite étale fibration and that $\mathcal{E}$ is stable. Then*

$$u_! \colon \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{B}, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$$

is monadic.

Proof. It follows from Lemma 20.2.5 that the functors

$$u_! \colon \mathrm{Fun}(\mathcal{B}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{A}, \mathcal{E}) \qquad \text{and} \qquad u^* \colon \mathrm{Fun}(\mathcal{A}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{B}, \mathcal{E})$$

are biadjoint. Combining Proposition 7.3.1-(1) and Corollary 7.10.8, we see that both respect cocartesian functors. Therefore, $u_! \colon \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{B}, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$ is biadjoint to u^*. Besides, Lemma 20.2.6 implies that $u_! \colon \mathrm{Fun}(\mathcal{B}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{A}, \mathcal{E})$ is conservative. Since $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$ is a full subcategory of $\mathrm{Fun}(\mathcal{A}, \mathcal{E})$, it follows that the same goes for the restriction of $u_!$ to cocartesian functors. At this point, the conclusion follows from Lurie-Barr-Beck [32, Theorem 4.7.3.5]. □

Corollary 7.10.10. *Let*

$$\begin{tikzcd} \mathcal{B} \arrow[r, "u"] \arrow[d, "q"] & \mathcal{A} \arrow[d, "p"] \\ \mathcal{Y} \arrow[r, "f"] & \mathcal{X} \end{tikzcd}$$

be a pullback square in $\mathbf{Cat}_\infty$, where p is a cocartesian fibration. Let $\mathcal{E}^\otimes$ be a presentably symmetric monoidal ∞-category. If f is a finite étale fibration and $\mathcal{E}$ is stable, then the comparison functor

$$\mathrm{Loc}(\mathcal{Y}) \otimes_{\mathrm{Loc}(\mathcal{Y})} \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{B}, \mathcal{E}) \ .$$

is an equivalence.

Proof. Using Corollary 7.10.9 as input, the same proof of Corollary 20.2.8 applies. □

8. Punctually split and Stokes functors

In this section, we explore the new features that appear when one specializes the exponential construction to the case of cocartesian fibrations *in posets*. Every such fibration $\mathcal{I}$ has an underlying discrete fibration $\mathcal{I}^{\mathrm{set}}$ (see Notation 5.2.2), and this allows to introduce punctually split functors. We analyze their role and explore this notion from the point of view of the exponential construction and discuss their basic functorialities. Finally, we introduce the main object of study of this paper: Stokes functors.

8.1. **Punctually split functors.** We fix a presentable ∞-category $\mathcal{E}$.

Definition 8.1.1. Let $p\colon \mathcal{I} \to \mathcal{X}$ be an object in $\mathbf{PosFib}$. Let $F \in \mathrm{Fun}(\mathcal{I}, \mathcal{E})$.

(1) For $x \in \mathcal{X}$, we say that F is *split at x* if $j_x^*(F)$ lies in the essential image of
$$i_{\mathcal{I}_x,!}\colon \mathrm{Fun}(\mathcal{I}_x^{\mathrm{set}}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{I}_x, \mathcal{E})$$

(2) We say that F is *punctually split* if it is split at every object $x \in \mathcal{X}$.

(3) We say that F is *split* if it lies in the essential image of the induction functor
$$i_{\mathcal{I},!}\colon \mathrm{Fun}(\mathcal{I}^{\mathrm{set}}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{I}, \mathcal{E})$$

We denote by $\mathrm{Fun}_{\mathrm{PS}}(\mathcal{I}, \mathcal{E})$ the full subcategory of $\mathrm{Fun}(\mathcal{I}, \mathcal{E})$ formed by punctually split functors.

Remark 8.1.2. It follows from Corollary 6.1.6 that split functors are punctually split.

Example 8.1.3. Let $p\colon \mathcal{I} \to \mathcal{X}$ be a cocartesian fibration in posets and let $a \in \mathcal{I}$ be an element. Write $\mathrm{ev}_a^{\mathcal{I}}\colon \{a\} \hookrightarrow \mathcal{I}$ for the canonical inclusion. Since $\mathrm{ev}_a^{\mathcal{I}}$ factors through $i_{\mathcal{I}}\colon \mathcal{I}^{\mathrm{set}} \to \mathcal{I}$, we see that for every $E \in \mathcal{E}$ the functor $\mathrm{ev}_{a,!}^{\mathcal{I}}(E) \in \mathrm{Fun}(\mathcal{I}, \mathcal{E})$ is split, and hence punctually split by Remark 8.1.2.

Definition 8.1.4. In the setting of Definition 8.1.1, a *splitting for F* is the given of a functor $F_0\colon \mathcal{I}^{\mathrm{set}} \to \mathcal{E}$ and an equivalence $\alpha\colon i_{\mathcal{I},!}(F_0) \simeq F$.

Warning 8.1.5. In general, splittings do not exist and even when they exist they are typically neither unique nor canonical.

The essential image construction of Lemma 5.2.1 allows to organize punctually split functors into a subfibration of the exponential fibration $\exp_{\mathcal{E}}(\mathcal{I}/\mathcal{X})$:

Definition 8.1.6. Let $p\colon \mathcal{I} \to \mathcal{X}$ be a cocartesian fibration in posets and let $i_{\mathcal{I}}\colon \mathcal{I}^{\mathrm{set}} \to \mathcal{I}$ be the canonical morphism. We define the *punctually split exponential fibration with coefficients in $\mathcal{E}$ associated to $p\colon \mathcal{I} \to \mathcal{X}$* as
$$\exp_{\mathcal{E}}^{\mathrm{PS}}(\mathcal{I}/\mathcal{X}) \coloneqq \mathrm{EssIm}(\mathcal{E}_!^{i_{\mathcal{I}}}) \ .$$

Remark 8.1.7. Notice that $\exp_{\mathcal{E}}^{\mathrm{PS}}(\mathcal{I}/\mathcal{X})$ defines an object in $\mathbf{CoCart}_{\mathcal{X}}$, but typically not in $\mathbf{PrFib}_{\mathcal{X}}^{\mathrm{L}}$ (see also Example 5.2.6). Lemma 5.2.1 shows that it is a sub-cocartesian fibration of $\exp_{\mathcal{E}}(\mathcal{I}/\mathcal{X})$. Under the specialization equivalence, we see that $\Sigma_{\mathcal{X}}(\exp_{\mathcal{E}}^{\mathrm{PS}}(\mathcal{I}/\mathcal{X}))$ coincides with the full subcategory of $\mathrm{Fun}(\mathcal{I}, \mathcal{E})$ spanned by punctually split functors.

Split functors provide a handy set of generators for $\mathrm{Fun}(\mathcal{A}, \mathcal{I})$:

Recollection 8.1.8. Let $\mathcal{A}$ be an ∞-category. For every $a \in \mathcal{A}$, write $\mathrm{ev}_a^{\mathcal{A}}\colon \{a\} \to \mathcal{A}$ be the canonical inclusion. It follows from the Yoneda lemma that the functor
$$\mathrm{ev}_{a,!}^{\mathcal{A}}\colon \mathbf{Spc} \to \mathrm{Fun}(\mathcal{A}, \mathbf{Spc})$$
is the unique colimit-preserving functor sending $*$ to $\mathrm{Map}_{\mathcal{A}}(a, -)$. The density of the Yoneda embedding implies therefore that $\mathrm{Fun}(\mathcal{A}, \mathbf{Spc})$ is generated under colimits by $\{\mathrm{ev}_{a,!}^{\mathcal{A}}(*)\}_{a \in \mathcal{A}}$. More

generally, let $\mathcal{E}$ be a presentable ∞-category generated under colimits by a set $\{E_\alpha\}_{\alpha\in I}$. Then under the identification
$$\mathrm{Fun}(\mathcal{A},\mathcal{E}) \simeq \mathrm{Fun}(\mathcal{A},\mathbf{Spc})\otimes\mathcal{E}$$
we see that $\mathrm{ev}_{a,!}(E_\alpha)\simeq \mathrm{ev}^{\mathcal{A}}_{a,!}(*)\otimes E_\alpha$ and therefore that $\{\mathrm{ev}^{\mathcal{A}}_{a,!}(E_\alpha)\}_{a\in\mathcal{A},\alpha\in I}$ generates $\mathrm{Fun}(\mathcal{A},\mathcal{E})$ under colimits.

Proposition 8.1.9. *Let $p\colon \mathcal{I}\to\mathcal{X}$ be a cocartesian fibration in posets. Then $\mathrm{Fun}(\mathcal{I},\mathcal{E})$ is generated under colimits by split functors.*

Proof. Combine Example 8.1.3 and Recollection 8.1.8. □

8.2. **Stokes functors.** We introduce here the fundamental object of this paper. We fix once more a presentable ∞-category $\mathcal{E}$.

Definition 8.2.1. Let $p\colon \mathcal{I}\to\mathcal{X}$ be a cocartesian fibration in posets. The ∞-category of $\mathcal{I}$-*Stokes functors with value in* $\mathcal{E}$ is by definition
$$\mathrm{St}_{\mathcal{I},\mathcal{E}} \coloneqq \Sigma^{\mathrm{cocart}}_{\mathcal{X}}(\exp^{\mathrm{PS}}_{\mathcal{E}}(\mathcal{I}/\mathcal{X}))\ .$$

Remark 8.2.2. Under the specialization equivalence (6.1.1), the ∞-category $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ coincides with full subcategory of $\mathrm{Fun}(\mathcal{I},\mathcal{E})$ spanned by functors $F\colon\mathcal{I}\to\mathcal{E}$ such that

(1) F is cocartesian (Definition 7.2.1).

(2) F is punctually split (Definition 8.1.1).

Example 8.2.3. Assume that the cocartesian fibration $p\colon\mathcal{I}\to\mathcal{X}$ is discrete, i.e. that its fibers are sets. Then the map $i_{\mathcal{I}}\colon \mathcal{I}^{\mathrm{set}}\to\mathcal{I}$ is an equivalence, so in this case every functor $F\colon\mathcal{I}\to\mathcal{E}$ is split (and hence punctually split). It follows from the above remark that in this case
$$\mathrm{St}_{\mathcal{I},\mathcal{E}}\simeq \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{I},\mathcal{E})\ .$$

The cocartesian condition can be used to transport a splitting defined at an object $x\in\mathcal{X}$ to a point y via a morphism $\gamma\colon x\to y$, as in the following lemma:

Lemma 8.2.4. *Let*
$$\begin{tikzcd} \mathcal{J} \arrow[d] & \mathcal{J}_{\mathcal{X}} \arrow[l, "u"'] \arrow[d] \\ \mathcal{Y} & \mathcal{X} \arrow[l, "f"'] \end{tikzcd}$$
be a pullback square in $\mathbf{Cat}_\infty$, whose vertical morphisms are cocartesian fibrations in posets. Let $\gamma\colon f(x)\to y$ be a morphism in $\mathcal{Y}$, with $x\in\mathcal{X}$. Let $F\colon\mathcal{J}_{\mathcal{X}}\to\mathcal{E}$ be a functor such that $u_!(F)\colon\mathcal{J}\to\mathcal{E}$ is cocartesian at γ and such that the unit $F\to u^*u_!(F)$ is an equivalence above x. If F is split at x, then $u_!(F)$ is split at y. In particular, when $f=\mathrm{id}_{\mathcal{X}}$ and F is cocartesian at $\gamma\colon x\to y$, if F is split at x then it is split at y as well.

Proof. Let $f_\gamma\colon\mathcal{J}_{f(x)}\to\mathcal{J}_y$ be the morphism of posets induced by $\gamma\colon f(x)\to y$. Since $u_!(F)\colon\mathcal{J}\to\mathcal{E}$ is cocartesian at γ, Proposition 7.2.3 implies the existence of an equivalence $f_{\gamma,!}((u_!(F))_{f(x)})\simeq F_y$. By assumption $F_x\simeq(u_!(F))_{f(x)}$. The conclusion thus follows. □

This leads to the following neat description of Stokes functors when $\mathcal{X}$ admits an initial object:

Proposition 8.2.5. *Let $p\colon\mathcal{I}\to\mathcal{X}$ be a cocartesian fibration in posets. If $\mathcal{X}$ admits an initial object x, then the adjunction*
$$j_{x,!}\colon\ \mathrm{Fun}(\mathcal{I}_x,\mathcal{E})\leftrightarrows\mathrm{Fun}(\mathcal{I},\mathcal{E})\colon j_x^*$$
restricts to an equivalence of ∞-categories between $\mathrm{St}_{\mathcal{I}_x,\mathcal{E}}$ and $\mathrm{St}_{\mathcal{I},\mathcal{E}}$.

Proof. Using Corollary 7.6.11, we see that both $j_{x,!}$ and j_x^* preserve cocartesian functors and that it restricts to an equivalence between $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{I}_x, \mathcal{E})$ and $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{I}, \mathcal{E})$. That j_x^* preserves punctually split functors follows directly from the definition. On the other hand, combining together Corollary 7.6.11 and Lemma 8.2.4 we see that $j_{x,!}$ also preserves the punctually split condition. The conclusion follows. ☐

Corollary 8.2.6. *Let $p\colon \mathcal{I} \to \mathcal{X}$ be an object of* **PosFib**. *Assume that $\mathcal{X}$ admits an initial object. Let $\mathcal{E}$ be a presentable ∞-category. Then, the induction*

$$i_{\mathcal{I}!}\colon \mathrm{St}_{\mathcal{I}^{\mathrm{set}},\mathcal{E}} \to \mathrm{St}_{\mathcal{I},\mathcal{E}}$$

is essentially surjective. That is, every Stokes functor $F\colon \mathcal{I} \to \mathcal{E}$ splits.

Proof. Let x be an initial object in $\mathcal{X}$. From Proposition 8.2.5, the horizontal arrows of the following commutative square

$$\begin{tikzcd} \mathrm{St}_{\mathcal{I}_x^{\mathrm{set}},\mathcal{E}} \arrow[r, "j_{x!}^{\mathrm{set}}"] \arrow[d, "i_{\mathcal{I}_x!}"] & \mathrm{St}_{\mathcal{I}^{\mathrm{set}},\mathcal{E}} \arrow[d, "i_{\mathcal{I}!}"] \\ \mathrm{St}_{\mathcal{I}_x,\mathcal{E}} \arrow[r, "j_{x!}"] & \mathrm{St}_{\mathcal{I},\mathcal{E}} \end{tikzcd}$$

are equivalences. On the other hand, the left vertical arrow is essentially surjective by definition. ☐

Warning 8.2.7. The splitting produced by Corollary 8.2.6 is not unique nor canonical.

8.3. **Functoriality for punctually split and Stokes functors.** Fix a morphism

$$\begin{tikzcd} \mathcal{J} \arrow[d] & \mathcal{J}_{\mathcal{X}} \arrow[l, "u"'] \arrow[r, "v"] \arrow[d] & \mathcal{I} \arrow[dl] \\ \mathcal{Y} & \mathcal{X} \arrow[l, "f"'] & \end{tikzcd}$$

in **PosFib**. We now show that the basic functorialities of pullback and induction are well behaved with respect to punctually split and Stokes functors. We start at the exponential level:

Proposition 8.3.1. *The functors*

$$\mathcal{E}^u\colon \exp_{\mathcal{E}}(\mathcal{J}/\mathcal{Y}) \to \exp_{\mathcal{E}}(\mathcal{J}_{\mathcal{X}}/\mathcal{X}) \qquad \textit{and} \qquad \mathcal{E}^v_!\colon \exp_{\mathcal{E}}(\mathcal{J}_{\mathcal{X}}/\mathcal{X}) \to \exp_{\mathcal{E}}(\mathcal{I}/\mathcal{X})$$

respect the punctually split sub-cocartesian fibrations and thus they induce the following commutative diagram:

$$\begin{tikzcd} \exp^{\mathrm{PS}}_{\mathcal{E}}(\mathcal{J}/\mathcal{Y}) \arrow[d, hook] & \exp^{\mathrm{PS}}_{\mathcal{E}}(\mathcal{J}_{\mathcal{X}}/\mathcal{X}) \arrow[l, "\mathcal{E}^u"'] \arrow[r, "\mathcal{E}^v_!"] \arrow[d, hook] & \exp^{\mathrm{PS}}_{\mathcal{E}}(\mathcal{I}/\mathcal{X}) \arrow[d, hook] \\ \exp_{\mathcal{E}}(\mathcal{J}/\mathcal{Y}) \arrow[d] & \exp_{\mathcal{E}}(\mathcal{J}_{\mathcal{X}}/\mathcal{X}) \arrow[l, "\mathcal{E}^u"'] \arrow[r, "\mathcal{E}^v_!"] \arrow[d] & \exp_{\mathcal{E}}(\mathcal{I}/\mathcal{X}) \arrow[dl] \\ \mathcal{Y} & \mathcal{X} \arrow[l] & \end{tikzcd} \tag{8.3.2}$$

whose left squares are pullbacks.

Proof. An object in $\exp_{\mathcal{E}}(\mathcal{J}_{\mathcal{X}}/\mathcal{X})$ is a pair (x, F), where $x \in \mathcal{X}$ and $F\colon (\mathcal{J}_{\mathcal{X}})_x \to \mathcal{E}$. The functor $\mathcal{E}^u$ takes (x, F) to $(f(x), F)$, where F is now seen as a functor from $\mathcal{I}_{f(x)} \simeq (\mathcal{J}_{\mathcal{X}})_x$ to $\mathcal{E}$. In particular,

$\mathcal{E}^u$ preserves and reflects the punctually spit condition, which shows that the top left square is both commutative and a pullback. On the other hand, the commutativity of

$$\begin{tikzcd} (\mathcal{J}_{\mathcal{X}}^{\mathrm{set}})_x \arrow[r, "v_x^{\mathrm{set}}"] \arrow[d, "i_{\mathcal{J}_{\mathcal{X}}}"] & \mathcal{I}_x^{\mathrm{set}} \arrow[d, "i_{\mathcal{I}}"] \\ (\mathcal{J}_{\mathcal{X}})_x \arrow[r, "v_x"] & \mathcal{I}_x \end{tikzcd}$$

immediately implies that $\mathcal{E}_!^v$ preserves the condition of being split at x. $\square$

Corollary 8.3.3. *In the above setting:*

(1) *Let $F \colon \mathcal{J} \to \mathcal{E}$ be a functor. Let $x \in \mathcal{X}$ be an object. Then, F is punctually split at $f(x)$ if and only if $u^*(F)$ is punctually split at x.*

(2) *if $G \colon \mathcal{J}_X \to \mathcal{E}$ is punctually split, then the same goes for $v_!(G) \colon \mathcal{I} \to \mathcal{E}$.*

In particular the functors

$$u^* \colon \operatorname{Fun}(\mathcal{J}, \mathcal{E}) \to \operatorname{Fun}(\mathcal{J}_{\mathcal{X}}, \mathcal{E}) \qquad \textit{and} \qquad v_! \colon \operatorname{Fun}(\mathcal{J}_{\mathcal{X}}, \mathcal{E}) \to \operatorname{Fun}(\mathcal{I}, \mathcal{E})$$

restrict to well-defined functors

$$u^* \colon \operatorname{Fun}_{\mathrm{PS}}(\mathcal{J}, \mathcal{E}) \to \operatorname{Fun}_{\mathrm{PS}}(\mathcal{J}_X, \mathcal{E}) \qquad \textit{and} \qquad v_! \colon \operatorname{Fun}_{\mathrm{PS}}(\mathcal{J}_X, \mathcal{E}) \to \operatorname{Fun}_{\mathrm{PS}}(\mathcal{I}, \mathcal{E}) \ .$$

Proof. Apply $\Sigma_{\mathcal{X}}$ to the commutative diagram (8.3.2) and use Proposition 6.1.2. $\square$

Corollary 8.3.4. *In the above setting:*

(1) *if $F \colon \mathcal{J} \to \mathcal{E}$ is a Stokes functor, the same goes for $u^*(F) \colon \mathcal{J}_{\mathcal{X}} \to \mathcal{E}$;*

(2) *if $G \colon \mathcal{J}_X \to \mathcal{E}$ is a Stokes functor, then the same goes for $v_!(G) \colon \mathcal{I} \to \mathcal{E}$.*

Thus, the functors

$$u^* \colon \operatorname{Fun}(\mathcal{J}, \mathcal{E}) \to \operatorname{Fun}(\mathcal{J}_{\mathcal{X}}, \mathcal{E}) \qquad \textit{and} \qquad v_! \colon \operatorname{Fun}(\mathcal{J}_{\mathcal{X}}, \mathcal{E}) \to \operatorname{Fun}(\mathcal{I}, \mathcal{E})$$

restrict to well-defined functors

$$u^* \colon \mathrm{St}_{\mathcal{J}, \mathcal{E}} \to \mathrm{St}_{\mathcal{J}_X, \mathcal{E}} \qquad \textit{and} \qquad v_! \colon \mathrm{St}_{\mathcal{J}_X, \mathcal{E}} \to \mathrm{St}_{\mathcal{I}, \mathcal{E}} \ .$$

Proof. Apply $\Sigma_{\mathcal{X}}^{\mathrm{cocart}}$ to the commutative diagram (8.3.2) and combine Proposition 7.3.1 and Corollary 8.3.3. $\square$

We conclude this section with the following generalization of Proposition 8.2.5:

Proposition 8.3.5. *Let*

$$\begin{tikzcd} \mathcal{J} \arrow[r, "g"] \arrow[d] & \mathcal{I} \arrow[d] \\ \mathcal{Y} \arrow[r, "f"] & \mathcal{X} \end{tikzcd}$$

be a pullback square in $\mathbf{Cat}_\infty$, where the vertical morphisms are cocartesian fibrations in posets. Assume that $f \colon \mathcal{Y} \to \mathcal{X}$ is a localization functor. Let $\mathcal{E}$ be a presentable ∞-category. Then, the following statements hold :

(1) *Let $F \in \operatorname{Fun}(\mathcal{I}, \mathcal{E})$. Then, F is a Stokes functor if and only if so is $g^*(F)$.*

(2) *Let $G \in \operatorname{Fun}(\mathcal{J}, \mathcal{E})$. If G is a Stokes functor, then so is $g_!(G)$.*

(3) *The adjunction $g_! \dashv g^*$ induces an equivalence of ∞-categories between $\mathrm{St}_{\mathcal{I}, \mathcal{E}}$ and $\mathrm{St}_{\mathcal{J}, \mathcal{E}}$.*

Proof. The claim (1) follows from Proposition 7.7.3-(1) and Corollary 8.3.3-(1). Let $G\colon \mathcal{J} \to \mathcal{E}$ be a Stokes functor. From Proposition 7.7.3-(2) the functor $g_!(G)$ is cocartesian. To check that $g_!(G)$ is punctually split amounts to show by (1) that $g^*(g_!(G))$ is punctually split. On the other hand, Proposition 7.7.3-(3) gives $g^*(g_!(G)) \simeq G$ and (2) is proved. The claim (3) then follows from Proposition 7.7.3-(3). □

8.4. **Stokes functors and (co)limits.** Stokes functors are poorly behaved with respect to limits and colimits, as the following next two lemmas are essentially the only stability properties one gets in general:

Proposition 8.4.1. *Let $p\colon \mathcal{I} \to \mathcal{X}$ be a cocartesian fibration in* sets*, seen as an object in* **PosFib**. *Then $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ is presentable and furthermore:*

(1) $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ is stable under colimits in $\mathrm{Fun}(\mathcal{I},\mathcal{E})$.

(2) Assume additionally that the fibers of p are finite and that $\mathcal{E}$ is presentable stable. Then $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ is stable under limits in $\mathrm{Fun}(\mathcal{I},\mathcal{E})$.

Proof. Via the equivalence $\mathrm{St}_{\mathcal{I},\mathcal{E}} \simeq \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{I},\mathcal{E})$ of Example 8.2.3, presentability follows from Corollary 7.2.4, statement (1) follows from Corollary 7.2.8 and statement (2) follows from Proposition 7.2.14. □

More in general, we have:

Lemma 8.4.2. *Let $p\colon \mathcal{I} \to \mathcal{X}$ be a cocartesian fibration in posets. Then $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ is closed under arbitrary coproducts in $\mathrm{Fun}(\mathcal{I},\mathcal{E})$.*

Proof. Thanks to Proposition 7.2.9, we know that cocartesian functors are closed under arbitrary colimits in $\mathrm{Fun}(\mathcal{I},\mathcal{E})$. Besides, for every $x \in \mathcal{X}$, the restriction functor $j_x^*\colon \mathrm{Fun}(\mathcal{I},\mathcal{E}) \to \mathrm{Fun}(\mathcal{I}_x,\mathcal{E})$ commute with all colimits as well. This reduces us to the case where $\mathcal{X}$ is a single point, and we have to prove that split functors are closed under coproducts. Let therefore $\{F_i\}_{i\in I}$ be a family of split functors and fix splittings

$$\alpha_i\colon i_{\mathcal{I},!}(V_i) \simeq F_i\ .$$

Since $i_{\mathcal{I},!}$ commutes with colimits, it immediately follows that $\coprod_{i\in I} V_i$ provides a splitting for $\coprod_{i\in I} F_i$. □

Definition 8.4.3. Let $p\colon \mathcal{I} \to \mathcal{X}$ be an object of **PosFib** and let $\mathcal{C} \subset \mathbf{Pr}^{\mathrm{L}}$ be a full subcategory. We say that $p\colon \mathcal{I} \to \mathcal{X}$ is $\mathcal{C}$-*bireflexive* if the full subcategory $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ of $\mathrm{Fun}(\mathcal{I},\mathcal{E})$ is closed under limits and colimits for every $\mathcal{E} \in \mathcal{C}$.

Example 8.4.4. If $\mathcal{C}$ only consists in a single category $\mathcal{E}$, we say that $p\colon \mathcal{I} \to \mathcal{X}$ is $\mathcal{E}$-bireflexive. If $\mathcal{C} \subset \mathbf{Pr}^{\mathrm{L}}$ is the collection of all presentable stable ∞-categories, we simply say that $p\colon \mathcal{I} \to \mathcal{X}$ is stably bireflexive.

Remark 8.4.5. Theorem 12.1.3 provides many geometrical examples of stably bireflexive cocartesian fibrations in posets.

Lemma 8.4.6. *Let $p\colon \mathcal{I} \to \mathcal{X}$ be an object of* **PosFib** *and let $\mathcal{E}$ be a presentable (stable) ∞-category such that $p\colon \mathcal{I} \to \mathcal{X}$ is $\mathcal{E}$-bireflexive. Then $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ is a localization of $\mathrm{Fun}(\mathcal{I},\mathcal{E})$, and in particular it is presentable (stable).*

Proof. Since $\mathcal{E}$ is presentable (stable), $\mathrm{Fun}(\mathcal{I},\mathcal{E})$ is presentable (stable) in virtue of [31, Proposition 5.5.3.6] and [32, Proposition 1.1.3.1]. Then, the conclusion follows from the ∞-categorical reflection theorem, see [42, Theorem 1.1]. □

Notation 8.4.7. In the setting of Lemma 8.4.6, the canonical inclusion $\mathrm{St}_{\mathcal{I},\mathcal{E}} \hookrightarrow \mathrm{Fun}(\mathcal{I},\mathcal{E})$ admits a left adjoint and a right adjoint, that we denote by $\mathrm{LSt}_{\mathcal{I},\mathcal{E}}$ and $\mathrm{RSt}_{\mathcal{I},\mathcal{E}}$ respectively.

Lemma 8.4.8. *Let $p\colon \mathcal{I} \to \mathcal{X}$ be an object of* $\mathbf{PosFib}$*. Let $\mathcal{E}$ be a presentable stable compactly generated ∞-category such that $p\colon \mathcal{I} \to \mathcal{X}$ is $\mathcal{E}$-bireflexive. Let $\{E_\alpha\}_{\alpha\in I}$ be a set of compact generators for $\mathcal{E}$. Then $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ is presentable stable compactly generated by the $\{\mathrm{LSt}_{\mathcal{I},\mathcal{E}}(\mathrm{ev}_{a,!}(E_\alpha))\}_{\alpha\in I, a\in\mathcal{I}}$ where the $\mathrm{ev}_a : \{a\} \to \mathcal{I}$ are the canonical inclusions.*

Proof. That $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ is presentable stable follows from Lemma 8.4.6. By Recollection 8.1.8, the $\{\mathrm{ev}_{a,!}(E_\alpha)\}_{\alpha\in I, a\in\mathcal{I}}$ are compact generators of $\mathrm{Fun}(\mathcal{I},\mathcal{E})$. Then Lemma 8.4.8 formally follows from the fact that $\mathrm{St}_{\mathcal{I},\mathcal{E}} \hookrightarrow \mathrm{Fun}(\mathcal{I},\mathcal{E})$ commutes with colimits. □

The following two lemmas are immediate consequences of Proposition 17.2.3.

Lemma 8.4.9. *Let $\mathcal{X}$ be an ∞-category. Let $p\colon \mathcal{I} \to \mathcal{J}$ be a morphism in $\mathbf{PosFib}^f_{\mathcal{X}}$. Let $\mathcal{E}$ be a presentable stable ∞-category such that $\mathcal{I} \to \mathcal{X}$ and $\mathcal{J} \to \mathcal{X}$ are $\mathcal{E}$-bireflexive. Then, $p_!\colon \mathrm{St}_{\mathcal{I},\mathcal{E}} \to \mathrm{St}_{\mathcal{J},\mathcal{E}}$ commutes with limits and colimits.*

Lemma 8.4.10. *Let $\mathcal{X}$ be an ∞-category. Let $p\colon \mathcal{I} \to \mathcal{J}$ be a morphism in $\mathbf{PosFib}^f_{\mathcal{X}}$. Let $\mathcal{E}$ be a presentable stable ∞-category such that $\mathcal{I} \to \mathcal{X}$ and $\mathcal{I}_p \to \mathcal{X}$ are $\mathcal{E}$-bireflexive. Then, $\mathrm{Gr}_p\colon \mathrm{St}_{\mathcal{I},\mathcal{E}} \to \mathrm{St}_{\mathcal{I}_p,\mathcal{E}}$ commutes with limits and colimits.*

Corollary 8.4.11. *Let (X,P) be an exodromic stratified space. Let $p\colon \mathcal{I} \to \mathcal{J}$ be a graduation morphism of cocartesian fibrations in finite posets over $\Pi_\infty(X,P)$. Let $\mathcal{E}$ be a presentable stable ∞-category and consider the pull-back square*

$$\begin{tikzcd} \mathrm{St}_{\mathcal{I},\mathcal{E}} \arrow[r, "p_!"] \arrow[d, "\mathrm{Gr}_p"] & \mathrm{St}_{\mathcal{J},\mathcal{E}} \arrow[d, "\mathrm{Gr}"] \\ \mathrm{St}_{\mathcal{I}_p,\mathcal{E}} \arrow[r, "\pi_!"] & \mathrm{St}_{\mathcal{J}^{\mathrm{set}},\mathcal{E}} \end{tikzcd}$$

supplied by Theorem 10.2.1. If all the above cocartesian fibrations in posets are $\mathcal{E}$-bireflexive, then the square is a pullback in $\mathbf{Pr}^{\mathrm{L,R}}$.

Proof. The ∞-categorical reflection theorem of [42, Theorem 1.1] implies that in this case all the ∞-categories of Stokes functors appearing in the above square are presentable. Then the conclusion follows combining Lemma 8.4.9 with Lemma 8.4.10. □

8.5. **Van Kampen for Stokes functors.** In Proposition 7.4.10 we proved a Van Kampen result for cocartesian functors. We now show that the same holds for Stokes functors:

Proposition 8.5.1 (Van Kampen for Stokes functors)**.** *Let $\mathcal{X}_\bullet\colon I \to \mathbf{Cat}_\infty$ be a diagram with colimit $\mathcal{X}$. Let $p\colon \mathcal{I} \to \mathcal{X}$ be a cocartesian fibration in posets and set*

$$\mathcal{I}_\bullet \coloneqq \mathcal{X}_\bullet \times_{\mathcal{X}} \mathcal{I}\colon I \to \mathbf{Cat}_\infty \ .$$

Let $\mathcal{E}$ be a presentable ∞-category. Then the equivalence of Lemma 7.4.1 restricts to an equivalence

$$\mathrm{St}_{\mathcal{I},\mathcal{E}} \simeq \lim_{i\in I} \mathrm{St}_{\mathcal{I}_i,\mathcal{E}} \ .$$

Proof. Using Corollary 8.3.4 in place of Proposition 7.3.1, we see that the natural map

$$\mathrm{Fun}(\mathcal{I},\mathcal{E}) \to \lim_{i\in I} \mathrm{Fun}(\mathcal{I}_i,\mathcal{E})$$

gives rise to the following commutative square:

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{I},\mathcal{E}) \arrow[r] & \lim_{i\in I} \mathrm{Fun}(\mathcal{I}_i,\mathcal{E}) \\ \mathrm{St}_{\mathcal{I},\mathcal{E}} \arrow[r] \arrow[u, hook] & \lim_{i\in I} \mathrm{St}_{\mathcal{I}_i,\mathcal{E}} \arrow[u, hook] \end{tikzcd} \tag{8.5.2}$$

It follows from Lemma 7.4.1 that the top horizontal functor is an equivalence. Thus, the bottom horizontal one is fully faithful. To conclude the proof, it is enough to prove that a functor $F\colon \mathcal{I} \to \mathcal{E}$ is Stokes if and only if for every $i \in I$, its image in $\mathrm{Fun}(\mathcal{I}_i, \mathcal{E})$ is Stokes. The "only if" follows from Corollary 8.3.4. For the converse, we have already shown in Proposition 7.4.10 that if each restriction of F is cocartesian then F was cocartesian to begin with. We are left to check that F is punctually split. Combining Corollary 7.2.6 and Lemma 7.4.9, we see that F is punctually split if and only if it is split at every object of $\mathcal{X}$ lying in the image of some structural map $f_i\colon \mathcal{X}_i \to \mathcal{X}$. However, if $x \in \mathcal{X}$ is in the image of f_i, then F is split at x thanks to Corollary 8.3.3. □

As a consequence of Van Kampen for Stokes functors, we can prove:

Corollary 8.5.3. *In the situation of Proposition 8.5.1, if furthermore $\mathcal{E}$ is stable and if $\mathcal{I}_i \to \mathcal{X}$ is $\mathcal{E}$-bireflexive for every $i \in I$, then $\mathcal{I} \to \mathcal{X}$ is $\mathcal{E}$-bireflexive and the limit of Proposition 8.5.1 can be computed inside $\mathbf{Pr}^{\mathrm{L,R}}$.*

Proof. Let $f\colon i \to j$ be a morphism in I. Since $\mathcal{I}_i$ and $\mathcal{I}_j$ are $\mathcal{E}$-bireflexive, it follows that $\mathrm{St}_{\mathcal{I}_i,\mathcal{E}}$ and $\mathrm{St}_{\mathcal{I}_j,\mathcal{E}}$ are presentable and that the transition functor

$$f^*\colon \mathrm{St}_{\mathcal{I}_j,\mathcal{E}} \to \mathrm{St}_{\mathcal{I}_i,\mathcal{E}}$$

commute with limits and colimits. Therefore, it admits both a left and a right adjoint. In particular, the diagram $\mathrm{St}_{\mathcal{I}_\bullet,\mathcal{E}}$ factors through $\mathbf{Pr}^{\mathrm{L,R}}$. Since limits in $\mathbf{Pr}^{\mathrm{L}}$ can be computed in $\mathbf{Cat}_\infty$, Proposition 8.5.1 implies that $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ is presentable and stable. Besides, since all transition maps in $\mathrm{St}_{\mathcal{I}_\bullet,\mathcal{E}}$ commute with limits, it automatically follows that the structural functors

$$\mathrm{St}_{\mathcal{I},\mathcal{E}} \to \mathrm{St}_{\mathcal{I}_i,\mathcal{E}}$$

commute with limits as well. Thus, $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ is closed under limits inside $\mathrm{Fun}(\mathcal{I}, \mathcal{E})$. On the other hand, Lemma 8.4.2 shows that $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ is closed under arbitrary coproducts inside $\mathrm{Fun}(\mathcal{I}, \mathcal{E})$. Since $\mathcal{E}$ is stable and we already showed that $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ is stable, closure under finite colimits is automatic. The conclusion follows. □

8.6. **Change of coefficients for punctually split and Stokes functors.** Fix a cocartesian fibration in posets $p\colon \mathcal{I} \to \mathcal{X}$ and let $f\colon \mathcal{E} \to \mathcal{E}'$ be a morphism in $\mathbf{Pr}^{\mathrm{L}}$. Recall from Section 6.5 that this induces a transformation

$$f^{\mathcal{I}/\mathcal{X}}\colon\ \exp_{\mathcal{E}}(\mathcal{I}/\mathcal{X}) \to \exp_{\mathcal{E}'}(\mathcal{I}/\mathcal{X})$$

in $\mathbf{PrFib}^{\mathrm{L}}$. We have:

Proposition 8.6.1. *The transformation $f^{\mathcal{I}/\mathcal{X}}$ respects the punctually split sub-cocartesian fibrations, and thus it induces a functor*

$$f^{\mathcal{I}/\mathcal{X}}\colon\ \exp^{\mathrm{PS}}_{\mathcal{E}}(\mathcal{I}/\mathcal{X}) \to \exp^{\mathrm{PS}}_{\mathcal{E}'}(\mathcal{I}/\mathcal{X})$$

in $\mathbf{PrFib}^{\mathrm{L}}_{\mathcal{X}}$. In particular, the induced functor

$$f\colon\ \mathrm{Fun}(\mathcal{I}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{I}, \mathcal{E}')$$

induces well defined functors

$$f\colon\ \mathrm{Fun}_{\mathrm{PS}}(\mathcal{I}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{I}, \mathcal{E}') \qquad \textit{and} \qquad f\colon \mathrm{St}_{\mathcal{I},\mathcal{E}} \to \mathrm{St}_{\mathcal{I},\mathcal{E}'}\ .$$

Proof. Since f commutes with colimits, it commutes with the formation of left Kan extensions. This immediately implies the first statement. Applying $\Sigma_{\mathcal{X}}$ and using Proposition 6.5.1, we deduce that $f\colon\ \mathrm{Fun}(\mathcal{I}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{I}, \mathcal{E}')$ preserves punctually split functors. In turn, this fact and Proposition 7.5.1 implies that f also preserves Stokes functors. □

Cocartesian functors exhibit a nice behavior with respect to the tensor product in $\mathbf{Pr}^{\mathrm{L}}$ (see Corollary 7.5.7). On the other hand $\exp_{\mathcal{E}}^{\mathrm{PS}}(\mathcal{I}/\mathcal{X})$ is typically not a presentable fibration, and $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ is typically not presentable. This prevents from formally deducing an analogue of Corollary 7.5.7 for Stokes functors: such a result will be true, but only in a more restrictive geometric setting, see Theorem 12.2.6. For the moment, let us simply collect a couple of elementary observations that will be needed later.

When bireflexivity holds, we can construct, for every pair of presentable ∞-categories $\mathcal{E}$ and $\mathcal{E}'$, a canonical comparison morphism $\mathrm{St}_{\mathcal{I},\mathcal{E}} \otimes \mathcal{E}' \to \mathrm{St}_{\mathcal{I},\mathcal{E}\otimes\mathcal{E}'}$. The key point is the following lemma:

Lemma 8.6.2. *Let $p\colon \mathcal{I} \to \mathcal{X}$ be a cocartesian fibration in posets. Let $\mathcal{E}$ and $\mathcal{E}'$ be presentable ∞-categories. Let $x \in \mathcal{X}$ be an object and let $F\colon \mathcal{I} \to \mathcal{E}$ be a functor that splits at x. Then for every object $E' \in \mathcal{E}'$, the functor*

$$F \otimes E' \in \mathrm{Fun}(\mathcal{I}, \mathcal{E}) \otimes \mathcal{E}' \simeq \mathrm{Fun}(\mathcal{I}, \mathcal{E} \otimes \mathcal{E}')$$

splits at x as well.

Proof. Let $i_x\colon \mathcal{I}_x \to \mathcal{I}$ be the canonical functor. For any presentable ∞-category $\mathcal{D}$, both functors

$$i_x^*\colon \mathrm{Fun}(\mathcal{I}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{I}_x, \mathcal{D}) \qquad \text{and} \qquad i_{\mathcal{I}_x,!}\colon \mathrm{Fun}(\mathcal{I}_x^{\mathrm{set}}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{I}_x, \mathcal{D})$$

commute with colimits, so we obtain the following canonical identifications:

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{I}, \mathbf{Spc}) \otimes \mathcal{D} \arrow[r, "i_x^*\otimes\mathcal{D}"] \arrow[d, "\wr"] & \mathrm{Fun}(\mathcal{I}_x, \mathbf{Spc}) \otimes \mathcal{D} \arrow[d, "\wr"] & \mathrm{Fun}(\mathcal{I}_x^{\mathrm{set}}, \mathbf{Spc}) \otimes \mathcal{D} \arrow[l, "i_{\mathcal{I}_x,!}\otimes\mathcal{D}"'] \arrow[d, "\wr"] \\ \mathrm{Fun}(\mathcal{I}, \mathcal{D}) \arrow[r, "i_x^*"] & \mathrm{Fun}(\mathcal{I}_x, \mathcal{D}) & \mathrm{Fun}(\mathcal{I}_x^{\mathrm{set}}, \mathcal{D}) \arrow[l, "i_{\mathcal{I}_x,!}"'] . \end{tikzcd}$$

This proves the lemma when $\mathcal{E} = \mathbf{Spc}$, and the general case follows from the associativity of the tensor product in $\mathbf{Pr}^{\mathrm{L}}$. $\square$

Construction 8.6.3. Let $p\colon \mathcal{I} \to \mathcal{X}$ be a cocartesian fibration in posets. Let $\mathcal{E}$ and $\mathcal{E}'$ be presentable ∞-categories such that $p\colon \mathcal{I} \to \mathcal{X}$ is $\{\mathcal{E}, \mathcal{E}'\}$-bireflexive. Consider the following solid commutative diagram:

$$\begin{tikzcd} \mathrm{St}_{\mathcal{I},\mathcal{E}} \otimes \mathcal{E}' \arrow[r, dashed] \arrow[d] & \mathrm{St}_{\mathcal{I},\mathcal{E}\otimes\mathcal{E}'} \arrow[d, hook] \\ \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{I}, \mathcal{E}) \otimes \mathcal{E}' \arrow[r, "\sim"] \arrow[d] & \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{I}, \mathcal{E} \otimes \mathcal{E}') \arrow[d, hook] \\ \mathrm{Fun}(\mathcal{I}, \mathcal{E}) \otimes \mathcal{E}' \arrow[r, "\sim"] & \mathrm{Fun}(\mathcal{I}, \mathcal{E} \otimes \mathcal{E}') \end{tikzcd} \tag{8.6.4}$$

in $\mathbf{Pr}^{\mathrm{L}}$. By definition, $\mathrm{St}_{\mathcal{I},\mathcal{E}} \otimes \mathcal{E}'$ is generated under colimits by objects of the form $F \otimes E'$, where $F\colon \mathcal{I} \to \mathcal{E}$ is a Stokes functor and $E' \in \mathcal{E}'$ is an object. Lemma 8.6.2 guarantees that such an object is mapped via the bottom horizontal equivalence into an object in $\mathrm{St}_{\mathcal{I},\mathcal{E}\otimes\mathcal{E}'}$. Since the right vertical arrows are fully faithful by definition, it follows that the dashed arrow exist.

Proposition 8.6.5. *Let $p\colon \mathcal{I} \to \mathcal{X}$ be a cocartesian fibration in posets. Let $\mathcal{E}$ and $\mathcal{E}'$ be presentable ∞-categories. Assume that:*

(1) The fibers of $\mathcal{I}$ are finite;

(2) $p\colon \mathcal{I} \to \mathcal{X}$ is $\mathcal{E}, \mathcal{E}'$-bireflexive.

Then the canonical comparison functor

$$\mathrm{St}_{\mathcal{I},\mathcal{E}} \otimes \mathcal{E}' \to \mathrm{St}_{\mathcal{I},\mathcal{E}\otimes\mathcal{E}'}$$

of Construction 8.6.3 is fully faithful.

Proof. Since the fibers of $\mathcal{I}$ are finite, Proposition 7.2.14 implies that $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{I},\mathcal{E})$ is closed under limits and colimits in $\mathrm{Fun}(\mathcal{I},\mathcal{E})$. By (2), it follows that $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ is closed under limits and colimits in $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{I},\mathcal{E})$ as well. Since in this situation $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ is presentable by Lemma 8.4.6, it follows that the inclusion of $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ into $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{I},\mathcal{E})$ has both a left and a right adjoint. Therefore, the functoriality of the tensor product in $\mathbf{Pr}^{\mathrm{L}}$ implies that the top left vertical arrow in the diagram (8.6.4) is fully faithful. On the other hand, the middle horizontal functor is an equivalence thanks to Corollary 7.5.7, so the conclusion follows. □

Definition 8.6.6. Let $p\colon \mathcal{I} \to \mathcal{X}$ be an object of **PosFib** and let $\mathcal{C} \subset \mathbf{Pr}^{\mathrm{L}}$ be a full subcategory stable under tensor product. We say that $p\colon \mathcal{I} \to \mathcal{X}$ is $\mathcal{C}$-*universal* if it is $\mathcal{C}$-bireflexive and the comparison map $\mathrm{St}_{\mathcal{I},\mathcal{E}} \otimes \mathcal{E}' \to \mathrm{St}_{\mathcal{I},\mathcal{E}\otimes\mathcal{E}'}$ of Construction 8.6.3 is an equivalence for every$\mathcal{E},\mathcal{E}' \in \mathcal{C}$. If $\mathcal{C} \subset \mathbf{Pr}^{\mathrm{L}}$ is the collection of all presentable stable ∞-categories, we simply say that $p\colon \mathcal{I} \to \mathcal{X}$ is stably universal.

Proposition 8.6.7. *Let $\mathcal{X}_\bullet\colon I \to \mathbf{Cat}_\infty$ be a diagram with colimit $\mathcal{X}$. Let $p\colon \mathcal{I} \to \mathcal{X}$ be a cocartesian fibration in posets and set*

$$\mathcal{I}_\bullet \coloneqq \mathcal{X}_\bullet \times_{\mathcal{X}} \mathcal{I}\colon I \to \mathbf{Cat}_\infty \ .$$

Assume that $\mathcal{I}_i \to \mathcal{X}$ is stably universal for every $i \in I$. Then $\mathcal{I} \to \mathcal{X}$ is stably universal.

Proof. Note that $\mathcal{I} \to \mathcal{X}$ is stably bireflexive by Corollary 8.5.3. For every presentable stable ∞-categories $\mathcal{E},\mathcal{E}'$, we have

$$\begin{aligned} \mathrm{St}_{\mathcal{I},\mathcal{E}} \otimes \mathcal{E}' &\simeq (\lim_{i\in I} \mathrm{St}_{\mathcal{I}_i,\mathcal{E}}) \otimes \mathcal{E}' && \text{Corollary 8.5.3} \\ &\simeq \lim_{i\in I} (\mathrm{St}_{\mathcal{I}_i,\mathcal{E}} \otimes \mathcal{E}') && \text{Lemma 7.5.5} \\ &\simeq \lim_{i\in I} \mathrm{St}_{\mathcal{I}_i,\mathcal{E}\otimes\mathcal{E}'} && \\ &\simeq \mathrm{St}_{\mathcal{I},\mathcal{E}\otimes\mathcal{E}'} && \text{Corollary 8.5.3} \end{aligned}$$

□

8.7. **Induced t-structures for Stokes functors.** Fix a presentable stable ∞-category $\mathcal{E}$ equipped with an induced t-structure $\tau = (\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$. In Section 7.9 we showed that cocartesian functors inherits a t-structure from τ, and we analyzed the basic properties. We now investigate the behavior with respect to Stokes functor.

We start with a couple of general facts concerning t-structures.

Construction 8.7.1. Let $\mathcal{C}$ and $\mathcal{D}$ be stable ∞-categories equipped with t-structures $\tau^{\mathcal{C}} = (\mathcal{C}_{\leqslant 0}, \mathcal{C}_{\geqslant 0})$ and $\tau^{\mathcal{D}} = (\mathcal{D}_{\leqslant 0}, \mathcal{D}_{\geqslant 0})$ and let

$$F\colon \mathcal{C} \to \mathcal{D}$$

be a right t-exact stable functor. For every object $C \in \mathcal{C}$, one has $F(\tau^{\mathcal{C}}_{\geqslant 0}(C)) \in \mathcal{D}_{\geqslant 0}$, and therefore the mapping space

$$\mathrm{Map}_{\mathcal{D}}\left(F(\tau^{\mathcal{C}}_{\geqslant 0}(C)), \tau^{\mathcal{D}}_{\leqslant -1}(F(C))\right)$$

is contractible. It follows that there exists the dashed morphisms making the diagram

$$\begin{tikzcd} F\big(\tau^{\mathcal{C}}_{\geqslant 0}(C)\big) \arrow[r] \arrow[d, dashed] & F(C) \arrow[r] \arrow[d, equal] & F\big(\tau^{\mathcal{C}}_{\leqslant -1}(C)\big) \arrow[d, dashed] \\ \tau^{\mathcal{D}}_{\geqslant 0}\big(F(C)\big) \arrow[r] & F(C) \arrow[r] & \tau^{\mathcal{D}}_{\leqslant -1}\big(F(C)\big) \end{tikzcd} \tag{8.7.2}$$

commutative.

Lemma 8.7.3. *In the situation of Construction 8.7.1, let $C \in \mathcal{C}$ be an object. If $F(\tau^{\mathcal{C}}_{\leqslant -1}(C)) \in \mathcal{D}_{\leqslant -1}$ then both canonical comparison maps*

$$F\big(\tau^{\mathcal{C}}_{\geqslant 0}(C)\big) \to \tau^{\mathcal{D}}_{\geqslant 0}\big(F(C)\big) \qquad \text{and} \qquad F\big(\tau^{\mathcal{C}}_{\leqslant -1}(C)\big) \to \tau^{\mathcal{D}}_{\leqslant -1}\big(F(C)\big)$$

are equivalences.

Proof. Since F is a stable functor, the top row of (8.7.2) is a fiber sequence in $\mathcal{D}$. By definition of t-structure, the same holds true for the bottom row. Set

$$K \coloneqq \mathrm{fib}\big(F\big(\tau^{\mathcal{C}}_{\geqslant 0}(C)\big) \to \tau^{\mathcal{D}}_{\geqslant 0}\big(F(C)\big)\big) \qquad \text{and} \qquad K' \coloneqq \mathrm{fib}\big(F\big(\tau^{\mathcal{C}}_{\leqslant -1}(C)\big) \to \tau^{\mathcal{D}}_{\leqslant -1}\big(F(C)\big)\big) \ .$$

We therefore obtain a fiber sequence

$$K \to 0 \to K' \ ,$$

which implies $K' \simeq K[1]$. Observe now that $K' \in \mathcal{D}_{\leqslant -1}$. At the same time,

$$K[1] \simeq \mathrm{cofib}\,\big(F\big(\tau^{\mathcal{C}}_{\geqslant 0}(C)\big) \to \tau^{\mathcal{D}}_{\geqslant 0}\big(F(C)\big)\big) \in \mathcal{D}_{\geqslant 0} \ .$$

Thus, it follows that $K' \in \mathcal{D}_{\geqslant 0} \cap \mathcal{D}_{\leqslant -1} = \{0\}$. Thus, both K and K' are zero, which implies that the comparison morphisms are equivalences. □

We now start analyzing the behavior of the standard t-structure on Stokes functors.

Recollection 8.7.4. Let $f \colon \mathcal{A} \to \mathcal{B}$ be a functor of ∞-categories. Then

$$f^* \colon \mathrm{Fun}(\mathcal{B}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{A}, \mathcal{E})$$

is t-exact with respect to the standard t-structures. In particular, $f_!$ is right t-exact and f_* is left t-exact.

Lemma 8.7.5. *Let $f \colon \mathcal{I} \to \mathcal{J}$ be a morphism of posets, where $\mathcal{I}$ is discrete and finite. Then*

$$f_! \colon \mathrm{Fun}(\mathcal{I}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{J}, \mathcal{E})$$

is t-exact.

Proof. Fix a functor $F \colon \mathcal{I} \to \mathcal{E}$ and an object $b \in \mathcal{J}$. By definition

$$f_!(F)_b \simeq \bigoplus_{f(a) \leq b} F_a \ ,$$

so the conclusion follows from the fact that both $\mathcal{E}_{\geqslant 0}$ and $\mathcal{E}_{\leqslant 0}$ are closed under finite sums. □

Corollary 8.7.6. *Let $\mathcal{I}$ be a finite poset and let $F \colon \mathcal{I} \to \mathcal{E}$ be a functor. If F is split, then so are $\tau_{\leqslant n}(F)$ and $\tau_{\geqslant n}(F)$ for every $n \in \mathbb{Z}$.*

Proof. It suffices to treat the case $n = 0$. Choose a functor $V \colon \mathcal{I}^{\mathrm{set}} \to \mathcal{E}$ with an equivalence $F \simeq i_{\mathcal{I},!}(V)$. Since $\mathcal{I}$ is finite, Lemma 8.7.5 implies that

$$\tau_{\leqslant 0}(i_{\mathcal{I},!}(V)) \simeq i_{\mathcal{I},!}(\tau_{\leqslant 0}(V)) \qquad \text{and} \qquad \tau_{\geqslant 0}(i_{\mathcal{I},!}(V)) \simeq i_{\mathcal{I},!}(\tau_{\geqslant 0}(V)) \ ,$$

whence the conclusion. □

Lemma 8.7.7. *Let $\mathcal{I}$ be a poset and let $F\colon \mathcal{I} \to \mathcal{E}$ be a functor. Let $(V, \phi\colon i_{\mathcal{I},!}(V) \simeq F)$ be a splitting for F. Let $n \in \mathbb{Z}$ be an integer. If F takes values in $\mathcal{E}_{\geqslant n}$ (resp. $\mathcal{E}_{\leqslant n}$), then the same goes for V.*

Proof. It suffices to consider the case $n = 0$. For $a \in \mathcal{I}$, ϕ induces

$$F_a \simeq \bigoplus_{b \leqslant a} V_b \ .$$

In particular, V_a is a retract of F_a. Since $F_a \in \mathcal{E}_{\geqslant 0}$ (resp. $F_a \in \mathcal{E}_{\leqslant 0}$), it follows that $V_a \in \mathcal{E}_{\geqslant 0}$ (resp. $V_a \in \mathcal{E}_{\leqslant 0}$) as well. □

Corollary 8.7.8. *Let $f\colon \mathcal{I} \to \mathcal{J}$ be a morphism of finite posets. Let $F\colon \mathcal{I} \to \mathcal{E}$ be a split functor and let $n \in \mathbb{Z}$ be an integer. If F takes values in $\mathcal{E}_{\leqslant n}$, then so does $f_!(F)$;*

Proof. It suffices to consider the case $n = 0$. Since F is split, we can find a functor $V\colon \mathcal{I}^{\mathrm{set}} \to \mathcal{E}$ and an equivalence $\phi\colon F \simeq i_{\mathcal{I},!}(V)$. Lemma 8.7.7 guarantees that V takes values in $\mathcal{E}_{\leqslant 0}$. Thus, we find

$$f_!(F) \simeq f_!(i_{\mathcal{I},!}(V)) \simeq i_{\mathcal{J},!}(f_!^{\mathrm{set}}(V)) \ ,$$

and the conclusion now follows from Lemma 8.7.5 applied to $i_{\mathcal{J}} \circ f^{\mathrm{set}}\colon \mathcal{I}^{\mathrm{set}} \to \mathcal{J}$. □

Notation 8.7.9. Given an ∞-category $\mathcal{A}$, we denote again by

$$\tau_{\geqslant n}\colon \mathrm{Fun}(\mathcal{A}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{A}, \mathcal{E}_{\geqslant n}) \qquad \text{and} \qquad \tau_{\leqslant n}\colon \mathrm{Fun}(\mathcal{A}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{A}, \mathcal{E}_{\leqslant n})$$

the induced truncation functors, given respectively by the compositions

$$\tau_{\geqslant n}(F) \coloneqq \tau_{\geqslant n} \circ F \qquad \text{and} \qquad \tau_{\leqslant n}(F) \coloneqq \tau_{\leqslant n} \circ F \ .$$

Lemma 8.7.10. *Let $f\colon \mathcal{I} \to \mathcal{J}$ be a morphism of finite posets and let $F\colon \mathcal{I} \to \mathcal{E}$ be a split functor. Then for every integer n, the canonical maps of Construction 8.7.1*

$$f_!(\tau_{\geqslant n}(F)) \to \tau_{\geqslant n}(f_!(F)) \qquad \text{and} \qquad f_!(\tau_{\leqslant n}(F)) \to \tau_{\leqslant n}(f_!(F))$$

are equivalences.

Proof. It suffices to consider the case $n = 0$. Since F is split, Corollary 8.7.6 guarantees that $\tau_{\leqslant -1}(F)$ is again split and takes values in $\mathcal{E}_{\leqslant -1}$. Therefore, Corollary 8.7.8 implies that $f_!(\tau_{\leqslant -1}(F))$ takes values in $\mathcal{E}_{\leqslant -1}$. At this point, the conclusion follows from Lemma 8.7.3. □

Proposition 8.7.11. *Let $p\colon \mathcal{I} \to \mathcal{X}$ be a cocartesian fibration in finite posets and let $\mathcal{E}$ be a stable presentable ∞-category equipped with an accessible t-structure $\tau = (\mathcal{E}_{\leqslant 0}, \mathcal{E}_{\geqslant 0})$. If $F\colon \mathcal{I} \to \mathcal{E}$ is a Stokes functor then for every integer $n \in \mathbb{Z}$, both $\tau_{\leqslant n}(F)$ and $\tau_{\geqslant n}(F)$ are again Stokes functors. In particular, $p\colon \mathcal{I} \to \mathcal{X}$ is $\mathcal{E}$-bireflexive, then $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ acquires a unique accessible t-structure such that the inclusion*

$$\mathrm{St}_{\mathcal{I},\mathcal{E}} \hookrightarrow \mathrm{Fun}(\mathcal{I}, \mathcal{E})$$

is t-exact. If in addition τ is compatible with filtered colimits, the same goes for the induced t-structure on $\mathrm{St}_{\mathcal{I},\mathcal{E}}$.

Proof. We know from Lemma 8.4.6 that $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ is presentable and stable. Since $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ is closed under limits and colimits in $\mathrm{Fun}(\mathcal{I}, \mathcal{E})$, the first half of the statement implies the existence of the desired t-structure, its accessibility and its compatibility with filtered colimits.

Let us therefore prove the first part. It suffices to consider the case $n = 0$. Let $F\colon \mathcal{I} \to \mathcal{E}$ be a Stokes functor. We first prove that $\tau_{\geqslant 0}(F)$ and $\tau_{\leqslant 0}(F)$ are punctually split. Fix an object $x \in \mathcal{X}$. Since $j_x^*\colon \mathrm{Fun}(\mathcal{I}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{I}_x, \mathcal{E})$ is t-exact, we find canonical equivalences

$$j_x^*(\tau_{\geqslant 0}(F)) \simeq \tau_{\geqslant 0}(j_x^*(F)) \qquad \text{and} \qquad j_x^*(\tau_{\leqslant 0}(F)) \simeq \tau_{\leqslant 0}(j_x^*(F)) \ .$$

Since $j_x^*(F)$ is split by assumption, Corollary 8.7.6 implies that the same goes for $\tau_{\leqslant 0}(j_x^*(F))$ and $\tau_{\geqslant 0}(j_x^*(F))$ as well, which proves the first claim.

We now prove that $\tau_{\geqslant 0}(F)$ and $\tau_{\leqslant 0}(F)$ are cocartesian. Let $\gamma\colon x \to y$ be a morphism in $\mathcal{X}$ and let $f_\gamma\colon \mathcal{I}_x \to \mathcal{I}_y$ be any straightening for $\mathcal{I}_\gamma \to \Delta^1$. By Lemma 8.7.10, the canonical comparison maps

$$f_{\gamma,!}(\tau_{\geqslant 0}(j_x^*(F))) \to \tau_{\geqslant 0}(f_{\gamma,!}(j_x^*(F))) \qquad \text{and} \qquad f_{\gamma,!}(\tau_{\leqslant 0}(j_x^*(F))) \to \tau_{\leqslant 0}(f_{\gamma,!}(j_x^*(F)))$$

are equivalences. Since F is cocartesian, the canonical map

$$f_{\gamma,!}(j_x^*(F)) \to j_y^*(F)$$

is an equivalence. The conclusion now follows from the t-exactness of both j_x^* and j_y^*. $\square$

Corollary 8.7.12. *In the setting of Proposition 8.7.11, one has a canonical equivalence:*

$$\mathrm{St}_{\mathcal{I},\mathcal{E}}^{\heartsuit} \simeq \mathrm{St}_{\mathcal{I},\mathcal{E}^{\heartsuit}} \ .$$

Proof. By definition of the standard t-structure on $\mathrm{Fun}(\mathcal{I},\mathcal{E})$, we have $\mathrm{Fun}(\mathcal{I},\mathcal{E})^{\heartsuit} \simeq \mathrm{Fun}(\mathcal{I},\mathcal{E}^{\heartsuit})$. Proposition 8.7.11 guarantees that a Stokes functor is connective (resp. coconnective) if and only if it is connective (resp. coconnective) as an object in $\mathrm{Fun}(\mathcal{I},\mathcal{E})$, so the conclusion follows. $\square$

Recollection 8.7.13. If $\mathcal{A}$ is a Grothendieck abelian category, we denote by $\mathsf{D}(\mathcal{A})$ the derived ∞-category of $\mathcal{A}$ (see [32, Definition 1.3.5.8]). By [32, Propositions 1.3.5.9 & 1.3.5.21] we see that $\mathsf{D}(\mathcal{A})$ is a presentable stable ∞-category equipped with an accessible t-structure $\tau = (\mathsf{D}(\mathcal{A})_{\geqslant 0}, \mathsf{D}(\mathcal{A})_{\leqslant 0})$ compatible with filtered colimits and such that $\mathcal{A} \simeq \mathsf{D}(\mathcal{A})^{\heartsuit}$.

Corollary 8.7.14. *Let $p\colon \mathcal{I} \to \mathcal{X}$ be an object of* **PosFib** *and let $\mathcal{A}$ be a Grothendieck abelian category such that $p\colon \mathcal{I} \to \mathcal{X}$ is $\mathsf{D}(\mathcal{A})$-bireflexive. Then $\mathrm{St}_{\mathcal{I},\mathcal{A}}$ is a Grothendieck abelian category.*

Proof. Lemma 8.4.6 implies that $\mathrm{St}_{\mathcal{I},\mathsf{D}(\mathcal{A})}$ is presentable and stable, while Proposition 8.7.11 guarantees that τ induces an accessible t-structure on $\mathrm{St}_{\mathcal{I},\mathsf{D}(\mathcal{A})}$ which is compatible with filtered colimits and such that the inclusion

$$\mathrm{St}_{\mathcal{I},\mathsf{D}(\mathcal{A})} \hookrightarrow \mathrm{Fun}(\mathcal{I},\mathsf{D}(\mathcal{A}))$$

is t-exact. Moreover, Corollary 8.7.12 and Recollection 8.7.13 imply that

$$\mathrm{St}_{\mathcal{I},\mathsf{D}(\mathcal{A})}^{\heartsuit} \simeq \mathrm{St}_{\mathcal{I},\mathcal{A}} \ .$$

Thus, it follows that $\mathrm{St}_{\mathcal{I},\mathcal{A}}$ is a Grothendieck abelian category. $\square$

Corollary 8.7.15. *Let $\mathcal{X}$ be an ∞-category and let $f\colon \mathcal{I} \to \mathcal{J}$ be a morphism between cocartesian fibrations in finite posets over $\mathcal{X}$. If $\mathcal{I} \to \mathcal{X}$ and $\mathcal{J} \to \mathcal{X}$ are $\mathcal{E}$-bireflexive, then the functor*

$$f_!\colon \mathrm{St}_{\mathcal{I},\mathcal{E}} \to \mathrm{St}_{\mathcal{J},\mathcal{E}}$$

is t-exact.

Proof. It follows from Proposition 8.7.11 and Recollection 8.7.4 that $f_!$ is right t-exact. Let $F \in (\mathrm{St}_{\mathcal{I},\mathcal{E}})_{\leqslant 0}$. We have to prove that $f_!(F)$ takes values in $\mathcal{E}_{\leqslant 0}$. Combining Corollaries 6.1.6 and 8.3.4, we can reduce ourselves to the case where $\mathcal{X}$ is reduced to a point, where the result follows from Corollary 8.7.8. $\square$

Remark 8.7.16. The inclusion $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{I},\mathcal{E}) \hookrightarrow \mathrm{Fun}(\mathcal{I},\mathcal{E})$ is typically not left t-exact and in general

$$\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{I},\mathcal{E})^{\heartsuit} \not\simeq \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{I},\mathcal{E}^{\heartsuit}) \ ,$$

as Example 7.9.5 shows. Notice however that the functor F considered there is not punctually split at 0. Similarly, if $f\colon \mathcal{I} \to \mathcal{J}$ is a morphism between cocartesian fibrations in finite posets, neither

$$f_!\colon \operatorname{Fun}(\mathcal{I}, \mathcal{E}) \to \operatorname{Fun}(\mathcal{J}, \mathcal{E})$$

nor its cocartesian variant are left t-exact. However, it becomes left t-exact once restricted to $\operatorname{St}_{\mathcal{I},\mathcal{E}}$, thanks to Corollary 8.7.15.

Corollary 8.7.17. *Let $\mathcal{I} \to \mathcal{X}$ be a cocartesian fibration in posets. If $\mathcal{X}$ admits an initial object x, then a Stokes functor $F\colon \mathcal{I} \to \mathcal{E}$ takes values in $\mathcal{E}^{\heartsuit}$ if and only if $j_x^*(F)\colon \mathcal{I}_x \to \mathcal{E}$ takes values in $\mathcal{E}^{\heartsuit}$.*

Proof. The "only if" direction simply follows from the t-exactness of $j_x^*\colon \operatorname{Fun}(\mathcal{I}, \mathcal{E}) \to \operatorname{Fun}(\mathcal{I}_x, \mathcal{E})$. For the "if" direction, we equivalently have to show that for every $y \in \mathcal{X}$, the restriction $j_y^*(F)$ takes values in $\mathcal{E}^{\heartsuit}$. Since x is initial in $\mathcal{X}$, we can find a morphism $\gamma\colon x \to y$. Choose any straightening $f_\gamma\colon \mathcal{I}_x \to \mathcal{I}_y$ for $\mathcal{I}_\gamma \to \Delta^1$. Since F is cocartesian, the canonical map

$$f_{\gamma,!}(j_x^*(F)) \to j_y^*(F)$$

is an equivalence. The conclusion now follows from Corollary 8.7.8. □

8.8. **Categorical actions on Stokes functors.** We use the terminology on categorical actions reviewed in Section 20 (see also Section 7.10, of which this section is the continuation). We fix a presentably symmetric monoidal and stable ∞-category $\mathcal{E}^{\otimes}$. In analogy to Proposition 7.10.1, we have:

Proposition 8.8.1. *Let $p\colon \mathcal{I} \to \mathcal{X}$ be a cocartesian fibration in posets. Then for every $L \in \operatorname{Loc}(\mathcal{X}; \mathcal{E})$ and every $G \in \operatorname{Fun}_{\mathrm{PS}}(\mathcal{I}, \mathcal{E})$, the functor*

$$p^*(L) \otimes G\colon \mathcal{I} \to \mathcal{E}$$

is again punctually split. In particular, if G is a Stokes functor, the same goes for $p^*(L) \otimes G$.

Proof. Fix $x \in \mathcal{X}$. Since the restrictions $j_x^*\colon \operatorname{Fun}(\mathcal{I}, \mathcal{E}) \to \operatorname{Fun}(\mathcal{I}_x, \mathcal{E})$ are $\mathcal{E}$-linear, we can assume without loss of generality that $\mathcal{X}$ is reduced at a single point. Choose a splitting $V\colon \mathcal{I}^{\mathrm{set}} \to \mathcal{E}$ for G. Lemma 20.1.3 implies that $i_{\mathcal{I},!}\colon \operatorname{Fun}(\mathcal{I}^{\mathrm{set}}, \mathcal{E}) \to \operatorname{Fun}(\mathcal{I}, \mathcal{E})$ is $\mathcal{E}$-linear. Therefore, for every $L \in \mathcal{E}$ we obtain

$$p^*(L) \otimes G \simeq p^*(L) \otimes i_{\mathcal{I},!}(V) \simeq i_{\mathcal{I},!}(p^{\mathrm{set},*}(L) \otimes V) \ ,$$

i.e. $p^*(L) \otimes G$ is split. □

Corollary 8.8.2. *Let $p\colon \mathcal{I} \to \mathcal{X}$ be a $\mathcal{E}$-bireflexive cocartesian fibration in posets. Then the categorical action of $\operatorname{Loc}(\mathcal{X}; \mathcal{E})$ on $\operatorname{Fun}(\mathcal{I}, \mathcal{E})$ restricts to a categorical action of $\operatorname{Loc}(\mathcal{X}; \mathcal{E})$ on $\operatorname{St}_{\mathcal{I},\mathcal{E}}$.*

Proof. This is obvious from Proposition 8.8.1. □

We now derive an analogue of Corollaries 20.2.8 and 7.10.10 in the setting of Stokes functors. We fix a pullback square

$$\begin{tikzcd} \mathcal{J} \arrow[r, "u"] \arrow[d, "q"] & \mathcal{I} \arrow[d, "p"] \\ \mathcal{Y} \arrow[r, "f"] & \mathcal{X} \end{tikzcd}$$

in $\mathbf{Cat}_\infty$, where p is a cocartesian fibration in posets. In addition, we assume that f is a finite étale fibration (see Definition 19.2.1) and that both $\mathcal{I}$ and $\mathcal{J}$ are $\mathcal{E}$-bireflexive. In this setting, Construction 20.2.1 supplies a canonical transformation

$$\mu\colon \operatorname{Loc}(\mathcal{Y}; \mathcal{E}) \otimes_{\operatorname{Loc}(\mathcal{X};\mathcal{E})} \operatorname{Fun}(\mathcal{I}, \mathcal{E}) \to \operatorname{Fun}(\mathcal{J}, \mathcal{E}) \ ,$$

and Proposition 7.10.1 and Corollary 8.8.2 imply that this action restricts to a well defined categorical action

$$\mu\colon \operatorname{Loc}(\mathcal{Y};\mathcal{E}) \otimes_{\operatorname{Loc}(\mathcal{X};\mathcal{E})} \operatorname{St}_{\mathcal{I},\mathcal{E}} \to \operatorname{St}_{\mathcal{J},\mathcal{E}} \ .$$

Lemma 8.8.3. *In the above setting, the functor*

$$u_!\colon \operatorname{Fun}(\mathcal{J},\mathcal{E}) \to \operatorname{Fun}(\mathcal{I},\mathcal{E})$$

respects Stokes functors and in particular it induces a well defined functor

$$u_!\colon \operatorname{St}_{\mathcal{J},\mathcal{E}} \to \operatorname{St}_{\mathcal{I},\mathcal{E}} \ .$$

Proof. We know from Corollary 7.10.8 that $u_!$ preserves cocartesian functors. It is therefore enough to prove that it preserves punctually split functors as well. Let therefore $F\colon \mathcal{J} \to \mathcal{E}$ be a punctually split functor. Fix $x \in \mathcal{X}$. For every $y \in \mathcal{Y}_x$, we have a splitting

$$V_y\colon \mathcal{J}_y^{\mathrm{set}} \to \mathcal{E}$$

for $j_y^*(F)$. Since f is a finite étale fibration, the same goes for u (see Lemma 19.2.2), and therefore

$$j_x^*(u_!(F)) \simeq \bigoplus_{y\in\mathcal{Y}_x} j_y^*(F) \ .$$

It follows from Lemma 8.4.2 that $\bigoplus_{y\in\mathcal{Y}_x} V_y$ provides a splitting for $j_x^*(u_!(F))$, whence the conclusion. □

Proposition 8.8.4. *In the above setting, the functor*

$$u_!\colon \operatorname{St}_{\mathcal{J},\mathcal{E}} \to \operatorname{St}_{\mathcal{I},\mathcal{E}}$$

is monadic.

Proof. As in the proof of Corollary 7.10.9, Lemmas 8.8.3 and 20.2.5 imply that $u_!$ and u^* are biadjoint. Besides, $u_!\colon \operatorname{Fun}(\mathcal{J},\mathcal{E}) \to \operatorname{Fun}(\mathcal{I},\mathcal{E})$ is conservative thanks to Lemma 20.2.6, so the same holds true for its restriction to the ∞-categories of Stokes functors. □

Corollary 8.8.5. *Let*

$$\begin{tikzcd} \mathcal{J} \arrow[r, "u"] \arrow[d, "q"] & \mathcal{I} \arrow[d, "p"] \\ \mathcal{Y} \arrow[r, "f"] & \mathcal{X} \end{tikzcd}$$

be a pullback square in $\mathbf{Cat}_\infty$*, where* p *is a cocartesian fibration in posets. Let* $\mathcal{E}^\otimes$ *be a presentably symmetric monoidal* ∞*-category. Assume that:*

(1) f *is a finite étale fibration;*

(2) $\mathcal{E}$ *is stable;*

(3) both $\mathcal{I}$ *and* $\mathcal{J}$ *are* $\mathcal{E}$*-bireflexive.*

Then, the comparison functor

$$\mu\colon \operatorname{Loc}(\mathcal{Y};\mathcal{E}) \otimes_{\operatorname{Loc}(\mathcal{X};\mathcal{E})} \operatorname{St}_{\mathcal{I},\mathcal{E}} \to \operatorname{St}_{\mathcal{J},\mathcal{E}}$$

is an equivalence.

Proof. Using Proposition 8.8.4 as input, the same proof of Corollary 20.2.8 applies. □

We conclude this section with the following result, which has been inspired by [4, Lemma 15.5] and that will play an important role later on:

Corollary 8.8.6 (Retraction lemma). *Let*

$$\mathcal{X}_\bullet \colon \mathbf{\Delta}_s^{\mathrm{op}} \to \mathbf{Cat}_\infty$$

be a semi-simplicial diagram with colimit $\mathcal{X}$. Let $\mathcal{I} \to \mathcal{X}$ be a cocartesian fibration in posets and set

$$\mathcal{I}_\bullet \coloneqq \mathcal{X}_\bullet \times_{\mathcal{X}} \mathcal{I} \ .$$

Let $\mathcal{E}^\otimes$ be a presentably symmetric monoidal stable ∞-category. Assume that:

(1) $\mathrm{Env}(\mathcal{X})$ *is compact in* $\mathbf{Spc}$*;*

(2) for every $[n] \in \mathbf{\Delta}_s$, the structural morphism $\mathcal{X}_n \to \mathcal{X}$ is a finite étale fibration;

(3) For every $[n] \in \mathbf{\Delta}_s$, $\mathcal{I}_n$ is $\mathcal{E}$-bireflexive;

Then there exists an integer $m \geqslant 0$ depending only on $\mathrm{Env}(\mathcal{X})$ *such that $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ is a retract of*

$$\lim_{[n]\in \mathbf{\Delta}_{s,\leq m}} \mathrm{St}_{\mathcal{I}_n,\mathcal{E}}$$

in $\mathbf{Pr}^{\mathrm{L}}$.

Proof. To begin with, Corollary 8.5.3 implies that $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ is presentable, stable and closed under limits and colimits in $\mathrm{Fun}(\mathcal{I},\mathcal{E})$ and that besides

$$\mathrm{St}_{\mathcal{I},\mathcal{E}} \simeq \lim_{i\in I} \mathrm{St}_{\mathcal{I}_i,\mathcal{E}} \ ,$$

the limit being computed in $\mathbf{Pr}^{\mathrm{L,R}}$.

For any integer $m \geqslant 0$, set

$$\mathcal{X}^{(m)} \coloneqq \operatorname*{colim}_{[n]\in \mathbf{\Delta}^{\mathrm{op}}_{s,\leq m}} \mathcal{X}_\bullet \ .$$

It automatically follows that

$$\mathcal{X} \simeq \operatorname*{colim}_{m\in \mathbb{N}^{\mathrm{op}}} \mathcal{X}^{(m)} \ ,$$

where the colimit is now filtered. Since the enveloping ∞-groupoid functor $\mathrm{Env} \colon \mathbf{Cat}_\infty \to \mathbf{Spc}$ is a left adjoint, we see that

$$\mathrm{Env}(\mathcal{X}) \simeq \operatorname*{colim}_{m\in \mathbb{N}} \mathrm{Env}(\mathcal{X}^{(m)}) \ .$$

Since $\mathrm{Env}(X)$ is compact, there exists an integer $m \geqslant 0$ such that $\mathrm{Env}(\mathcal{X})$ is a retract of $\mathrm{Env}(\mathcal{X}^{(m)})$. As a consequence, we see that $\mathrm{Loc}(\mathcal{X};\mathcal{E})$ is a retract of $\mathrm{Loc}(\mathcal{X}^{(m)};\mathcal{E})$. In particular,

$$\mathrm{St}_{\mathcal{I},\mathcal{E}} \simeq \mathrm{Loc}(\mathcal{X};\mathcal{E}) \otimes_{\mathrm{Loc}(\mathcal{X};\mathcal{E})} \mathrm{St}_{\mathcal{I},\mathcal{E}}$$

is a retract of

$$\mathrm{Loc}(\mathcal{X}^{(m)};\mathcal{E}) \otimes_{\mathrm{Loc}(\mathcal{X};\mathcal{E})} \mathrm{St}_{\mathcal{I},\mathcal{E}} \simeq \Big(\lim_{[n]\in \mathbf{\Delta}_{s,\leq m}} \mathrm{Loc}(\mathcal{X}_n;\mathcal{E}) \Big) \otimes_{\mathrm{Loc}(\mathcal{X};\mathcal{E})} \mathrm{St}_{\mathcal{I};\mathcal{E}} \ .$$

Notice that the diagram $\mathrm{Loc}(\mathcal{X}_\bullet;\mathcal{E})$ takes values in $\mathbf{Pr}^{\mathrm{L,R}}$. Therefore, Lemma 7.5.5 supplies a canonical equivalence

$$\Big(\lim_{[n]\in \mathbf{\Delta}_{s,\leq m}} \mathrm{Loc}(\mathcal{X}_n;\mathcal{E}) \Big) \otimes_{\mathrm{Loc}(\mathcal{X};\mathcal{E})} \mathrm{St}_{\mathcal{I};\mathcal{E}} \simeq \lim_{[n]\in \mathbf{\Delta}_{s,\leq m}} \mathrm{Loc}(\mathcal{X}_n;\mathcal{E}) \otimes_{\mathrm{Loc}(\mathcal{X};\mathcal{E})} \mathrm{St}_{\mathcal{I};\mathcal{E}} \ .$$

Since each $\mathcal{X}_n \to \mathcal{X}$ is a finite étale fibration, Corollary 8.8.5 supplies a canonical equivalence

$$\mathrm{Loc}(\mathcal{X}_n;\mathcal{E}) \otimes_{\mathrm{Loc}(\mathcal{X};\mathcal{E})} \mathrm{St}_{\mathcal{I};\mathcal{E}} \simeq \mathrm{St}_{\mathcal{I}_n;\mathcal{E}} \ .$$

Thus, the conclusion follows. $\square$

9. Graduation

In this section we keep working with cocartesian fibrations in posets, but we restrict to stable coefficients. This allows to introduce a new fundamental operation for Stokes functors: *graduation*. Intuitively, this allows to break a Stokes functor in more elementary pieces, and it will be the key ingredient needed to develop the theory of level morphisms and level induction.

9.1. **Relative graduation.** Let $\mathcal{X}$ be an ∞-category. Starting with a morphism $p\colon \mathcal{I} \to \mathcal{J}$ in $\mathbf{PosFib}_{\mathcal{X}}$, we can perform the following two constructions:

Construction 9.1.1. Consider the fiber product

$$\begin{tikzcd} \mathcal{I}_p \arrow[r] \arrow[d, "\pi"] & \mathcal{I} \arrow[d, "p"] \\ \mathcal{J}^{\mathrm{set}} \arrow[r, "i_{\mathcal{J}}"] & \mathcal{J} \end{tikzcd}$$

Notice that $\mathcal{I}_p^{\mathrm{set}} \to \mathcal{I}^{\mathrm{set}}$ is an equivalence. When $\mathcal{X}$ is reduced to a point, we can identify $\mathcal{I}_p$ with the poset $(\mathcal{I}, \leq_p)$, where

$$a \leq_p a' \overset{\text{def.}}{\iff} p(a) = p(a') \text{ and } a \leq a' \ .$$

In other words, if $p(a) \neq p(a')$, then a and a' are incomparable with respect to $\leq_p$.

Construction 9.1.2. Let

$$\Upsilon = \Upsilon_{\mathcal{J}} \coloneqq \mathrm{St}^{\mathrm{co}}_{\mathcal{X}}(\mathcal{J})\colon \mathcal{X} \to \mathbf{Cat}_\infty$$

be the straightening of $p_{\mathcal{J}}\colon \mathcal{J} \to \mathcal{X}$. Consider the composition

$$\Upsilon^{\Delta^1} \coloneqq \mathrm{Fun}(\Delta^1, \Upsilon(-))\colon \mathcal{X} \to \mathbf{Cat}_\infty,$$

and write $\mathcal{J}^{\Delta^1} \coloneqq \mathrm{Un}^{\mathrm{co}}_{\mathcal{X}}(\Upsilon^{\Delta^1})$ for the associated cocartesian fibration. The source and identity functors

$$\mathrm{Fun}(\Delta^1, \Upsilon(-)) \to \Upsilon(-) \qquad \text{and} \qquad \Upsilon(-) \to \mathrm{Fun}(\Delta^1, \Upsilon(-))$$

induce morphisms of cocartesian fibration in posets over $\mathcal{X}$

$$s\colon \mathcal{J}^{\Delta^1} \to \mathcal{J} \qquad \text{and} \qquad \mathrm{id}\colon \mathcal{J} \to \mathcal{J}^{\Delta^1}$$

Consider the pullback diagram

$$\begin{tikzcd} \mathcal{I}_{\leq} \arrow[r, "\sigma"] \arrow[d] & \mathcal{I} \arrow[d, "p"] \\ \mathcal{J}^{\Delta^1} \arrow[r, "s"] & \mathcal{J} \ . \end{tikzcd}$$

Objects of $\mathcal{I}_{\leq}$ are triples (x, a, b) where $a \in \mathcal{I}_x$, $b \in \mathcal{J}_x$ and where $p(a) \leq b$ in $\mathcal{J}_x$. We also consider the full subcategory $i_{\mathcal{I},<}\colon \mathcal{I}_< \hookrightarrow \mathcal{I}_{\leq}$ spanned by objects (x, a, b) with $p(a) < b$. When $\mathcal{I}$ is clear from the context, we simply write $i_<$ instead of $i_{\mathcal{I},<}$.

Remark 9.1.3. The target functor $\mathrm{Fun}(\Delta^1, \Upsilon(-)) \to \Upsilon(-)$ induces a morphism of cocartesian fibration in posets $t\colon \mathcal{J}^{\Delta^1} \to \mathcal{J}$. Let $\tau\colon \mathcal{I}_{\leq} \to \mathcal{J}$ be the composition of $\mathcal{I}_{\leq} \to \mathcal{J}^{\Delta^1}$ with $t\colon \mathcal{J}^{\Delta^1} \to \mathcal{J}$. Then, one checks that if $\mathcal{X}$ is a point, the induced functor $(\sigma, \tau)\colon \mathcal{I}_{\leq} \to \mathcal{I} \times \mathcal{J}$ is fully-faithful.

In general, $\mathcal{I}_<$ is no longer a cocartesian fibration. To remedy this, we introduce the following:

Definition 9.1.4. Let $\mathcal{X}$ be an ∞-category. Let $p\colon \mathcal{I} \to \mathcal{J}$ be a morphism in **PosFib** over $\mathcal{X}$. We say that $p\colon \mathcal{I} \to \mathcal{J}$ is a *graduation morphism* if the cocartesian fibration $\mathcal{J}^{\mathrm{set}} \to \mathcal{X}$ is locally constant in the sense of Definition 19.1.4.

The following lemma is simply a matter of unraveling the definitions:

Lemma 9.1.5. *Let $\mathcal{X}$ be an ∞-category. Let $p\colon \mathcal{I} \to \mathcal{J}$ be a graduation morphism over $\mathcal{X}$. Then, $i_<\colon \mathcal{I}_< \to \mathcal{I}_\leq$ is a cocartesian subfibration of $\mathcal{I}_\leq$ over $\mathcal{X}$.*

Consider the following diagram with pull-back squares:

$$\begin{tikzcd} & & \mathcal{I}_< \arrow[d, "i_<"] & \\ \mathcal{I}_p \arrow[rr, "i_p"] \arrow[d] & & \mathcal{I}_\leq \arrow[r, "\sigma"] \arrow[d] & \mathcal{I} \arrow[d, "p"] \\ \mathcal{J}^{\mathrm{set}} \arrow[r, "i_{\mathcal{J}}"] & \mathcal{J} \arrow[r, "\mathrm{id}"] & \mathcal{J}^{\Delta^1} \arrow[r, "s"] & \mathcal{J} . \end{tikzcd} \tag{9.1.6}$$

We fix a presentable stable ∞-category $\mathcal{E}$ and write

$$\varepsilon_<\colon i_{<!}i_<^* \to \mathrm{id}_{\mathrm{Fun}(\mathcal{I}_\leq,\mathcal{E})}$$

for the counit of the adjunction $i_{<!}\colon \mathrm{Fun}(\mathcal{I}_<,\mathcal{E}) \leftrightarrows \mathrm{Fun}(\mathcal{I}_\leq,\mathcal{E})\colon i_<^*$.

Definition 9.1.7. The *graduation functor relative to* $p\colon \mathcal{I} \to \mathcal{J}$ (or *p-graduation functor*)

$$\mathrm{Gr}_p\colon \mathrm{Fun}(\mathcal{I},\mathcal{E}) \to \mathrm{Fun}(\mathcal{I}_p,\mathcal{E})$$

is the cofiber

$$\mathrm{Gr}_p \coloneqq \mathrm{cofib}\left(i_p^*\varepsilon_<\sigma^*\colon i_p^* \circ i_{<!} \circ i_<^* \circ \sigma^* \to i_p^* \circ \sigma^*\right) .$$

Notation 9.1.8. When $p = \mathrm{id}$, we note Gr for $\mathrm{Gr}_{\mathrm{id}}$.

In the following basic example, we recall that if $p\colon \mathcal{I} \to \mathcal{J}$ is a morphism of posets and if $b \in \mathcal{J}$, we put $\mathcal{I}_{\leq b} = \mathcal{I}_{/b} \coloneqq \mathcal{I} \times_{\mathcal{J}} \mathcal{J}_{/b}$. Since $\mathcal{J}$ is a poset, the canonical morphism $\mathcal{J}_{/b} \to \mathcal{J}$ is fully-faithful. Thus, the canonical morphism $\mathcal{I}_{/b} \to \mathcal{I}$ identifies $\mathcal{I}_{/b}$ with the full subcategory of $\mathcal{I}$ spanned by objects $a \in \mathcal{I}$ such that $p(a) \leq b$. Similarly, $\mathcal{I}_{<b} \coloneqq \mathcal{I} \times_{\mathcal{J}} \mathcal{J}_{<b}$ is the full subcategory of $\mathcal{I}$ spanned by objects $a \in \mathcal{I}$ such that $p(a) < b$.

Example 9.1.9. Let $p\colon \mathcal{I} \to \mathcal{J}$ be a morphism of posets. Let $\mathcal{E}$ be a presentable stable ∞-category. Let $V\colon \mathcal{I}^{\mathrm{set}} \to \mathcal{E}$ be a functor and put $F \coloneqq i_{\mathcal{I}!}(V)$. Let $a \in \mathcal{I}_p$. Then, there is a canonical equivalence

$$(\mathrm{Gr}_p(F))_a \simeq \bigoplus_{\substack{a' \leq a \\ p(a')=p(a)}} V_{a'} .$$

Proof. We have

$$(i_p^* \circ i_{<!} \circ i_<^* \circ \sigma^*(F))_a \simeq \operatorname*{colim}_{(a',b)\in(\mathcal{I}_<)_{/(a,p(a))}} F_{a'} \simeq \operatorname*{colim}_{(a',a'',b)\in\mathcal{C}} V_{a''}$$

where $\mathcal{C}$ is the full subcategory of $\mathcal{I} \times \mathcal{J} \times \mathcal{I}^{\mathrm{set}}$ spanned by triples (a', b, a'') such that $a'' \leq a' \leq a$ and $p(a') < b \leq p(a)$. Let $\mathcal{D}$ be the full subcategory of $\mathcal{I} \times \mathcal{I}^{\mathrm{set}}$ spanned by pairs (a', a'') such that $a'' \leq a' \leq a$ and $p(a') < p(a)$. Let $\mathcal{A}$ be the subset of $\mathcal{I}^{\mathrm{set}}$ formed by the a'' such that $a'' \leq a$ and $p(a'') < p(a)$. Consider the commutative diagram

$$\begin{tikzcd} \mathcal{D} \arrow[r, "f"] \arrow[d, "p_2"] & \mathcal{C} \arrow[d, "p_3"] \\ \mathcal{A} \arrow[r] & \mathcal{I}^{\mathrm{set}} \end{tikzcd}$$

where $f\colon \mathcal{D} \to \mathcal{C}$ is given by $(a', a'') \mapsto (a', p(a), a'')$. We claim that f is cofinal. Indeed, for every triple $(a', b, a'') \in \mathcal{C}$, $\mathcal{D}_{(a',b,a'')/} \coloneqq \mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{(a',b,a'')/}$ is the subposet of $\mathcal{D}$ spanned by pairs (α', a'')

with $a' \leq \alpha'$. Observe that (a', a'') is a minimal element of $\mathcal{D}_{(a',b,a'')/}$, which is thus weakly contractible. Hence, the claimed cofinality follows from [31, Theorem 4.1.3.1]. Thus,

$$(i_p^* \circ i_{<!} \circ i_<^* \circ \sigma^*(F))_a \simeq \operatorname*{colim}_{\mathcal{D}} V|_{\mathcal{D}}$$

We also claim that $p_2 \colon \mathcal{D} \to \mathcal{A}$ is cofinal. Indeed, if $a'' \in \mathcal{A}$, $\mathcal{D}_{a''/} := \mathcal{D} \times_{\mathcal{A}} \mathcal{A}_{a''/}$ is the subposet of $\mathcal{D}$ spanned by couples (α', a''). Observe that (a'', a'') is a minimal element of $\mathcal{D}_{a''/}$, which is thus weakly contractible. Hence, the claimed cofinality follows from [31, Theorem 4.1.3.1]. Thus,

$$(i_p^* \circ i_{<!} \circ i_<^* \circ \sigma^*(F))_a \simeq \operatorname*{colim}_{\mathcal{A}} V|_{\mathcal{A}} \simeq \bigoplus_{\substack{a' \leq a \\ p(a') < p(a)}} V_{a'}$$

On the other hand,

$$(i_p^* \circ \sigma^*(F))_a \simeq F_a \simeq \bigoplus_{a' \leq a} V_{a'}$$

Example 9.1.9 thus follows. $\square$

In particular when $p \colon \mathcal{I} \to \mathcal{J}$ is the identity of $\mathcal{I}$, we obtain:

Example 9.1.10. In the setting of Example 9.1.9, let $V \colon \mathcal{I}^{\mathrm{set}} \to \mathcal{E}$ be a functor and put $F := i_{\mathcal{I}!}(V)$. Let $a \in \mathcal{I}^{\mathrm{set}}$. Then, there is a canonical equivalence

$$(\mathrm{Gr}\, F)_a \simeq V_a$$

Example 9.1.11. Let $\mathcal{X}$ be an ∞-category. Let $p \colon \mathcal{I} \to \mathcal{J}$ be a graduation morphism over $\mathcal{X}$ and assume that $\mathcal{J} = \mathcal{J}^{\mathrm{set}}$. Then, $\mathcal{I}_p = \mathcal{I}$ and $\mathrm{Gr}_p \colon \operatorname{Fun}(\mathcal{I}_p, \mathcal{E}) \to \operatorname{Fun}(\mathcal{I}, \mathcal{E})$ is the identity functor.

Proposition 9.1.12. *Let $\mathcal{X}$ be an ∞-category. Let $p \colon \mathcal{I} \to \mathcal{J}$ be a graduation morphism over $\mathcal{X}$. Let $\mathcal{E}$ be a presentable stable ∞-category. Then $\mathrm{Gr}_p \colon \operatorname{Fun}(\mathcal{I}, \mathcal{E}) \to \operatorname{Fun}(\mathcal{I}_p, \mathcal{E})$ commutes with colimits. In particular, Gr_p admits a right adjoint*

$$\mathrm{Gr}_p^* \colon \operatorname{Fun}(\mathcal{I}_p, \mathcal{E}) \to \operatorname{Fun}(\mathcal{I}, \mathcal{E})$$

that can be explicitly computed as

$$\mathrm{Gr}_p^* \simeq \operatorname{fib}\big(\sigma_* \circ i_{p,*} \xrightarrow{\eta_<} \sigma_* \circ i_{<,*} \circ i_<^* \circ i_{p,*}\big) \ ,$$

where $\eta_<$ is the unit of the adjunction $i_<^* \dashv i_{<,*}$.

Proof. The first half follows immediately from the fact that Gr_p is a composition of functors commuting with colimits. The second half simply follows from the Yoneda lemma. $\square$

Remark 9.1.13. For an explicit description of the right adjoint Gr_p^* in the spirit of Example 9.1.9, see Proposition 9.3.11.

Under extra finitness conditions, Proposition 9.1.12 has a counterpart for limits. Before stating it, we introduce the following

Definition 9.1.14. We define $\mathbf{PosFib}^f$ as the full subcategory of $\mathbf{PosFib}$ spanned by cocartesian fibrations in posets $p : \mathcal{I} \to \mathcal{X}$ such that for every $x \in \mathcal{X}$, the poset $\mathcal{I}_x$ is finite.

Proposition 9.1.15. *Let $\mathcal{X}$ be an ∞-category. Let $p \colon \mathcal{I} \to \mathcal{J}$ be a graduation morphism in $\mathbf{PosFib}^f$ over $\mathcal{X}$. Let $\mathcal{E}$ be a presentable stable ∞-category. Then $\mathrm{Gr}_p \colon \operatorname{Fun}(\mathcal{I}, \mathcal{E}) \to \operatorname{Fun}(\mathcal{I}_p, \mathcal{E})$ commutes with limits.*

Proof. Follows immediately from Proposition 17.2.3 and the fact that in a stable ∞-category, the cofiber functor commutes with limits in virtue of Lemma 17.2.1. $\square$

Proposition 9.1.16. *Let $\mathcal{X}$ be an ∞-category. Let $p\colon \mathcal{I} \to \mathcal{J}$ be a graduation morphism in* $\mathbf{PosFib}^f$ *over $\mathcal{X}$. Let $\mathcal{E}$ be a presentable stable ∞-category. Then $\mathrm{Gr}_p\colon \mathrm{Fun}(\mathcal{I},\mathcal{E}) \to \mathrm{Fun}(\mathcal{I}_p,\mathcal{E})$ is conservative.*

Proof. From Corollary 6.1.6, we can suppose that $\mathcal{X}$ is a point. Let $f\colon F \to G$ be a morphism in $\mathrm{Fun}(\mathcal{I},\mathcal{E})$ such that $\mathrm{Gr}_p(f)\colon \mathrm{Gr}_p(F) \to \mathrm{Gr}_p(G)$ is an isomorphism. Let $A \subset \mathcal{I}$ be the subset of elements a such that f is not an isomorphism at a. We argue by contradiction and assume that A is not empty. Since $\mathcal{I}$ is finite, A admits a minimal element a. If $\mathcal{C}_a \coloneqq \mathcal{I}_< \times_{\mathcal{I}_\leq} (\mathcal{I}_\leq)_{/(a,p(a))}$, there is a morphism of cofibre sequences

$$\begin{tikzcd} \operatorname*{colim}_{\mathcal{C}_a} F|_{\mathcal{C}_a} \arrow[r] \arrow[d] & F_a \arrow[r] \arrow[d, "f_a"] & \mathrm{Gr}_p(F)_a \arrow[d, "\mathrm{Gr}(f)_a"] \\ \operatorname*{colim}_{\mathcal{C}_a} G|_{\mathcal{C}_a} \arrow[r] & G_a \arrow[r] & \mathrm{Gr}_p(G)_a \end{tikzcd}$$

By definition, an object of $\mathcal{C}_a$ is a couple $(b,c) \in \mathcal{I} \times \mathcal{J}$ with $p(b) < c$ such that $b \leq a$ and $c \leq p(a)$. In particular $p(b) < p(a)$, so that $b < a$. That is, the above colimit over $\mathcal{C}_a$ only features values of f at elements $b \in \mathcal{I}$ strictly smaller than a. Thus, the left vertical arrow is an equivalence by the minimality of a. The right vertical arrow is an equivalence by assumption. Hence $f_a\colon F_a \to G_a$ is an equivalence. Contradiction. □

Remark 9.1.17. Note that Proposition 9.1.16 fail if the finiteness assumption on $\mathcal{I}$ is dropped. If $\mathcal{I} = \mathbb{Z}$ and if F is the functor constant to a non zero object in $\mathcal{E}$, then we have $F \neq 0$ and $\mathrm{Gr}(F) \simeq 0$.

9.2. **Exponential graduation.** The graduation functor introduced in Section 9.1 should be understood as the global counterpart of the exponential graduation, which we now discuss. Fix a presentable stable ∞-category $\mathcal{E}$. For every ∞-category $\mathcal{X}$ and every graduation morphism $p\colon \mathcal{I} \to \mathcal{J}$ in **PosFib** over $\mathcal{X}$, we can apply $\exp_{\mathcal{E}}(-/\mathcal{X})$ to the diagram (9.1.6). This yields the following commutative diagram

$$\begin{tikzcd} & & \exp_{\mathcal{E}}(\mathcal{I}_</\mathcal{X}) \arrow[d, "\mathcal{E}_!^{i_<}"] & \\ \exp_{\mathcal{E}}(\mathcal{I}_p/\mathcal{X}) \arrow[rr, "\mathcal{E}_!^{i_p}"] \arrow[d] & & \exp_{\mathcal{E}}(\mathcal{I}_\leq/\mathcal{X}) \arrow[r, "\mathcal{E}_!^{\sigma}"] \arrow[d] & \exp_{\mathcal{E}}(\mathcal{I}/\mathcal{X}) \arrow[d] \\ \exp_{\mathcal{E}}(\mathcal{J}^{\mathrm{set}}/\mathcal{X}) \arrow[r, "\mathcal{E}_!^{i_{\mathcal{J}}}"] & \exp_{\mathcal{E}}(\mathcal{J}/\mathcal{X}) \arrow[r, "\mathcal{E}_!^{\mathrm{id}}"] & \exp_{\mathcal{E}}(\mathcal{J}^{\Delta^1}/\mathcal{X}) \arrow[r, "\mathcal{E}_!^{s}"] & \exp_{\mathcal{E}}(\mathcal{J}/\mathcal{X}) \end{tikzcd}$$

in $\mathbf{PrFib}^{\mathrm{L}}_{\mathcal{X}}$. Recall from Lemma 6.4.1 the existence of right adjoints $\mathcal{E}^{\sigma,*}$, $\mathcal{E}^{i_<,*}$ and $\mathcal{E}^{i_p,*}$ for $\mathcal{E}_!^{\sigma}$, $\mathcal{E}_!^{i_<}$ and $\mathcal{E}_!^{i_p}$, respectively.

Definition 9.2.1. In the above setup, the *exponential graduation relative to p* is the functor

$$\mathrm{expGr}_p \coloneqq \mathrm{cofib}(\mathcal{E}^{i_p,*} \circ \mathcal{E}_!^{i_<} \circ \mathcal{E}^{i_<,*} \circ \mathcal{E}^{\sigma,*} \to \mathcal{E}^{i_p,*} \circ \mathcal{E}^{\sigma,*}) ,$$

where the morphism is induced by the counit of the adjunction $\mathcal{E}_!^{i_<} \dashv \mathcal{E}^{i_<,*}$.

The following result summarizes the local and the global behavior of the exponential graduation functor:

Proposition 9.2.2. *Keep the same notations as above. Then:*

(1) for every $x \in \mathcal{X}$, the diagram

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{I}_x, \mathcal{E}) \arrow[r, "\mathrm{Gr}_{p_x}"] \arrow[d] & \mathrm{Fun}((\mathcal{I}_p)_x, \mathcal{E}) \arrow[d] \\ \exp_{\mathcal{E}}(\mathcal{I}/\mathcal{X}) \arrow[r, "\mathrm{expGr}_p"] & \mathrm{Fun}(\mathcal{I}_p, \mathcal{E}) \end{tikzcd}$$

commutes.

(2) The diagram

$$\begin{tikzcd} \mathrm{Fun}_{/\mathcal{X}}(\mathcal{X}, \exp_{\mathcal{E}}(\mathcal{I}/\mathcal{X})) \arrow[r, "\Sigma_{\mathcal{X}}(\mathrm{expGr}_p)"] \arrow[d, "\mathrm{sp}_{\mathcal{I}}"] & \mathrm{Fun}_{/\mathcal{X}}(\mathcal{X}, \exp_{\mathcal{E}}(\mathcal{I}_p/\mathcal{X})) \arrow[d, "\mathrm{sp}_{\mathcal{I}_p}"] \\ \mathrm{Fun}(\mathcal{I}, \mathcal{E}) \arrow[r, "\mathrm{Gr}_p"] & \mathrm{Fun}(\mathcal{I}_p, \mathcal{E}) \end{tikzcd}$$

commutes.

Proof. Statement (1) immediately follows from Corollary 6.1.6 applied to $\mathcal{E}_!^{i_<}$ and the fact that the adjunctions $\mathcal{E}_!^{\sigma} \dashv \mathcal{E}^{\sigma,*}$, $\mathcal{E}_!^{i_<} \dashv \mathcal{E}^{i_<,*}$ and $\mathcal{E}_!^{i_p} \dashv \mathcal{E}^{i_p,*}$ are relative to $\mathcal{X}$, see Lemma 6.4.1. On the other hand, statement (2) is a direct consequence of Proposition 6.4.2. □

Our next goal is to understand the behavior of expGr with the exponential functoriality for morphisms in **PosFib**. We start analyzing cartesian morphisms. Consider therefore a diagram

(9.2.3)
$$\begin{tikzcd} \mathcal{I} \arrow[d, "p"] & \mathcal{I}_{\mathcal{X}} \arrow[l, "u"'] \arrow[d, "q"] \\ \mathcal{J} \arrow[d] & \mathcal{J}_{\mathcal{X}} \arrow[l, "u'"'] \arrow[d] \\ \mathcal{Y} & \mathcal{X} \arrow[l, "f"'] \end{tikzcd}$$

whose squares are pullbacks and where $\mathcal{I} \to \mathcal{Y}$ and $\mathcal{J} \to \mathcal{Y}$ are cocartesian fibrations in posets. We also assume that $\mathcal{J}^{\mathrm{set}} \to \mathcal{X}$ is locally constant. We have:

Proposition 9.2.4. *In the above setting, the diagram*

(9.2.5)
$$\begin{tikzcd} \exp_{\mathcal{E}}(\mathcal{I}_p/\mathcal{Y}) & \exp_{\mathcal{E}}((\mathcal{I}_{\mathcal{X}})_{p_{\mathcal{X}}}/\mathcal{X}) \arrow[l, "\mathcal{E}^{u_p}"'] \\ \exp_{\mathcal{E}}(\mathcal{I}/\mathcal{Y}) \arrow[u, "\mathrm{expGr}_p"] & \exp_{\mathcal{E}}(\mathcal{I}_{\mathcal{X}}/\mathcal{X}) \arrow[l, "\mathcal{E}^u"'] \arrow[u, "\mathrm{expGr}_q"'] \end{tikzcd}$$

is canonically commutative, and it is therefore a pullback.

Proof. Unraveling the definitions, we see that an object in $\exp_{\mathcal{E}}(\mathcal{I}_{\mathcal{X}}/\mathcal{X})$ can be identified with a pair (F, x), where $x \in \mathcal{X}$ and $F \colon (\mathcal{I}_{\mathcal{X}})_x \to \mathcal{E}$ is a functor. Under the canonical identification $(\mathcal{I}_{\mathcal{X}})_x \simeq \mathcal{I}_{f(x)}$, the functor $\mathcal{E}^u$ sends (F, x) to $(F, f(x))$. At this point, the commutativity follows from Corollary 6.1.6 and Proposition 9.2.2-(1), while Proposition 3.2.6-(1) immediately implies that the square in consideration is a pullback. □

Corollary 9.2.6. *In the above setting, the natural transformation*

$$\mathrm{Gr}_q \circ u^* \to u_p^* \circ \mathrm{Gr}_p$$

between functors from $\mathrm{Fun}(\mathcal{I}, \mathcal{E})$ *to* $\mathrm{Fun}((\mathcal{I}_{\mathcal{X}})_q, \mathcal{E})$ *is an equivalence.*

Proof. We can see (9.2.5) as a commutative square in $\mathbf{PrFib}^{\mathrm{L}}$. The statement then follows applying Σ and invoking Proposition 6.1.2-(1) and Proposition 9.2.2-(2). $\square$

Remark 9.2.7. As a particular case of Corollary 9.2.6, we see that relative graduation commutes with restriction over an object of $\mathcal{X}$.

9.3. **Graduation and induction.** Our next task is to understand how graduation behaves with respect to morphisms in $\mathbf{PosFib}_{\mathcal{X}}$ for a fixed ∞-category $\mathcal{X}$. In other words, we are interested in seeing to which extent (exponential) graduation and (exponential) induction intertwine with each other. Our starting point is the following. Let $\mathcal{X}$ be an ∞-category and let

$$
\begin{tikzcd}
\mathcal{I} \arrow[r, "f"] \arrow[d, "p"] & \mathcal{K} \arrow[d, "q"] \\
\mathcal{J} \arrow[r, "g"] & \mathcal{L}
\end{tikzcd}
\tag{9.3.1}
$$

be a commutative diagram in **PosFib** over $\mathcal{X}$. We make the following running

Assumption 9.3.2.

(1) Both $p : \mathcal{I} \to \mathcal{J}$ and $q : \mathcal{K} \to \mathcal{L}$ are graduation morphisms.

(2) For every $x \in \mathcal{X}$, the map $g_x^{\mathrm{set}} : \mathcal{J}_x^{\mathrm{set}} \to \mathcal{L}_x^{\mathrm{set}}$ is injective.

The second half of this assumption guarantees that if $a, b \in \mathcal{J}_x$ are such that $a < b$, then $g(a) < g(b)$ as well. Thus, the above assumption guarantees the existence and commutativity of the following diagram:

$$
\begin{tikzcd}
 & & & \mathcal{I}_{<} \arrow[dl, "f_{<}"'] \arrow[dd, "i_{\mathcal{I},<}"] & & \\
 & & \mathcal{K}_{<} \arrow[dd, "i_{\mathcal{K},<}"] & & & \\
 & \mathcal{I}_p \arrow[rr, "i_p"] \arrow[dl, "f_{p,q}"'] \arrow[dd] & & \mathcal{I}_{\leq} \arrow[rr, "\sigma_{\mathcal{I}}"] \arrow[dl, "f_{\leq}"'] \arrow[dd] & & \mathcal{I} \arrow[dl, "f"'] \arrow[dd, "p"] \\
\mathcal{K}_q \arrow[rr, "i_q"] \arrow[dd] & & \mathcal{K}_{\leq} \arrow[rr, "\sigma_{\mathcal{K}}"] \arrow[dd] & & \mathcal{K} \arrow[dd, "q"] & \\
 & \mathcal{J}^{\mathrm{set}} \arrow[rr] \arrow[dl] & & \mathcal{J}^{\Delta^1} \arrow[rr] \arrow[dl] & & \mathcal{J} \arrow[dl, "g"] \\
\mathcal{L}^{\mathrm{set}} \arrow[rr] & & \mathcal{L}^{\Delta^1} \arrow[rr] & & \mathcal{L} &
\end{tikzcd}
\tag{9.3.3}
$$

Fix a stable presentable ∞-category $\mathcal{E}$ and consider the induced natural transformation

$$
f_{p,q!} \circ \mathrm{Gr}_p \to \mathrm{Gr}_q \circ f_! \tag{9.3.4}
$$

of functors from $\mathrm{Fun}(\mathcal{I}, \mathcal{E})$ to $\mathrm{Fun}(\mathcal{K}_q, \mathcal{E})$. The goal of this section is to establish the following:

Proposition 9.3.5. *Let $\mathcal{E}$ be a stable presentable ∞-category and let $F \in \mathrm{Fun}(\mathcal{I}, \mathcal{E})$. Under Assumption 9.3.2, the natural transformation* (9.3.4) *is an equivalence and the diagram*

$$\begin{tikzcd} \exp_{\mathcal{E}}(\mathcal{I}/\mathcal{X}) \arrow[r, "\mathcal{E}_!^f"] \arrow[d, "\exp\mathrm{Gr}_p"] & \exp_{\mathcal{E}}(\mathcal{K}/\mathcal{X}) \arrow[d, "\exp\mathrm{Gr}_q"] \\ \exp_{\mathcal{E}}(\mathcal{I}_p/\mathcal{X}) \arrow[r, "\mathcal{E}_!^{f_{p,q}}"] & \exp_{\mathcal{E}}(\mathcal{K}_q/\mathcal{X}) \end{tikzcd}$$

commutes.

We first deal with the natural transformation (9.3.4), and we start by the following particular case:

Lemma 9.3.6. *Let $\mathcal{X}$ be an ∞-category. Let $\mathcal{E}$ be a presentable stable ∞-category. Let $p\colon \mathcal{I} \to \mathcal{J}$ be a graduation morphism in* **PosFib** *over $\mathcal{X}$. Consider the commutative diagram*

$$\begin{tikzcd} \mathcal{I}^{\mathrm{set}} \arrow[r, "i_{\mathcal{I}}"] \arrow[d, "p\circ i_{\mathcal{I}}"] & \mathcal{I} \arrow[d, "p"] \\ \mathcal{J} \arrow[r, "\mathrm{id}"] & \mathcal{J} \end{tikzcd}$$

Then, the induced natural transformation $i_{\mathcal{I}_p!} \to \mathrm{Gr}_p \circ i_{\mathcal{I}!}$ is an equivalence.

Proof. Combining Corollary 6.1.6 and Corollary 9.2.6, we can suppose that $\mathcal{X}$ is a point. Let $V\colon \mathcal{I}^{\mathrm{set}} \to \mathcal{E}$ be a functor. Then, for every $a \in \mathcal{I}_p$, we have

$$(i_{\mathcal{I}_p!}(V))_a \simeq \bigoplus_{\substack{a' \leq a \\ p(a')=p(a)}} V_{a'}$$

Then, Lemma 9.3.6 follows from the computation performed in Example 9.1.9. □

Corollary 9.3.7. *Under the assumptions of Lemma 9.3.6, for every punctually split functor $F\colon \mathcal{I} \to \mathcal{E}$, the graduation $\mathrm{Gr}_p(F)\colon \mathcal{I}_p \to \mathcal{E}$ is punctually split.*

Proof. From Corollary 9.2.6, we are left to treat the case where $\mathcal{X}$ is a point. In this case, the statement follows from Lemma 9.3.6. □

Corollary 9.3.8. *Let $\mathcal{I} \to \mathcal{X}$ be an object of* $\mathbf{PosFib}^f$ *such that $\mathcal{I}^{\mathrm{set}} \to \mathcal{X}$ is locally constant. Let $\mathcal{E}$ be a presentable stable ∞-category. Let $F\colon \mathcal{I} \to \mathcal{E}$ be a functor. Then, the following are equivalent:*

(1) the canonical morphism $i_{\mathcal{I}}^*(F) \to \mathrm{Gr}(F)$ admits a section $\sigma\colon \mathrm{Gr}(F) \to i_{\mathcal{I}}^*(F)$;

(2) the functor F split.

If these conditions are satisfied, the morphism

$$\tau\colon i_{\mathcal{I}!}\,\mathrm{Gr}(F) \to F$$

induced by σ is an equivalence.

Proof. Assume that (1) holds. To prove (2), it is enough to show that τ is an equivalence. By Proposition 9.1.16, it is enough to show that

$$\mathrm{Gr}(\tau)\colon\ \mathrm{Gr}\, i_{\mathcal{I}!}\,\mathrm{Gr}(F) \to \mathrm{Gr}(F)$$

is an equivalence. Then, Corollary 9.3.8 follows from Lemma 9.3.6. Assume that F split and let us write $F \simeq i_{\mathcal{I},!}(V)$ where $V\colon \mathcal{I} \to \mathcal{E}$ is a functor. By Lemma 9.3.6, the canonical morphism from (1) reads as $i_{\mathcal{I}}^* i_{\mathcal{I},!}(V) \to V$. Then, the unit transformation $V \to i_{\mathcal{I}}^* i_{\mathcal{I},!}(V)$ does the job. □

We are now ready for:

Proof of Proposition 9.3.5. Combining Corollary 6.1.6 and Corollary 9.2.6, we can assume that $\mathcal{X}$ is a point. Recall moreover from Proposition 8.1.9 that $\mathrm{Fun}(\mathcal{I}, \mathcal{E})$ is generated under colimits by punctually split functors. Since both source and target of (9.3.4) commute with colimits, it is enough to check that the canonical morphism

$$f_{p,q!}(\mathrm{Gr}_p(F)) \to \mathrm{Gr}_q(f_!(F))$$

is an equivalence when F is punctually split. We can therefore assume that $F \simeq i_{\mathcal{I}!}(V)$ for some functor $V \colon \mathcal{I}^{\mathrm{set}} \to \mathcal{E}$. Thus, we can compute:

$$\begin{aligned} f_{p,q!}(\mathrm{Gr}_p(F)) &\simeq f_{p,q!}(\mathrm{Gr}_p(i_{\mathcal{I}!}(V))) & \\ &\simeq f_{p,q!} \circ i_{\mathcal{I}_p!}(V) & \text{By Lemma 9.3.6} \\ &\simeq i_{\mathcal{K}_q!}(f_!^{\mathrm{set}}(V)) & \\ &\simeq \mathrm{Gr}_q(i_{\mathcal{K}!}(f_!^{\mathrm{set}}(V))) & \text{By Lemma 9.3.6} \\ &\simeq \mathrm{Gr}_p \circ f_! \circ i_{\mathcal{K}!}(V) & \\ &\simeq \mathrm{Gr}_p \circ f_!(F) \ . & \end{aligned}$$

Thus, (9.3.4) is an equivalence. As for the second half of the statement, observe that applying $\exp_{\mathcal{E}}(-/\mathcal{X})$ to the diagram (9.3.4) supplies a canonical natural transformation

$$\alpha \colon \mathcal{E}_!^{f_{p,q}} \circ \exp\mathrm{Gr}_p \to \exp\mathrm{Gr}_q \circ \mathcal{E}_!^{f} \ .$$

To prove that it is an equivalence, it is enough to prove that its restriction α_x is an equivalence for every $x \in \mathcal{X}$. Combining Proposition 6.1.2-(2) and Proposition 9.2.2-(1), we see that α_x coincides with the natural transformation (9.3.4), so the conclusion follows from what we have already proven. ☐

We store the following particular cases of Proposition 9.3.5 for later use.

Corollary 9.3.9. *Let $\mathcal{X}$ be an ∞-category. Let $p \colon \mathcal{I} \to \mathcal{J}$ be a graduation morphism in* **PosFib** *over $\mathcal{X}$. Consider the commutative square*

$$\begin{tikzcd} \mathcal{I} \arrow[r, "p"] \arrow[d, "p"'] & \mathcal{J} \arrow[d, "\mathrm{id}"] \\ \mathcal{J} \arrow[r, "\mathrm{id}"] & \mathcal{J} \end{tikzcd}$$

Let $\pi \colon \mathcal{I}_p \to \mathcal{J}^{\mathrm{set}}$ be the morphism induced by p. Let $\mathcal{E}$ be a presentable stable ∞-category. Then, for every functor $F \colon \mathcal{I} \to \mathcal{E}$, the canonical morphism

$$\pi_!(\mathrm{Gr}_p(F)) \to \mathrm{Gr}(p_!(F))$$

is an equivalence.

Corollary 9.3.10. *Let $\mathcal{X}$ be an ∞-category. Let $i \colon \mathcal{I} \hookrightarrow \mathcal{J}$ be a fully faithful functor in* **PosFib** *over $\mathcal{X}$. Consider the commutative square*

$$\begin{tikzcd} \mathcal{I} \arrow[r, hook, "i"] \arrow[d, "\mathrm{id}"'] & \mathcal{J} \arrow[d, "\mathrm{id}"] \\ \mathcal{I} \arrow[r, hook, "i"] & \mathcal{J} \ . \end{tikzcd}$$

Let $\mathcal{E}$ be a presentable stable ∞-category. Then, for every functor $F \colon \mathcal{I} \to \mathcal{E}$, the canonical morphism

$$i_!^{\mathrm{set}}(\mathrm{Gr}(F)) \to \mathrm{Gr}(i_!(F))$$

is an equivalence.

Proposition 9.3.11. *Let $p\colon \mathcal{I} \to \mathcal{J}$ be a morphism of posets. Let $\mathcal{E}$ be a presentable stable ∞-category. Then, for every $F \in \mathrm{Fun}(\mathcal{I}_p, \mathcal{E})$ and every $a, b \in \mathcal{I}$ with $a \leq b$, we have canonical equivalences*

$$(\mathrm{Gr}_p^*(F))(a \leq b) \simeq \begin{cases} F(a \leq b) & \text{if } p(a) = p(b) \\ 0\colon F_a \to F_b & \text{if } p(a) < p(b) \end{cases}$$

Proof. Let $i\colon p^{-1}(p(a)) \to \mathcal{I}$ and $j\colon p^{-1}(p(a)) \to \mathcal{I}_p$ be the inclusions. From Proposition 9.3.5 and Example 9.1.11 applied to the commutative square

$$\begin{tikzcd} p^{-1}(p(a)) \arrow[r, "i"] \arrow[d] & \mathcal{I} \arrow[d, "p"] \\ * \arrow[r, "p(a)"] & \mathcal{J} \end{tikzcd}$$

there is a canonical equivalence of functors $j_! \simeq \mathrm{Gr}_p \circ i_!$. Passing to right adjoints gives a canonical equivalence $i^* \circ \mathrm{Gr}_p^* \simeq j^*$. This proves the first claim. Let $a, b \in \mathcal{I}$ with $a \leq b$ and $p(a) < p(b)$. We want to show that

$$\alpha \coloneqq (\mathrm{Gr}_p^*(F))(a \leq b)\colon (\mathrm{Gr}_p^*(F))_a \to (\mathrm{Gr}_p^*(F))_b$$

is the zero morphism. This amounts to show that for every $V \in \mathcal{E}$, the morphism

$$\mathrm{Map}(V, \alpha)\colon\ \mathrm{Map}(V, \mathrm{ev}_a^{\mathcal{I},*} \mathrm{Gr}_p^*(F)) \to \mathrm{Map}(V, \mathrm{ev}_b^{\mathcal{I},*} \mathrm{Gr}_p^*(F))$$

is the zero morphism. By adjunction, this amounts to show that

$$\mathrm{Map}(\beta, F)\colon\ \mathrm{Map}(\mathrm{Gr}_p \circ \mathrm{ev}_{a,!}^{\mathcal{I}}(V), F) \to \mathrm{Map}(\mathrm{Gr}_p \circ \mathrm{ev}_{b,!}^{\mathcal{I}}(V), F)$$

is the zero morphism, where

$$\beta : \mathrm{Gr}_p \circ \mathrm{ev}_{b,!}^{\mathcal{I}}(V) \to \mathrm{Gr}_p \circ \mathrm{ev}_{a,!}^{\mathcal{I}}(V)$$

is the induced morphism in $\mathrm{Fun}(\mathcal{I}_p, \mathcal{E})$. We are thus left to show that β is the zero morphism. From Proposition 9.3.5, β identifies with a morphism of the form $\mathrm{ev}_{b,!}^{\mathcal{I}_p}(V) \to \mathrm{ev}_{a,!}^{\mathcal{I}_p}(V)$. Let $c \in \mathcal{I}_p$. Since $p(a) < p(b)$, then either $p(c) \neq p(a)$ or $p(c) \neq p(b)$. In the first case, a and c cannot be compared in $\mathcal{I}_p$, so that $\mathrm{ev}_{a,!}^{\mathcal{I}_p}(V)$ sends c to 0. In the second case, b and c cannot be compared in $\mathcal{I}_p$, so that $\mathrm{ev}_{b,!}^{\mathcal{I}_p}(V)$ sends c to 0. Hence, in both cases β is zero when evaluated at c. Thus, β is the zero morphism. $\square$

Proposition 9.3.12. *Let $\mathcal{X}$ be an ∞-category. Let $p\colon \mathcal{I} \to \mathcal{J}$ be a graduation morphism over $\mathcal{X}$. Let $a \in \mathcal{I}$. Let $\mathcal{E}$ be a presentable stable ∞-category. Then the triangle*

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{I}_p, \mathcal{E}) \arrow[rr, "\mathrm{Gr}_p^*"] \arrow[dr, "\mathrm{ev}_a^{\mathcal{I}_p,*}"'] & & \mathrm{Fun}(\mathcal{I}, \mathcal{E}) \arrow[dl, "\mathrm{ev}_a^{\mathcal{I},*}"] \\ & \mathcal{E} & \end{tikzcd}$$

is canonically commutative.

Proof. Equivalently, it is enough to check that the canonical map

$$\mathrm{ev}_{a!}^{\mathcal{I}_p} \to \mathrm{Gr}_p \circ \mathrm{ev}_{a!}^{\mathcal{I}}$$

is an equivalence. Since $\mathrm{ev}_a^{\mathcal{I}_p}$ factors through $\mathcal{I}^{\mathrm{set}} \to \mathcal{I}_p$ and $\mathrm{ev}_a^{\mathcal{I}}$ factors through $\mathcal{I}^{\mathrm{set}} \to \mathcal{I}$, the statement follows directly from Proposition 9.3.5, in the form of the special case treated in Lemma 9.3.6. $\square$

Corollary 9.3.13. *Let $\mathcal{X}$ be an ∞-category. Let $p\colon \mathcal{I} \to \mathcal{J}$ be a graduation morphism over $\mathcal{X}$. Then, $\mathrm{Gr}_p^*\colon \mathrm{Fun}(\mathcal{I}_p, \mathcal{E}) \to \mathrm{Fun}(\mathcal{I}, \mathcal{E})$ commutes with colimits. In particular, $\mathrm{Gr}_p\colon \mathrm{Fun}(\mathcal{I}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{I}_p, \mathcal{E})$ preserves compact objects.*

Proof. Immediate from Proposition 9.3.12 and the fact that the functors $\mathrm{ev}_a^{\mathcal{I},*}, a \in \mathcal{I}$ are jointly conservative and commute with colimits. □

9.4. **Graduation and cocartesian functors.**

Proposition 9.4.1. *Let $\mathcal{X}$ be an ∞-category. Let $p\colon \mathcal{I} \to \mathcal{J}$ be a graduation morphism in* **PosFib** *over $\mathcal{X}$. Let $\mathcal{E}$ be a presentable stable ∞-category. Then, the functor*

$$\mathrm{expGr}_p\colon \exp_{\mathcal{E}}(\mathcal{I}/\mathcal{X}) \to \exp_{\mathcal{E}}(\mathcal{I}_p/\mathcal{X})$$

preserves cocartesian edges. If in addition both $\mathcal{I}$ and $\mathcal{J}$ belong to $\mathbf{PosFib}^f$*, then expGr_p reflects cocartesian edges as well.*

Warning 9.4.2. Since the adjoints $\mathcal{E}^{\sigma,*}$, $\mathcal{E}^{i_<,*}$ and $\mathcal{E}^{i_p,*}$ do *not* preserve cocartesian edges, it is a priori not obvious that expGr_p defines a morphism of cocartesian fibrations over $\mathcal{X}$.

Proof of Proposition 9.4.1. Unraveling the definitions, we have to prove the following statement. Let $\gamma\colon x \to y$ be any morphism in $\mathcal{X}$ and fix compatible straightenings $f_\gamma\colon \mathcal{I}_x \to \mathcal{I}_y$, $g_\gamma\colon \mathcal{J}_x \to \mathcal{J}_y$ making the diagram

(9.4.3)
$$\begin{tikzcd} \mathcal{I}_x \arrow[r, "f_\gamma"] \arrow[d, "p_x"] & \mathcal{I}_y \arrow[d, "p_y"] \\ \mathcal{J}_x \arrow[r, "g_\gamma"] & \mathcal{J}_y \end{tikzcd}$$

commutative. Then we have to prove that for every pair of functors $F_x\colon \mathcal{I}_x \to \mathcal{E}$ and $F_y\colon \mathcal{I}_y \to \mathcal{E}$ and every map $\alpha\colon (F_x, x) \to (F_y, y)$ in $\exp_{\mathcal{E}}(\mathcal{I}/\mathcal{X})$ lying over γ, if the canonically induced morphism

$$\overline{\alpha}\colon f_{\gamma,!}(F_x) \to F_y$$

is an equivalence then the same goes for the map

(9.4.4)
$$\overline{\beta}\colon (f_\gamma)_{p_x,p_y!}(\mathrm{Gr}_{p_x}(F)) \to \mathrm{Gr}_{p_y}(F_y)$$

induced by the morphism $\beta \coloneqq \mathrm{expGr}_p(\alpha)\colon (\mathrm{Gr}_{p_x}(F_x), x) \to (\mathrm{Gr}_{p_y}(F_y), y)$ in $\exp_{\mathcal{E}}(\mathcal{I}_p/\mathcal{X})$. Notice that, since $\mathcal{J}^{\mathrm{set}} \to \mathcal{X}$ is locally constant, the underlying map $g_\gamma^{\mathrm{set}}\colon \mathcal{J}_x^{\mathrm{set}} \to \mathcal{J}_y^{\mathrm{set}}$ is a bijection. In particular, Assumption 9.3.2 is satisfied, and we therefore find a natural transformation $(f_\gamma)_{p_x,p_y!} \circ \mathrm{Gr}_{p_x} \to \mathrm{Gr}_{p_y} \circ f_{\gamma!}$ making the diagram

$$\begin{tikzcd} (f_\gamma)_{p_x,p_y!}(\mathrm{Gr}_{p_x}(F_x)) \arrow[r, "\overline{\beta}"] \arrow[d] & \mathrm{Gr}_{p_y}(F_y) \\ \mathrm{Gr}_{p_y}(f_{\gamma!}(F_x)) \arrow[ur, "\mathrm{Gr}_{p_y}(\overline{\alpha})"'] & \end{tikzcd},$$

commutative. Now, Proposition 9.3.5 guarantees that the vertical arrow is an equivalence, so $\overline{\beta}$ is an equivalence if and only if $\mathrm{Gr}_{p_y}(\overline{\alpha})$ is. This immediately proves the first half of the statement, and the second half follows from the conservativity of Gr_{p_y}, that holds when $\mathcal{I}$ and $\mathcal{J}$ are in $\mathbf{PosFib}^f$ thanks to Proposition 9.1.16. □

Corollary 9.4.5. *In the setting of Proposition 9.4.1, the functor* $\mathrm{Gr}_p\colon \mathrm{Fun}(\mathcal{I},\mathcal{E}) \to \mathrm{Fun}(\mathcal{I}_p,\mathcal{E})$ *preserves cocartesian functors. If in addition* $\mathcal{I}$ *and* $\mathcal{J}$ *belong to* $\mathbf{PosFib}^f$*, then the resulting commutative square*

$$\begin{tikzcd} \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{I},\mathcal{E}) \arrow[r, hook] \arrow[d, "\mathrm{Gr}_p"] & \mathrm{Fun}(\mathcal{I},\mathcal{E}) \arrow[d, "\mathrm{Gr}_p"] \\ \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{I}_p,\mathcal{E}) \arrow[r, hook] & \mathrm{Fun}(\mathcal{I}_p,\mathcal{E}) \end{tikzcd}$$

is a pullback.

Proof. Thanks to Proposition 9.4.1, we see that expGr_p is a morphism in $\mathbf{PrFib}^{\mathrm{L}}$. Applying $\Sigma^{\mathrm{cocart}}_{\mathcal{X}}$, we see that Gr_p preserves cocartesian functors. As for the pullback statement, since both horizontal functors are fully faithful, it amounts to check that if $F\colon \mathcal{I} \to \mathcal{E}$ is such that $\mathrm{Gr}_p(F)$ is cocartesian, then the same goes for F. Via the specialization equivalence (6.1.1), we can equivalently see F as a section $\mathrm{sp}_{\mathcal{I}}(F)\colon \mathcal{X} \to \exp_{\mathcal{E}}(\mathcal{I}/\mathcal{X})$ of the structural map of $\exp_{\mathcal{E}}(\mathcal{I}/\mathcal{X})$. Using Proposition 9.2.2-(2), we see that the problem at hand becomes showing that $\mathrm{sp}_{\mathcal{I}}(F)$ is a cocartesian section if and only if $\mathrm{expGr}_p \circ \mathrm{sp}_{\mathcal{I}}(F)$ is a cocartesian section of $\exp_{\mathcal{E}}(\mathcal{I}_p/\mathcal{X})$, and this latter statement follows directly from the second half of Proposition 9.4.1. □

In fact, we can extract from the proof of Corollary 9.4.5 the following more precise statement:

Corollary 9.4.6. *In the setting of Proposition 9.4.1, let* $\gamma\colon x \to y$ *be a morphism in* $\mathcal{X}$ *and let* $F\colon \mathcal{I} \to \mathcal{E}$ *be a functor. Then if* F *is cocartesian at* γ*, the same goes for* $\mathrm{Gr}_p(F)\colon \mathcal{I}_p \to \mathcal{E}$*. The converse holds provided that both* $\mathcal{I}$ *and* $\mathcal{J}$ *belong to* $\mathbf{PosFib}^f$*.*

Proof. Passing to the other side of the specialization equivalence (6.1.1) and invoking Proposition 9.2.2-(2), we have to prove that a section $s\colon \mathcal{X} \to \exp_{\mathcal{E}}(\mathcal{I}/\mathcal{X})$ takes γ to a cocartesian morphism if and only if $\mathrm{expGr}_p \circ s$ takes γ to a cocartesian morphism in $\exp_{\mathcal{E}}(\mathcal{I}_p/\mathcal{X})$. As this statement is obviously implied by Proposition 9.4.1, the conclusion follows. □

Combining together Corollary 9.3.7 and Corollary 9.4.5 we obtain:

Corollary 9.4.7. *Let* $\mathcal{X}$ *be an* ∞*-category. Let* $p\colon \mathcal{I} \to \mathcal{J}$ *be a graduation morphism in* $\mathbf{PosFib}_{\mathcal{X}}$*. Let* $\mathcal{E}$ *be a presentable stable* ∞*-category. Then, for every cocartesian punctually split functor* $F\colon \mathcal{I} \to \mathcal{E}$*, its* p*-graduation* $\mathrm{Gr}_p(F)\colon \mathcal{I}_p \to \mathcal{E}$ *is cocartesian and punctually split.*

We conclude this section with the following handy consequence:

Corollary 9.4.8. *Let* $\mathcal{X}$ *be an* ∞*-category. Let* $p\colon \mathcal{I} \to \mathcal{J}$ *be a graduation morphism in* $\mathbf{PosFib}_{\mathcal{X}}$*. Let* $\mathcal{E}$ *be a presentable stable* ∞*-category. Then the functor*

$$\mathrm{expGr}_p\colon \exp_{\mathcal{E}}(\mathcal{I}/\mathcal{X}) \to \exp_{\mathcal{E}}(\mathcal{I}_p/\mathcal{X})$$

admits a right adjoint expGr^*_p *relative to* $\mathcal{X}$*. In particular, for every* $x \in \mathcal{X}$*, the diagram*

$$\begin{tikzcd} \mathrm{Fun}((\mathcal{I}_x)_{p_x},\mathcal{E}) \arrow[r, "\mathrm{Gr}^*_{p_x}"] \arrow[d] & \mathrm{Fun}(\mathcal{I}_x,\mathcal{E}) \arrow[d] \\ \exp_{\mathcal{E}}(\mathcal{I}_p/\mathcal{X}) \arrow[r, "\mathrm{expGr}^*_p"] & \exp_{\mathcal{E}}(\mathcal{I}/\mathcal{X}) \end{tikzcd}$$

commutes. In addition, the diagram

$$\begin{tikzcd} \mathrm{Fun}_{/\mathcal{X}}(\mathcal{X},\exp_{\mathcal{E}}(\mathcal{I}_p/\mathcal{X})) \arrow[r, "\Sigma_{\mathcal{X}}(\mathrm{expGr}^*_p)"] \arrow[d, "\mathrm{sp}_{\mathcal{I}_p}"] & \mathrm{Fun}_{/\mathcal{X}}(\mathcal{X},\exp_{\mathcal{E}}(\mathcal{I}/\mathcal{X})) \arrow[d, "\mathrm{sp}_{\mathcal{I}}"] \\ \mathrm{Fun}(\mathcal{I}_p,\mathcal{E}) \arrow[r, "\mathrm{Gr}^*_p"] & \mathrm{Fun}(\mathcal{I},\mathcal{E}) \end{tikzcd}$$

commutes as well.

Proof. Since expGr_p preserves cocartesian edges by Proposition 9.4.1, [32, Proposition 7.3.2.6] shows that it is enough to prove that for every x, the induced functor on the fibers at x

$$(\mathrm{expGr}_p)_x\colon \mathrm{Fun}((\mathcal{I}_x)_{p_x}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{I}_x, \mathcal{E})$$

admits a right adjoint. By Proposition 9.2.2-(1), we see that $(\mathrm{expGr}_p)_x$ canonically coincides with Gr_{p_x}, so the existence of the right adjoint is guaranteed by Proposition 9.1.12. This proves at the same time the commutativity of the first diagram. As for the second, it simply follows from the uniqueness of the adjoints, the fact that the specialization functors are equivalences and Proposition 9.2.2-(2). □

Proposition 9.4.9. *Let $\mathcal{X}$ be an ∞-category. Let $p\colon \mathcal{I} \to \mathcal{J}$ be a graduation morphism over $\mathcal{X}$. Let $\mathcal{E}$ be a presentable stable ∞-category. Then, the graduation functor relative to p (Definition 9.1.7)*

$$\mathrm{Gr}_p\colon \mathrm{Fun}(\mathcal{I}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{I}_p, \mathcal{E})$$

preserves the category of Stokes functors. In other words, it restricts to a functor

$$\mathrm{Gr}_p\colon \mathrm{St}_{\mathcal{I},\mathcal{E}} \to \mathrm{St}_{\mathcal{I}_p,\mathcal{E}}.$$

Proof. This follows from Corollary 9.4.7, Corollary 9.2.6 and Lemma 9.3.6. □

Corollary 9.4.10. *Let $\mathcal{I} \to \mathcal{X}$ be an object of* **PosFib** *such that $\mathcal{I}^{\mathrm{set}} \to \mathcal{X}$ is locally constant. Let $\mathcal{E}$ be a presentable stable ∞-category. Then, the following square*

$$\begin{tikzcd} \mathrm{St}_{\mathcal{I}^{\mathrm{set}},\mathcal{E}} \arrow[r, "i_{\mathcal{I},!}"] \arrow[d, hook] & \mathrm{St}_{\mathcal{I},\mathcal{E}} \arrow[d, hook] \\ \mathrm{Fun}(\mathcal{I}^{\mathrm{set}}, \mathcal{E}) \arrow[r, "i_{\mathcal{I},!}"] & \mathrm{Fun}(\mathcal{I}, \mathcal{E}) \end{tikzcd}$$

is a pullback

Proof. Let $F\colon \mathcal{I} \to \mathcal{E}$ be a split Stokes functor. Let $V\colon \mathcal{I}^{\mathrm{set}} \to \mathcal{E}$ such that $F \simeq i_{\mathcal{I},!}(V)$. By Lemma 9.3.6, we have

$$\mathrm{Gr}(F) \simeq \mathrm{Gr}(i_{\mathcal{I},!}(V)) \simeq V \ .$$

By Corollary 9.4.5, we deduce that $V\colon \mathcal{I}^{\mathrm{set}} \to \mathcal{E}$ is cocartesian. Since V is automatically punctually split, Corollary 9.4.10 thus follows.

□

9.5. **Essential image of a fully-faithul induction.** The following propositions describe the essential image of a fully-faithul induction in terms of graduation.

Lemma 9.5.1. *Let $\mathcal{X}$ be an ∞-category. Let $i\colon \mathcal{I} \to \mathcal{J}$ be a fully faithful functor in* **PosFib** *over $\mathcal{X}$. Let $\mathcal{E}$ be a presentable stable ∞-category. Then, the functor*

$$i_!\colon \mathrm{Fun}_{\mathrm{PS}}(\mathcal{I}, \mathcal{E}) \to \mathrm{Fun}_{\mathrm{PS}}(\mathcal{J}, \mathcal{E})$$

is fully faithful. Let $F \in \mathrm{Fun}_{\mathrm{PS}}(\mathcal{J}, \mathcal{E})$. Then, the following statements are equivalent :

1. *F lies in the essential image of $i_!\colon \mathrm{Fun}_{\mathrm{PS}}(\mathcal{I}, \mathcal{E}) \to \mathrm{Fun}_{\mathrm{PS}}(\mathcal{J}, \mathcal{E})$.*
2. *$i^*(F)$ lies in $\mathrm{Fun}_{\mathrm{PS}}(\mathcal{I}, \mathcal{E})$ and the counit map $i_!(i^*(F)) \to F$ is an equivalence.*
3. *$(\mathrm{Gr}\, F)_a \simeq 0$ for every $a \in \mathcal{J}^{\mathrm{set}}$ not in the essential image of $i^{\mathrm{set}}\colon \mathcal{I}^{\mathrm{set}} \to \mathcal{J}^{\mathrm{set}}$.*

Proof. Since $i\colon \mathcal{I} \to \mathcal{J}$ is fully faithful, so is $i_!\colon \mathrm{Fun}(\mathcal{I}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{J}, \mathcal{E})$. In particular, the unit of $i_! \dashv i^*$ is an equivalence. The fact that (1) implies (2) is then obvious. The statement (2) trivially implies (1). To show the equivalence with (3), we can suppose from Corollary 6.1.6 and Corollary 9.2.6 that $\mathcal{X}$ is a point. If (2) holds, the sought-after vanishing follows from Example 9.1.10. Suppose that (3) holds. Let us write $F = i_{\mathcal{J}!}(V)$ where $V\colon \mathcal{J}^{\mathrm{set}} \to \mathcal{E}$. From Example 9.1.10, $V_a \simeq 0$ for every $a \in \mathcal{J}^{\mathrm{set}} \setminus \mathcal{I}^{\mathrm{set}}$. If $W = V|_{\mathcal{I}^{\mathrm{set}}}$, we thus have $V \simeq i_!^{\mathrm{set}}(W)$. Hence,

$$F = i_{\mathcal{J}!}(V) \simeq i_{\mathcal{J}!} \circ i_!^{\mathrm{set}}(W) \simeq i_! \circ i_{\mathcal{I}!}(W)$$

which proves (1), thus finishing the proof of Lemma 9.5.1. □

Proposition 9.5.2. *Let $\mathcal{X}$ be an ∞-category. Let $i\colon \mathcal{I} \hookrightarrow \mathcal{J}$ be a fully faithful functor in* **PosFib** *over $\mathcal{X}$. Let $\mathcal{E}$ be a presentable stable ∞-category. Let $F \in \mathrm{St}_{\mathcal{J},\mathcal{E}}$. Then, the following statements are equivalent :*

(1) *F lies in the essential image of $i_!\colon \mathrm{St}_{\mathcal{I},\mathcal{E}} \to \mathrm{St}_{\mathcal{J},\mathcal{E}}$.*

(2) *$i^*(F)$ lies in $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ and the counit map $i_!(i^*(F)) \to F$ is an equivalence.*

(3) *$(\mathrm{Gr}\, F)_a \simeq 0$ for every $a \in \mathcal{J}^{\mathrm{set}}$ not in the essential image of $i^{\mathrm{set}}\colon \mathcal{I}^{\mathrm{set}} \to \mathcal{J}^{\mathrm{set}}$.*

Proof. The equivalence between (1) and (2) follows as in Lemma 9.5.1. Assume that (1) holds. Then (3) holds in virtue of Corollary 9.3.10. Assume that (3) holds. We are doing to show that (2) holds. Since F is punctually split, Lemma 9.5.1 implies that $i^*(F)$ is punctually split and that the counit map $i_!(i^*(F)) \to F$ is an equivalence. Hence, we are left to show that $i^*(F)$ is cocartesian. To do this, we can suppose that $\mathcal{X} = \Delta^1$. In that case, consider the commutative square

$$\begin{tikzcd} \mathcal{I}_x \arrow[r, hook, "j_x"] \arrow[d, hook, "i_x"] & \mathcal{I} \arrow[d, hook, "i"] \\ \mathcal{J}_x \arrow[r, hook, "j_x"] & \mathcal{J} \end{tikzcd} .$$

By Proposition 8.2.5, the counit map $j_{x,!}j_x^*(F) \to F$ is an equivalence. By Corollary 9.2.6, the split functor $j_x^*(F)\colon \mathcal{J}_x \to \mathcal{E}$ satisfies the conditions of Lemma 9.5.1-(3). Thus, there exists $G\colon \mathcal{I}_x \to \mathcal{E}$ such that $j_x^*(F) \simeq i_{x,!}(G)$. Hence, we have

$$i^*(F) \simeq i^* j_{x,!} j_x^*(F) \simeq i^* j_{x,!} i_{x,!}(G) \simeq i^* i_! j_{x,!}(G) \simeq j_{x,!}(G)$$

where the last equivalence follows from the fully faithfulness of $i\colon \mathcal{I} \hookrightarrow \mathcal{J}$. Then $i^*(F)$ is cocartesian by Proposition 8.2.5. □

Corollary 9.5.3. *Let $\mathcal{X}$ be an ∞-category. Let $i\colon \mathcal{I} \hookrightarrow \mathcal{J}$ be a fully faithful morphism in* **PosFib** *over $\mathcal{X}$. Let $\mathcal{E}$ be a presentable ∞-category. Assume that*

$$i_{\mathcal{J},!}\colon \mathrm{St}_{\mathcal{J}^{\mathrm{set}},\mathcal{E}} \to \mathrm{St}_{\mathcal{J},\mathcal{E}}$$

is essentially surjective (resp. fully faithful). Then, so is

$$i_{\mathcal{I},!}\colon \mathrm{St}_{\mathcal{I}^{\mathrm{set}},\mathcal{E}} \to \mathrm{St}_{\mathcal{I},\mathcal{E}} \ .$$

Proof. Consider the commutative square

$$\begin{tikzcd} \mathrm{St}_{\mathcal{I}^{\mathrm{set}},\mathcal{E}} \arrow[r, hook, "i_!^{\mathrm{set}}"] \arrow[d, "i_{\mathcal{I},!}"] & \mathrm{St}_{\mathcal{J}^{\mathrm{set}},\mathcal{E}} \arrow[d, "i_{\mathcal{J},!}"] \\ \mathrm{St}_{\mathcal{I},\mathcal{E}} \arrow[r, hook, "i_!"] & \mathrm{St}_{\mathcal{J},\mathcal{E}} \end{tikzcd}$$

whose horizontal arrows are fully faithful since $i\colon \mathcal{I} \hookrightarrow \mathcal{J}$ is fully faithful. In particular, if $i_{\mathcal{J},!}$ is fully faithful so is $i_{\mathcal{I},!}$. Assume that $i_{\mathcal{J},!}$ is essentially surjective. Let $F\colon \mathcal{I} \to \mathcal{E}$ be a Stokes functor. Write $i_!(F) \simeq i_{\mathcal{J},!}(V)$ where $V\colon \mathcal{J}^{\mathrm{set}} \to \mathcal{E}$ is Stokes. By Lemma 9.3.6, we have

$$\operatorname{Gr} i_!(F) \simeq \operatorname{Gr} i_{\mathcal{J},!}(V) \simeq V \simeq \operatorname{Gr} V \ .$$

By Proposition 9.5.2, we deduce that $(\operatorname{Gr} V)_a \simeq 0$ for every $a \in \mathcal{J}^{\mathrm{set}}$ not in the essential image of $i^{\mathrm{set}}\colon \mathcal{I}^{\mathrm{set}} \to \mathcal{J}^{\mathrm{set}}$. By Proposition 9.5.2 again, there is a Stokes functor $W\colon \mathcal{I}^{\mathrm{set}} \to \mathcal{E}$ such that $V \simeq i^{\mathrm{set}}_!(W)$. Then,

$$i_! i_{\mathcal{I},!}(W) \simeq i_{\mathcal{J},!} i^{\mathrm{set}}_!(W) \simeq i_!(F) \ .$$

Since $i\colon \mathcal{I} \hookrightarrow \mathcal{J}$ is fully faithful, we deduce that $F \simeq i_{\mathcal{J},!}(W)$. The proof of Corollary 9.5.3 is thus complete. □

9.6. **Graduation and t-structures.** We now explore the properties of the relative graduation with respect to the t-structures of Proposition 8.7.11.

Proposition 9.6.1. *Let $\mathcal{X}$ be an ∞-category and let $p\colon \mathcal{I} \to \mathcal{J}$ be a graduation morphism of cocartesian fibrations in finite posets over $\mathcal{X}$. Let $\mathcal{E}$ be a presentable stable ∞-category equipped with an accessible t-structure $\tau = (\mathcal{E}_{\leqslant 0}, \mathcal{E}_{\geqslant 0})$. If $\mathcal{I}$, $\mathcal{I}_p$ are $\mathcal{E}$-bireflexive, then the relative graduation functor*

$$\operatorname{Gr}_p\colon \operatorname{St}_{\mathcal{I},\mathcal{E}} \to \operatorname{St}_{\mathcal{I}_p,\mathcal{E}}$$

is t-exact.

Proof. The very definition of Gr_p (see Definition 9.1.7) and Recollection 8.7.4 imply together that Gr_p is right t-exact. Let now $F \in (\operatorname{St}_{\mathcal{I},\mathcal{E}})_{\leqslant 0}$. To check that $\operatorname{Gr}_p(F) \in (\operatorname{St}_{\mathcal{I}_p,\mathcal{E}})_{\leqslant 0}$, it suffices to show that for every $x \in \mathcal{X}$ one has

$$j_x^*(\operatorname{Gr}_p(F)) \in \operatorname{Fun}((\mathcal{I}_p)_x, \mathcal{E}) \ .$$

By Corollary 9.2.6 and Remark 9.2.7 we have a canonical equivalence

$$j_x^*(\operatorname{Gr}_p(F)) \simeq \operatorname{Gr}_{p_x}(j_x^*(F)) \ .$$

We can therefore assume that $\mathcal{X}$ is reduced to a point. Since F is punctually split, we can find a functor $V\colon \mathcal{I}^{\mathrm{set}} \to \mathcal{E}$. Lemma 8.7.7 guarantees that V takes values in $\mathcal{E}_{\leqslant 0}$. Since $\mathcal{I}_x$ is finite and $\mathcal{E}_{\leqslant 0}$ is closed under finite sums, the conclusion follows from the fomula given in Example 9.1.9. □

Corollary 9.6.2. *In the setting of Proposition 9.6.1, a Stokes functor $F\colon \mathcal{I} \to \mathcal{E}$ is connective (resp. coconnective) with respect to the induced t-structure on $\operatorname{St}_{\mathcal{I},\mathcal{E}}$ if and only if $\operatorname{Gr}_p(F)$ is connective (resp. coconnective).*

Proof. It follows combining t-exactness and conservativity of Gr_p, see Proposition 9.6.1 and Proposition 9.1.16. □

9.7. **Splitting criterion.** The goal of this subsection is to establish a splitting criterion (Corollary 9.7.17), to be used in the essential surjectivity part of the proof of Theorem 15.2.1.

Construction 9.7.1. Let $\mathcal{X}$ be an ∞-category. Let $i\colon \mathcal{I} \hookrightarrow \mathcal{J}$ and $k\colon \mathcal{K} \hookrightarrow \mathcal{J}$ be fully faithful functors in $\mathbf{PosFib}^f$ over $\mathcal{X}$ such that $\mathcal{J}^{\mathrm{set}} = \mathcal{I}^{\mathrm{set}} \sqcup \mathcal{K}^{\mathrm{set}}$. In particular, for every functor $G\colon \mathcal{J}^{\mathrm{set}} \to \mathcal{E}$, we have

$$G \simeq i^{\mathrm{set}}_! i^{\mathrm{set}\,*}(G) \oplus k^{\mathrm{set}}_! k^{\mathrm{set}\,*}(G) \ .$$

We denote by $\Delta(G)$ the split cofiber sequence

$$i^{\mathrm{set}}_! i^{\mathrm{set}\,*}(G) \to G \to k^{\mathrm{set}}_! k^{\mathrm{set}\,*}(G) \ . \tag{9.7.2}$$

We assume that $\mathcal{J}^{\mathrm{set}} \to \mathcal{X}$ is locally constant. Let $\mathcal{E}$ be a presentable stable ∞-category. Let $F\colon \mathcal{J} \to \mathcal{E}$ be a functor. We suppose that the canonical morphism $i^{\mathrm{set},*} i_{\mathcal{J}}^*(F) \to i^{\mathrm{set}\,*}\operatorname{Gr}(F)$ admits a section

$$\sigma\colon i^{\mathrm{set}\,*}\operatorname{Gr}(F) \to i^{\mathrm{set},*} i_{\mathcal{J}}^*(F) \ . \tag{9.7.3}$$

By adjunction, σ yields a morphism

$$\tau\colon i_{\mathcal{J}!} i_!^{\mathrm{set}} i^{\mathrm{set}\,*}\operatorname{Gr}(F) \to F$$

in $\operatorname{Fun}(\mathcal{J}, \mathcal{E})$. We denote by $\Delta(F,\sigma)$ the following cofiber sequence

$$i_{\mathcal{J}!} i_!^{\mathrm{set}} i^{\mathrm{set}\,*}\operatorname{Gr}(F) \xrightarrow{\ \tau\ } F \longrightarrow F^{\backslash \mathcal{I}} \ . \tag{9.7.4}$$

Remark 9.7.5. By Corollary 9.2.6 and Corollary 6.1.6, observe that the formation of $F^{\backslash \mathcal{I}}$ commutes with pull-back.

Lemma 9.7.6. *In the setting of Construction 9.7.1, the canonical morphism*

$$k_!^{\mathrm{set}} k^{\mathrm{set},*}\operatorname{Gr}(F) \to \operatorname{Gr}(F^{\backslash \mathcal{I}})$$

is an equivalence.

Proof. Since Gr commutes with colimits, applying Gr to (9.7.4) yields a cofiber sequence

$$\operatorname{Gr} i_! i_{\mathcal{J}!} i^{\mathrm{set}\,*}\operatorname{Gr}(F) \to \operatorname{Gr}(F) \to \operatorname{Gr}(F^{\backslash \mathcal{I}}) \ .$$

By Lemma 9.3.6 and Corollary 9.3.10, we have $\operatorname{Gr} i_! i_{\mathcal{J}!} i^{\mathrm{set}\,*}\operatorname{Gr}(F) \simeq i_!^{\mathrm{set}} i^{\mathrm{set},*}\operatorname{Gr}(F)$. Since we have

$$\operatorname{Gr}(F) \simeq i_!^{\mathrm{set}} i^{\mathrm{set},*}\operatorname{Gr}(F) \oplus k_!^{\mathrm{set}} k^{\mathrm{set},*}\operatorname{Gr}(F) \ ,$$

Lemma 9.7.6 thus follows. □

Lemma 9.7.7. *In the setting of Construction 9.7.1, the following hold:*

1. *If F is cocartesian, so is $F^{\backslash \mathcal{I}}$.*
2. *If F is punctually split, so is $F^{\backslash \mathcal{I}}$.*
3. *If F split,the cofiber sequences $\Delta(F,\sigma)$ and $i_{\mathcal{J},!}\Delta(\operatorname{Gr}(F))$ are equivalent. In particular, $F^{\backslash \mathcal{I}}$ split.*

Proof. Item (1) follows immediately from the stability of $\operatorname{Fun}^{\mathrm{cocart}}(\mathcal{J}, \mathcal{E})$ under colimits (Proposition 7.2.9). By Remark 9.7.5, the formation of $F^{\backslash \mathcal{I}}$ commutes with pull-back. Hence, (3) implies (2). We now prove (3) and assume that F split. By Corollary 9.3.8, the canonical morphism $i_{\mathcal{J}}^*(F) \to \operatorname{Gr}(F)$ admits a section $\iota \oplus \kappa$. Then, the vertical arrows of the commutative square

$$\begin{tikzcd} i_!^{\mathrm{set}} i^{\mathrm{set},*} i_{\mathcal{J}}^*(F) \arrow[r] \arrow[d] & i_{\mathcal{J}}^*(F) \arrow[d] \\ i_!^{\mathrm{set}} i^{\mathrm{set},*}\operatorname{Gr}(F) \arrow[r] \arrow[u, dashed, bend left, "\sigma\oplus 0"] & \operatorname{Gr}(F) \arrow[u, dashed, bend right, "\sigma\oplus\kappa"'] \end{tikzcd}$$

admit sections represented by dashed arrows. By adjunction, there is a commutative square

$$\begin{tikzcd} F \arrow[r, "\mathrm{id}"] & F \\ i_{\mathcal{J},!} i_!^{\mathrm{set}} i^{\mathrm{set},*}\operatorname{Gr}(F) \arrow[r] \arrow[u, "\tau"] & i_{\mathcal{J},!}\operatorname{Gr}(F) \arrow[u] \end{tikzcd}$$

whose right vertical arrow is an equivalence in virtue of Corollary 9.3.8. Item (3) is thus proved. □

Corollary 9.7.8. *In the setting of Construction 9.7.1, assume that $F\colon \mathscr{J} \to \mathcal{E}$ punctally split. Then, $F^{\backslash \mathcal{I}}$ lies in the essential image of $k_!\colon \mathrm{Fun}(\mathcal{K}, \mathcal{E}) \to \mathrm{Fun}(\mathscr{J}, \mathcal{E})$.*

Proof. By Lemma 9.7.7, we know that $F^{\backslash \mathcal{I}}$ punctually split. By Lemma 9.7.6, we have $(\mathrm{Gr}\, F^{\backslash \mathcal{I}})(a) \simeq 0$ for every $a \in \mathcal{I}$. Then, Corollary 9.7.8 follows from Lemma 9.5.1. $\square$

Construction 9.7.9. In the setting of Construction 9.7.1, let $l\colon \mathcal{L} \hookrightarrow \mathcal{K}$ and $m\colon \mathcal{M} \hookrightarrow \mathcal{K}$ be fully faithful functors in $\mathbf{PosFib}^f$ over $\mathcal{X}$ such that $\mathcal{K}^{\mathrm{set}} = \mathcal{L}^{\mathrm{set}} \sqcup \mathcal{M}^{\mathrm{set}}$. We suppose that the canonical morphism $l^{\mathrm{set}\,*} i_{\mathscr{J}}^*(F) \to l^{\mathrm{set}\,*}\,\mathrm{Gr}(F)$ admits a section

$$\lambda\colon l^{\mathrm{set}\,*}\,\mathrm{Gr}(F) \to l^{\mathrm{set}\,*} i_{\mathscr{J}}^*(F) \ . \tag{9.7.10}$$

Let $\iota\colon \mathcal{I} \cup \mathcal{L} \hookrightarrow \mathscr{J}$ be the full subcategory of $\mathcal{I}$ spanned by objects of $\mathcal{I}$ and $\mathcal{L}$. Then, the vertical arrows of the commutative square

$$\begin{tikzcd} \iota^* i_{\mathscr{J}}^*(F) \arrow[r] \arrow[d] & \iota^* i_{\mathscr{J}}^*(F^{\backslash \mathcal{I}}) \arrow[d] \\ \iota^*\,\mathrm{Gr}(F) \arrow[r] \arrow[u, dashed, bend left, "\sigma\oplus\lambda"] & \iota^*\,\mathrm{Gr}(F^{\backslash \mathcal{I}}) \arrow[u, dashed, bend right, "0\oplus\lambda"'] \end{tikzcd}$$

admit sections represented by dashed arrows. By adjunction, we thus deduce a morphism of cofiber sequence

$$\Delta(F, \sigma \oplus \lambda) \to \Delta(F^{\backslash \mathcal{I}}, 0 \oplus \lambda) \ . \tag{9.7.11}$$

Lemma 9.7.12. *In the setting of Construction 9.7.9, the natural transformation*

$$F^{\backslash \mathcal{I}\cup\mathcal{L}} \to (F^{\backslash \mathcal{I}})^{\backslash \mathcal{I}\cup\mathcal{L}}$$

deduced from (9.7.11) *is an equivalence.*

Proof. Immediate from Lemma 9.7.6 and Proposition 9.1.16. $\square$

Notation 9.7.13. We denote by $\alpha_{\mathcal{I},\mathcal{L}}\colon F^{\backslash \mathcal{I}} \to F^{\backslash \mathcal{I}\cup\mathcal{L}}$ the canonical morphism obtained by composing $F^{\backslash \mathcal{I}} \to (F^{\backslash \mathcal{I}})^{\backslash \mathcal{I}\cup\mathcal{L}}$ with the inverse of $F^{\backslash \mathcal{I}\cup\mathcal{L}} \to (F^{\backslash \mathcal{I}})^{\backslash \mathcal{I}\cup\mathcal{L}}$ supplied by Lemma 9.7.12.

Lemma 9.7.14. *In the setting of Construction 9.7.9, assume that $F^{\backslash \mathcal{I}}$ split. Then there is a commutative square*

$$\begin{tikzcd} i_{\mathscr{J},!}\,\mathrm{Gr}(F^{\backslash \mathcal{I}}) \arrow[r] \arrow[d, "i_{\mathscr{J},!}\,\mathrm{Gr}(\alpha_{\mathcal{I},\mathcal{L}})"] & F^{\backslash \mathcal{I}} \arrow[d, "\alpha_{\mathcal{I},\mathcal{L}}"] \\ i_{\mathscr{J},!}\,\mathrm{Gr}(F^{\backslash \mathcal{I}\cup\mathcal{L}}) \arrow[r] & F^{\backslash \mathcal{I}\cup\mathcal{L}} \end{tikzcd}$$

whose horizontal arrows are equivalences.

Proof. Recall that $\mathscr{J}^{\mathrm{set}} = \mathcal{I}^{\mathrm{set}} \sqcup \mathcal{L}^{\mathrm{set}} \sqcup \mathcal{M}^{\mathrm{set}}$. Consider the commutative diagram

$$\begin{tikzcd} i_{\mathscr{J}}^*(\alpha_{\mathcal{I},\mathcal{L}})\colon i_{\mathscr{J}}^*(F^{\backslash \mathcal{I}}) \arrow[r] \arrow[d] & i_{\mathscr{J}}^*(F^{\backslash \mathcal{I}})^{\backslash \mathcal{I}\cup\mathcal{L}} \arrow[r, "\sim"] \arrow[d] & i_{\mathscr{J}}^*(F^{\backslash \mathcal{I}\cup\mathcal{L}}) \arrow[d] \\ \mathrm{Gr}(\alpha_{\mathcal{I},\mathcal{L}})\colon\ \mathrm{Gr}(F^{\backslash \mathcal{I}}) \arrow[r] \arrow[u, dashed, bend right, "0\oplus\lambda\oplus\mu'"'] & \mathrm{Gr}(F^{\backslash \mathcal{I}})^{\backslash \mathcal{I}\cup\mathcal{L}} \arrow[r, "\sim"] \arrow[u, dashed, bend right, "0\oplus 0\oplus\mu'"'] & \mathrm{Gr}(F^{\backslash \mathcal{I}\cup\mathcal{L}}) \ . \arrow[u, dashed, bend right, "0\oplus 0\oplus\mu"'] \end{tikzcd}$$

Since $F^{\backslash \mathcal{I}}$ split, Lemma 9.7.7 ensures that so do $(F^{\backslash \mathcal{I}})^{\backslash \mathcal{I}\cup\mathcal{L}}$. By Lemma 9.7.12, the functor $F^{\backslash \mathcal{I}\cup\mathcal{L}}$ split as well. By Corollary 9.3.8, we thus deduce the existence of the section μ represented as a dashed arrow. Since the right horizontal arrows are equivalences, there exists a section μ' making the right square commutative. On the other hand, Lemma 9.7.6 implies that $\mathrm{Gr}(F^{\backslash \mathcal{I}})^{\backslash \mathcal{I}\cup\mathcal{L}}$ is a direct factor of $\mathrm{Gr}(F^{\backslash \mathcal{I}})$. Since $F^{\backslash \mathcal{I}}$ split, we deduce from Lemma 9.7.7 the existence of a section λ making the left square commutative. Then, Lemma 9.7.14 follows from Corollary 9.3.8. □

Lemma 9.7.15. *In the setting of Construction 9.7.9, the morphism*

$$F \to F^{\backslash \mathcal{I}} \times_{F^{\backslash \mathcal{I}\cup\mathcal{L}}} F^{\backslash \mathcal{L}} \tag{9.7.16}$$

is an equivalence.

Proof. Follows immediately from Lemma 9.7.6 and Proposition 9.1.16. □

Corollary 9.7.17. *In the setting of Construction 9.7.9, the following are equivalent:*

1. *the functor F split;*
2. *the functors $F^{\backslash \mathcal{I}}$ and $F^{\backslash \mathcal{L}}$ split.*

Proof. If (1) holds, so do (2) in virtue of Lemma 9.7.7. Assume that (2) holds. From Lemma 9.7.15, we are left to show that $F^{\backslash \mathcal{I}} \times_{F^{\backslash \mathcal{I}\cup\mathcal{L}}} F^{\backslash \mathcal{L}}$ split. Since $F^{\backslash \mathcal{I}}$ and $F^{\backslash \mathcal{L}}$ split, Lemma 9.7.14 ensures that the diagram

$$F^{\backslash \mathcal{I}} \xrightarrow{\alpha_{\mathcal{I},\mathcal{L}}} F^{\backslash \mathcal{I}\cup\mathcal{L}} \xleftarrow{\alpha_{\mathcal{L},\mathcal{I}}} F^{\backslash \mathcal{L}}$$

is equivalent to

$$i_{\mathcal{J},!}\,\mathrm{Gr}(F^{\backslash \mathcal{I}}) \xrightarrow{i_{\mathcal{J},!}\,\mathrm{Gr}(\alpha_{\mathcal{I},\mathcal{L}})} i_{\mathcal{J},!}\,\mathrm{Gr}(F^{\backslash \mathcal{I}\cup\mathcal{L}}) \xleftarrow{i_{\mathcal{J},!}\,\mathrm{Gr}(\alpha_{\mathcal{L},\mathcal{I}})} F^{\backslash \mathcal{L}} \ .$$

Since the induction functor $i_{\mathcal{J},!}$ commutes with finite limits, Corollary 9.7.17 thus follows. □

10. Level structures

We now introduce an axiomatization of the notion of level structure from the theory of good meromorphic flat bundles [37]. The key concept is that of level morphism for a morphism of cocartesian fibrations in posets.

10.1. Level morphisms.

We start with the following pair of definitions:

Definition 10.1.1. A morphism of posets $p\colon \mathcal{I} \to \mathcal{J}$ is a *level morphism* if it is surjective and for every $a, b \in \mathcal{I}$, we have

$$p(a) < p(b) \text{ in } \mathcal{J} \Rightarrow a < b \text{ in } \mathcal{I} \ .$$

Definition 10.1.2. Let $\mathcal{X}$ be an ∞-category and let $p\colon \mathcal{I} \to \mathcal{J}$ be a morphism in $\mathbf{PosFib}_{\mathcal{X}}$. We say that p is a *level morphism* if for every $x \in \mathcal{X}$, the induced morphism $p_x\colon \mathcal{I}_x \to \mathcal{J}_x$ is a level morphism.

Example 10.1.3. Let $\mathcal{I} \to \mathcal{X}$ be an object of $\mathbf{PosFib}$. Then, the morphisms of cocartesian fibrations $\mathrm{id}_{\mathcal{I}}\colon \mathcal{I} \to \mathcal{I}$ and $\mathcal{I} \to \mathcal{X} \times * \simeq \mathcal{X}$ are level morphisms.

Remark 10.1.4. The class of level morphisms is stable under pullback.

Construction 10.1.5. Fix an ∞-category $\mathcal{X}$ and let $p\colon \mathcal{I} \to \mathcal{J}$ be a level graduation morphism in $\mathbf{PosFib}_{\mathcal{X}}$. Fix also a presentable stable ∞-category $\mathcal{E}$. Recall from Construction 9.1.1 the following pullback diagram

$$\begin{tikzcd} \mathcal{I}_p \arrow[r] \arrow[d, "\pi"] & \mathcal{I} \arrow[d, "p"] \\ \mathcal{J}^{\mathrm{set}} \arrow[r] & \mathcal{J} \end{tikzcd} ,$$

as well as the commutative diagram

$$\begin{tikzcd} \exp_{\mathcal{E}}(\mathcal{I}/\mathcal{X}) \arrow[r, "\mathcal{E}^p_!"] \arrow[d, "\exp\mathrm{Gr}_p"] & \exp_{\mathcal{E}}(\mathcal{J}/\mathcal{X}) \arrow[d, "\exp\mathrm{Gr}"] \\ \exp_{\mathcal{E}}(\mathcal{I}_p/\mathcal{X}) \arrow[r, "\mathcal{E}^\pi_!"] & \exp_{\mathcal{E}}(\mathcal{J}^{\mathrm{set}}/\mathcal{X}) \ . \end{tikzcd}$$

supplied by Proposition 9.3.5. It induces a canonical transformation

$$\phi_p \colon \exp_{\mathcal{E}}(\mathcal{I}/\mathcal{X}) \to \exp_{\mathcal{E}}(\mathcal{J}/\mathcal{X}) \times_{\exp_{\mathcal{E}}(\mathcal{J}^{\mathrm{set}}/\mathcal{X})} \exp_{\mathcal{E}}(\mathcal{I}_p/\mathcal{X})$$

in $\mathbf{PrFib}^{\mathrm{L}}_{\mathcal{X}}$. Observe as well that combining Propositions 3.2.6-(2) and Proposition 9.4.1, we see that all the functors in the above square preserve cocartesian edges. Thus, the same goes for ϕ_p. Since $\Sigma_{\mathcal{X}} \colon \mathbf{PrFib}^{\mathrm{L}}_{\mathcal{X}} \to \mathbf{Pr}^{\mathrm{L}}$ is a right adjoint, $\Sigma_{\mathcal{X}}(\phi_p)$ is a functor

$$\Sigma_{\mathcal{X}}(\phi_p) \colon \mathrm{Fun}(\mathcal{I}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{J}, \mathcal{E}) \times_{\mathrm{Fun}(\mathcal{J}^{\mathrm{set}}, \mathcal{E})} \mathrm{Fun}(\mathcal{I}_p, \mathcal{E}) \ ,$$

and Propositions 6.4.2 and 9.2.2-(2) imply that it canonically coincides with the functor induced by the commutative diagram

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{I}, \mathcal{E}) \arrow[r, "p_!"] \arrow[d, "\mathrm{Gr}_p"] & \mathrm{Fun}(\mathcal{J}, \mathcal{E}) \arrow[d, "\mathrm{Gr}"] \\ \mathrm{Fun}(\mathcal{I}_p, \mathcal{E}) \arrow[r, "\pi_!"] & \mathrm{Fun}(\mathcal{J}^{\mathrm{set}}, \mathcal{E}) \ . \end{tikzcd}$$

Proposition 10.1.6. *The functors ϕ_p and $\Sigma_{\mathcal{X}}(\phi_p)$ are fully faithful.*

Proof. Thanks to Proposition 21.1.1, we are immediately reduced to prove the statement when $\mathcal{X}$ is a point. In this case, unraveling the definitions, we have to check that for every pair of functors $F, G \colon \mathcal{I} \to \mathcal{E}$ the square

(10.1.7)
$$\begin{tikzcd} \mathrm{Map}_{\mathrm{Fun}(\mathcal{I},\mathcal{E})}(F, G) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{Fun}(\mathcal{J},\mathcal{E})}(p_!(F), p_!(G)) \arrow[d] \\ \mathrm{Map}_{\mathrm{Fun}(\mathcal{I}_p,\mathcal{E})}(\mathrm{Gr}_p(F), \mathrm{Gr}_p(G)) \arrow[r] & \mathrm{Map}_{\mathrm{Fun}(\mathcal{J}^{\mathrm{set}},\mathcal{E})}(\mathrm{Gr}\, p_!(F), \mathrm{Gr}\, p_!(G)) \end{tikzcd}$$

is a pullback. Notice that the collection of functors F for which the statement is true is closed under colimits. Invoking Proposition 8.1.9 and Example 8.1.3 we can therefore assume without loss of generality that $F \simeq \mathrm{ev}^{\mathcal{I}}_{a,!}(E)$ for some $a \in \mathcal{I}$ and some $E \in \mathcal{E}$. Notice that

$$p_!\big(\mathrm{ev}^{\mathcal{I}}_{a,!}(E)\big) \simeq \mathrm{ev}^{\mathcal{J}}_{p(a),!}(E)$$

and that Lemma 9.3.6 supplies canonical identifications

$$\mathrm{Gr}_p\left(\mathrm{ev}^{\mathcal{I}}_{a,!}(E)\right) \simeq \mathrm{ev}^{\mathcal{I}_p}_{a,!}(E) \qquad \text{and} \qquad \mathrm{Gr}\left(\mathrm{ev}^{\mathcal{J}}_{p(a),!}(E)\right) \simeq \mathrm{ev}^{\mathcal{J}^{\mathrm{set}}}_{p(a),!}(E) \ .$$

Thus (10.1.7) can be rewritten as follows:

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{E}}(E, G_a) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{E}}(E, (p_!(G))_a) \arrow[d] \\ \mathrm{Map}_{\mathcal{E}}(E, \mathrm{Gr}_p(G)_a) \arrow[r] & \mathrm{Map}_{\mathcal{E}}(E, \mathrm{Gr}(p_!(G))_a) \ , \end{tikzcd}$$

and to prove that it is a pullback becomes equivalent to prove that for every $a \in \mathcal{I}$ and every $G \colon \mathcal{I} \to \mathcal{E}$, the square

$$\begin{tikzcd} G_a \arrow[r] \arrow[d] & (p_!(G))_a \arrow[d] \\ \mathrm{Gr}_p(G)_a \arrow[r] & \mathrm{Gr}(p_!(G))_a \end{tikzcd} \tag{10.1.8}$$

is a pullback in $\mathcal{E}$. Since $\mathcal{E}$ is stable, we see that the collection of functors G for which the above square is a pullback is closed under colimits. Invoking once again Proposition 8.1.9 and Example 8.1.3, we can suppose that $G \simeq \mathrm{ev}^{\mathcal{I}}_{b,!}(M)$, for some $b \in \mathcal{I}$ and $M \in \mathcal{E}$. We now proceed by analysis case-by-case:

(1) *Case $p(a) < p(b)$ or $p(a)$ and $p(b)$ incomparable.* Since p is a level morphism, this implies respectively that $a < b$ or that a and b are incomparable. In either cases, (10.1.8) becomes

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd}$$

which is indeed a pullback.

(2) *Case $p(a) > p(b)$.* Since p is a level morphism, this implies that $a > b$. Then (10.1.8) becomes

$$\begin{tikzcd} M \arrow[r, "\mathrm{id}_M"] \arrow[d] & M \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd}$$

which is indeed a pullback.

(3) *Case $p(a) = p(b)$.* We then distinguish two further cases:

(i) *Case $a \geq b$.* Then (10.1.8) becomes

$$\begin{tikzcd} M \arrow[r, "\mathrm{id}_M"] \arrow[d, "\mathrm{id}_M"] & M \arrow[d, "\mathrm{id}_M"] \\ M \arrow[r, "\mathrm{id}_M"] & M \end{tikzcd} ,$$

which is indeed a pullback.

(ii) *Case $a < b$ or a and b incomparable.* Then (10.1.8) becomes

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & M \arrow[d, "\mathrm{id}_M"] \\ 0 \arrow[r] & M \end{tikzcd} ,$$

which is indeed a pullback.

Thus, the conclusion follows. □

10.2. **Level induction.** The goal of this subsection is to prove the following result:

Theorem 10.2.1. *Let $\mathcal{X}$ be an ∞-category and let $p\colon \mathcal{I} \to \mathcal{J}$ be a level graduation morphism in $\mathbf{PosFib}_{\mathcal{X}}$. Then the square*

$$\begin{tikzcd} \exp_{\mathcal{E}}^{\mathrm{PS}}(\mathcal{I}/\mathcal{X}) \arrow[r, "\mathcal{E}_!^p"] \arrow[d, "\exp\mathrm{Gr}_p"] & \exp_{\mathcal{E}}^{\mathrm{PS}}(\mathcal{J}/\mathcal{X}) \arrow[d, "\exp\mathrm{Gr}"] \\ \exp_{\mathcal{E}}^{\mathrm{PS}}(\mathcal{I}_p/\mathcal{X}) \arrow[r, "\mathcal{E}_!^\pi"] & \exp_{\mathcal{E}}^{\mathrm{PS}}(\mathcal{J}^{\mathrm{set}}/\mathcal{X}) \end{tikzcd}$$

is a pullback square in $\mathbf{CoCart}_{\mathcal{X}}$. In particular, the induced square

$$\begin{tikzcd} \mathrm{St}_{\mathcal{I},\mathcal{E}} \arrow[r, "p_!"] \arrow[d, "\mathrm{Gr}_p"] & \mathrm{St}_{\mathcal{J},\mathcal{E}} \arrow[d, "\mathrm{Gr}"] \\ \mathrm{St}_{\mathcal{I}_p,\mathcal{E}} \arrow[r] & \mathrm{St}_{\mathcal{J}^{\mathrm{set}},\mathcal{E}} \end{tikzcd}$$

is a pullback square in $\mathbf{Cat}_\infty$.

Proof. The second half follows directly from the first since $\Sigma_{\mathcal{X}}^{\mathrm{cocart}}\colon \mathbf{CoCart}_{\mathcal{X}} \to \mathbf{Cat}_\infty$ is a right adjoint. Moreover, the straightening/unstraightening equivalence immediately reduces the proof of the first half to the case where $\mathcal{X}$ is a point. In this case, we have to show that the top horizontal arrow of the commutative square

$$\begin{tikzcd} \mathrm{Fun}^{\mathrm{PS}}(\mathcal{I},\mathcal{E}) \arrow[r] \arrow[d] & \mathrm{Fun}^{\mathrm{PS}}(\mathcal{J},\mathcal{E}) \times_{\mathrm{Fun}^{\mathrm{PS}}(\mathcal{J}^{\mathrm{set}},\mathcal{E})} \mathrm{Fun}^{\mathrm{PS}}(\mathcal{I}_p,\mathcal{E}) \arrow[d] \\ \mathrm{Fun}(\mathcal{I},\mathcal{E}) \arrow[r] & \mathrm{Fun}(\mathcal{J},\mathcal{E}) \times_{\mathrm{Fun}(\mathcal{J}^{\mathrm{set}},\mathcal{E})} \mathrm{Fun}(\mathcal{I}_p,\mathcal{E}) \ . \end{tikzcd}$$

is an equivalence. Note that the vertical arrows are fully faithful. From Proposition 10.1.6, the bottom arrow is fully faithful. Thus, so is the top horizontal arrow. We are thus left to show essentially surjectivity. From Lemma 9.3.6, the lateral faces of the following cube

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{I}^{\mathrm{set}},\mathcal{E}) \arrow[rr, "p_!^{\mathrm{set}}"] \arrow[dr, "i_{\mathcal{I}!}"] \arrow[dd] & & \mathrm{Fun}(\mathcal{J}^{\mathrm{set}},\mathcal{E}) \arrow[dr, "i_{\mathcal{J}!}"] \arrow[dd] & \\ & \mathrm{Fun}^{\mathrm{PS}}(\mathcal{I},\mathcal{E}) \arrow[rr, "p_!"] \arrow[dd, "\mathrm{Gr}_p"] & & \mathrm{Fun}^{\mathrm{PS}}(\mathcal{J},\mathcal{E}) \arrow[dd, "\mathrm{Gr}"] \\ \mathrm{Fun}(\mathcal{I}^{\mathrm{set}},\mathcal{E}) \arrow[rr, "p_!^{\mathrm{set}}"] \arrow[dr, "i_{\mathcal{I}_p!}"] & & \mathrm{Fun}(\mathcal{J}^{\mathrm{set}},\mathcal{E}) \arrow[dr, "\mathrm{id}"] & \\ & \mathrm{Fun}^{\mathrm{PS}}(\mathcal{I}_p,\mathcal{E}) \arrow[rr, "\pi_!"] & & \mathrm{Fun}(\mathcal{J}^{\mathrm{set}},\mathcal{E}) \end{tikzcd}$$

are commutative. Hence, all faces are commutative. We thus obtain a commutative square

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{I}^{\mathrm{set}},\mathcal{E}) \arrow[r] \arrow[d, "i_{\mathcal{I}!}"] & \mathrm{Fun}(\mathcal{I}^{\mathrm{set}},\mathcal{E}) \times_{\mathrm{Fun}(\mathcal{J}^{\mathrm{set}},\mathcal{E})} \mathrm{Fun}(\mathcal{J}^{\mathrm{set}},\mathcal{E}) \arrow[d] \\ \mathrm{Fun}^{\mathrm{PS}}(\mathcal{I},\mathcal{E}) \arrow[r] & \mathrm{Fun}^{\mathrm{PS}}(\mathcal{I}_p,\mathcal{E}) \times_{\mathrm{Fun}(\mathcal{J}^{\mathrm{set}},\mathcal{E})} \mathrm{Fun}^{\mathrm{PS}}(\mathcal{J},\mathcal{E}) \end{tikzcd}$$

Since $i_{\mathcal{I}_p!}\colon \mathrm{Fun}(\mathcal{I}^{\mathrm{set}},\mathcal{E}) \to \mathrm{Fun}^{\mathrm{PS}}(\mathcal{I}_p,\mathcal{E})$ and $i_{\mathcal{J}!}\colon \mathrm{Fun}(\mathcal{J}^{\mathrm{set}},\mathcal{E}) \to \mathrm{Fun}^{\mathrm{PS}}(\mathcal{J},\mathcal{E})$ are essentially surjective by definition, we deduce that so is the right vertical arrow of the above square. Since the top horizontal arrow is an equivalence, the conclusion follows. $\square$

10.3. **Level induction and Stokes detection.**

Construction 10.3.1. Fix an ∞-category $\mathcal{X}$ and let $p\colon \mathcal{I} \to \mathcal{J}$ be a level graduation morphism in $\mathbf{PosFib}_{\mathcal{X}}$. Fix also a presentable stable ∞-category $\mathcal{E}$. We consider the following commutative cube:

$$\begin{tikzcd}
& \mathrm{Fun}(\mathcal{I},\mathcal{E}) \arrow[rr, "p_!"] \arrow[dd, "\mathrm{Gr}_p"] & & \mathrm{Fun}(\mathcal{J},\mathcal{E}) \arrow[dd, "\mathrm{Gr}"] \\
\mathrm{St}_{\mathcal{I},\mathcal{E}} \arrow[ur, hook] \arrow[rr, "p_!"] \arrow[dd, "\mathrm{Gr}_p"] & & \mathrm{St}_{\mathcal{J},\mathcal{E}} \arrow[ur, hook] \arrow[dd, "\mathrm{Gr}"] & \\
& \mathrm{Fun}(\mathcal{I}_p,\mathcal{E}) \arrow[rr, "\pi_!"] & & \mathrm{Fun}(\mathcal{J}^{\mathrm{set}},\mathcal{E}) \\
\mathrm{St}_{\mathcal{I}_p,\mathcal{E}} \arrow[ur, hook] \arrow[rr, "\pi_!"] & & \mathrm{St}_{\mathcal{J}^{\mathrm{set}},\mathcal{E}} \arrow[ur, hook] &
\end{tikzcd}$$

Passing to fiber products on the front and back squares, we obtain the following commutative square:

$$\begin{tikzcd}
\mathrm{St}_{\mathcal{I},\mathcal{E}} \arrow[r, "L_{\mathrm{St}}"] \arrow[d, hook, "i"] & \mathrm{St}_{\mathcal{J},\mathcal{E}} \times_{\mathrm{St}_{\mathcal{J}^{\mathrm{set}},\mathcal{E}}} \mathrm{St}_{\mathcal{I}_p,\mathcal{E}} \arrow[d, hook, "j"] \\
\mathrm{Fun}(\mathcal{J},\mathcal{E}) \arrow[r, "L_{\mathrm{Fil}}"] & \mathrm{Fun}(\mathcal{J},\mathcal{E}) \times_{\mathrm{Fun}(\mathcal{J}^{\mathrm{set}},\mathcal{E})} \mathrm{Fun}(\mathcal{I}_p,\mathcal{E}) \ .
\end{tikzcd} \tag{10.3.2}$$

Since

$$\mathcal{C} \coloneqq \mathrm{Fun}(\mathcal{J},\mathcal{E}) \times_{\mathrm{Fun}(\mathcal{J}^{\mathrm{set}},\mathcal{E})} \mathrm{Fun}(\mathcal{I}_p,\mathcal{E})$$

is a finite limit in $\mathbf{Cat}_\infty$ whose transitions functors commute with filtered colimits, filtered colimits in $\mathcal{C}$ are computed objectwise. Since $\mathcal{E}$ is stable, we deduce that colimits in $\mathcal{C}$ are computed objectwise. Hence, since $p_!\colon \mathrm{Fun}(\mathcal{I},\mathcal{E}) \to \mathrm{Fun}(\mathcal{J},\mathcal{E})$ and $\mathrm{Gr}_p\colon \mathrm{Fun}(\mathcal{I},\mathcal{E}) \to \mathrm{Fun}(\mathcal{I}_p,\mathcal{E})$ commute with colimits, so does L_{Fil}. Thus, L_{Fil} admits a right adjoint

$$R_{\mathrm{Fil}}\colon \mathrm{Fun}(\mathcal{J},\mathcal{E}) \times_{\mathrm{Fun}(\mathcal{J}^{\mathrm{set}},\mathcal{E})} \mathrm{Fun}(\mathcal{I}_p,\mathcal{E}) \to \mathrm{Fun}(\mathcal{I},\mathcal{E}) \ .$$

Remark 10.3.3. By abstract nonsense, R_{Fil} sends a triple $G = (F_1, F_2, \alpha)$ to the pullback square

$$\begin{tikzcd}
R_{\mathrm{Fil}}(G) \arrow[r] \arrow[d] & p^*(F_1) \arrow[d] \\
\mathrm{Gr}_p^*(F_2) \arrow[r, "\alpha"] & \mathrm{Gr}_p^* \pi^* \mathrm{Gr}(F_1)
\end{tikzcd}$$

in $\mathrm{Fun}(\mathcal{I},\mathcal{E})$.

From Theorem 10.2.1, the functor L_{St} in (10.3.2) is an equivalence. Let R_{St} be an inverse. Then for every $G \in \mathrm{St}_{\mathcal{J},\mathcal{E}} \times_{\mathrm{St}_{\mathcal{J}^{\mathrm{set}},\mathcal{E}}} \mathrm{St}_{\mathcal{I}_p,\mathcal{E}}$, the chain of equivalences

$$\begin{aligned}
\mathrm{Map}(R_{\mathrm{St}}(G), R_{\mathrm{Fil}}(G)) &\simeq \mathrm{Map}(G, L_{\mathrm{Fil}}(R_{\mathrm{Fil}}(G))) && \text{By Proposition 10.1.6} \\
&\simeq \mathrm{Map}(R_{\mathrm{Fil}}(G), R_{\mathrm{Fil}}(G))
\end{aligned}$$

gives rise to a canonical morphism

$$R_{\mathrm{St}}(G) \to R_{\mathrm{Fil}}(G) \ . \tag{10.3.4}$$

Proposition 10.3.5. *Let $\mathcal{X}$ be an ∞-category. Let $p\colon \mathcal{I} \to \mathcal{J}$ be a level graduation morphism in $\mathbf{PosFib}^f$ over $\mathcal{X}$. Let $\mathcal{E}$ be a presentable stable ∞-category. Let $F\colon \mathcal{I} \to \mathcal{E}$ be a functor. Then the following are equivalent :*

(1) F *is a Stokes functor.*

(2) $\mathrm{Gr}_p(F)\colon \mathcal{I}_p \to \mathcal{E}$ *and* $p_!(F)\colon \mathcal{J} \to \mathcal{E}$ *are Stokes functors.*

Proof. That (1) implies (2) follows from Corollary 8.3.4 and Proposition 9.4.9. Assume that (2) holds. Then $L_{\mathrm{Fil}}(F)$ lies in $\mathrm{St}_{\mathcal{I},\mathcal{E}} \times_{\mathrm{St}_{\mathcal{J}^{\mathrm{set}},\mathcal{E}}} \mathrm{St}_{\mathcal{I}_p,\mathcal{E}}$. From (10.3.4) applied to $G \coloneqq L_{\mathrm{Fil}}(F)$, there is a zig-zag

$$R_{\mathrm{St}}(L_{\mathrm{Fil}}(F)) \to R_{\mathrm{Fil}}(L_{\mathrm{Fil}}(F)) \leftarrow F$$

whose right arrow is an equivalence in virtue of Proposition 10.1.6. Hence, there is a canonical morphism

$$\alpha\colon R_{\mathrm{St}}(L_{\mathrm{Fil}}(F)) \to F\ .$$

Since $R_{\mathrm{St}}(L_{\mathrm{Fil}}(F))$ is a Stokes functor, we are left to show that α is an equivalence. Since $\mathrm{Gr}_p(\alpha)\colon \mathrm{Gr}_p(R_{\mathrm{St}}(L_{\mathrm{Fil}}(F))) \to \mathrm{Gr}_p(F)$ identifies canonically with the identity of $\mathrm{Gr}_p(F)$, we conclude from Proposition 9.1.16 by conservativity of Gr_p. □

Part 3. Geometric aspects

In this part we approach the study of Stokes functors in the geometric setting. In Section 11.3 we introduce the fundamental notion of elementarity and its variants. Working under the assumption of the existence of ramified piecewise elementary level structures, we establish the main categorical and geometrical properties of the ∞-category of Stokes functors: presentability and stability (see Theorem 12.1.1), non-commutative smoothness (see Theorem 12.3.5) and the representability of the derived moduli stack of Stokes structures (see Theorem 13.1.4). In Section 15 we develop a criterion to test the elementarity of a Stokes stratified space, which is inspired by [36, Proposition 3.16] and relates the elementarity property with the geometry of the Stokes loci. Finally, in Section 16 we analyze the Stokes stratified spaces arising from the study of flat bundles, and establish the existence of ramified piecewise linear level structures for them.

11. Stokes analytic stratified spaces

We start deepening our analysis of the category **StStrat** of Stokes (analytic) stratified spaces introduced in Section 4, and introducing the key notion of elementary morphisms.

11.1. Functorialities of Stokes stratified spaces.

Recall from Definition 4.1.1 and Remark 4.1.5 that a Stokes stratified space is a triple $(X, P, \mathcal{I})$ where (X, P) is an exodromic stratified space and $\mathcal{I} \to \Pi_\infty(X, P)$ is a cocartesian fibration in posets.

Definition 11.1.1. If $\mathcal{C} \subset \mathrm{Mor}(\mathbf{ExStrat})$ is a class of morphisms, we say that a morphism $(X, P, \mathcal{I}) \to (Y, Q, \mathcal{J})$ in **StStrat** lies in $\mathcal{C}$ if the induced morphism of analytic stratified spaces $(X, P) \to (Y, Q)$ lies in $\mathcal{C}$.

Example 11.1.2. The most relevant classes for our purposes are those of proper morphisms, refinements and Galois cover.

Fix a presentable ∞-category $\mathcal{E}$. Recall from Definitions 5.1.2 and 5.2.3 that to every Stokes stratified space $(X, P, \mathcal{I})$ we can attach two P-hyperconstructible sheaves with values in $\mathbf{Cat}_\infty$:

$$\mathfrak{Fil}_{\mathcal{I},\mathcal{E}}, \mathfrak{St}_{\mathcal{I},\mathcal{E}} \in \mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathbf{Cat}_\infty) \ .$$

Construction 11.1.3. Let $f\colon (Y, Q, \mathcal{J}) \to (X, P, \mathcal{I})$ be a morphism in **StStrat**. Recall that f amounts to the datum of a morphism of stratified spaces $f\colon (Y, Q) \to (X, P)$ and a commutative diagram

$$\begin{tikzcd}
\mathcal{I} \arrow[d] & \mathcal{I}_Y \arrow[l, "u_f"'] \arrow[r, "v_f"] \arrow[d] & \mathcal{J} \arrow[dl] \\
\Pi_\infty(X,P) & \Pi_\infty(Y,Q) \arrow[l, "\Pi_\infty(f)"'] &
\end{tikzcd}$$

where the square is cartesian. Applying Corollary 8.3.4 yields the following commutative diagram

$$\begin{tikzcd}
\exp^{\mathrm{PS}}_{\mathcal{E}}(\mathcal{I}/\Pi_\infty(X,P)) \arrow[d, hook] & \exp^{\mathrm{PS}}_{\mathcal{E}}(\mathcal{I}_Y/\Pi_\infty(Y,Q)) \arrow[l, "\mathcal{E}^{u_f}"'] \arrow[r, "\mathcal{E}^{v_f}_!"] \arrow[d, hook] & \exp^{\mathrm{PS}}_{\mathcal{E}}(\mathcal{J}/\Pi_\infty(Y,Q)) \arrow[d, hook] \\
\exp_{\mathcal{E}}(\mathcal{I}/\Pi_\infty(X,P)) \arrow[d] & \exp_{\mathcal{E}}(\mathcal{I}_Y/\Pi_\infty(Y,Q)) \arrow[l, "\mathcal{E}^{u_f}"'] \arrow[r, "\mathcal{E}^{v_f}_!"] \arrow[d] & \exp_{\mathcal{E}}(\mathcal{J}/\Pi_\infty(Y,Q)) \arrow[dl] \\
\Pi_\infty(X,P) & \Pi_\infty(Y,Q) \arrow[l, "\Pi_\infty(f)"'] &
\end{tikzcd} \tag{11.1.4}$$

The functoriality of the exodromy equivalence with coefficients in $\mathbf{Pr}^{\mathrm{L}}$ recalled in Remark 2.3.7 shows that the middle row zig-zag induces transformations

$$\mathfrak{u}_f^*\colon f^{*,\mathrm{hyp}}(\mathfrak{Fil}_{\mathcal{I},\mathcal{E}}) \to \mathfrak{Fil}_{\mathcal{I}_Y,\mathcal{E}} \qquad \text{and} \qquad \mathfrak{v}_{f,!}\colon \mathfrak{Fil}_{\mathcal{I}_Y,\mathcal{E}} \to \mathfrak{Fil}_{\mathcal{J},\mathcal{E}}$$

in $\mathrm{Cons}_Q^{\mathrm{hyp}}(Y;\mathbf{Pr}^{\mathrm{L}})$. Similarly the functoriality of the exodromy equivalence with coefficients in $\mathbf{CAT}_\infty$ recalled in Recollection 2.3.5 shows that the top row zig-zag induces transformations

$$\mathfrak{u}_f^*\colon f^{*,\mathrm{hyp}}(\mathfrak{St}_{\mathcal{I},\mathcal{E}}) \to \mathfrak{St}_{\mathcal{I}_Y,\mathcal{E}} \qquad \text{and} \qquad \mathfrak{v}_{f,!}\colon \mathfrak{St}_{\mathcal{I}_Y,\mathcal{E}} \to \mathfrak{St}_{\mathcal{J},\mathcal{E}}$$

in $\mathrm{Cons}_Q^{\mathrm{hyp}}(Y;\mathbf{CAT}_\infty)$. Notice that the commutativity of (11.1.4) shows that these natural transformation are compatible with the natural inclusion of $\mathfrak{St}_{(-),\mathcal{E}}$ into $\mathfrak{Fil}_{(-),\mathcal{E}}$.

Proposition 11.1.5. *Let $f\colon (Y,Q,\mathcal{J}) \to (X,P,\mathcal{I})$ be a morphism in* **StStrat** *(see Remark 4.1.6). Then the canonical morphisms*

$$\mathfrak{u}_f^*\colon f^{*,\mathrm{hyp}}(\mathfrak{Fil}_{\mathcal{I},\mathcal{E}}) \to \mathfrak{Fil}_{\mathcal{I}_Y,\mathcal{E}} \qquad \textit{and} \qquad \mathfrak{u}_f^*\colon f^{*,\mathrm{hyp}}(\mathfrak{St}_{\mathcal{I},\mathcal{E}}) \to \mathfrak{St}_{\mathcal{I}_Y,\mathcal{E}}$$

are equivalences. If in addition f is cartesian, then the morphisms

$$\mathfrak{v}_{f,!}\colon \mathfrak{Fil}_{\mathcal{I}_Y,\mathcal{E}} \to \mathfrak{Fil}_{\mathcal{J},\mathcal{E}} \qquad \textit{and} \qquad \mathfrak{v}_{f,!}\colon \mathfrak{St}_{\mathcal{I}_Y,\mathcal{E}} \to \mathfrak{St}_{\mathcal{J},\mathcal{E}}$$

are equivalences.

Proof. Since the exodromy equivalence with coefficients in $\mathbf{Pr}^{\mathrm{L}}$ and in $\mathbf{CAT}_\infty$ is functorial by Recollection 2.3.5 and Remark 2.3.7, the first statement follows directly from the fact that the left squares in (11.1.4) are pullback, see Corollary 8.3.4. The second statement follows from the functoriality of $\exp_{\mathcal{E}}$, since when f is cartesian $v_f\colon \mathcal{I}_Y \to \mathcal{J}$ is itself an equivalence. □

Corollary 11.1.6. *Let $(X,P,\mathcal{I}) \in$* **StStrat**. *For every $x \in X$, the stalk of $\mathfrak{St}_{\mathcal{I},\mathcal{E}}$ at x is canonically identified with $\mathrm{St}_{\mathcal{I}_x,\mathcal{E}}$, i.e. with the essential image of $i_{\mathcal{I}_x,!}\colon \mathrm{Fun}(\mathcal{I}_x^{\mathrm{set}},\mathcal{E}) \to \mathrm{Fun}(\mathcal{I}_x,\mathcal{E})$.*

By design, $\mathfrak{St}_{\mathcal{I},\mathcal{E}}$ satisfies hyperdescent. The next proposition shows that actually more is true.

Proposition 11.1.7. *Let $(X,P,\mathcal{I}) \in$* **StStrat** *and let $\mathcal{E}$ be a presentable ∞-category. Then, the following holds:*

(1) for every étale hypercover $U_\bullet$ of X such that (U_n,P) is exodromic for every $[n] \in \mathbf{\Delta}_s$, the canonical functor

$$\mathrm{St}_{\mathcal{I},\mathcal{E}} \to \lim_{[n]\in\mathbf{\Delta}_s^{\mathrm{op}}} \mathrm{St}_{\mathcal{I}_{U_n},\mathcal{E}}$$

is an equivalence.

(2) If furthermore $\mathcal{E}$ is stable and $(U_n,P,\mathcal{I}_{U_n})$ is $\mathcal{E}$-bireflexive for every $[n] \in \mathbf{\Delta}_s$, then so is $(X,P,\mathcal{I})$ and the above limit can be computed in $\mathbf{Pr}^{\mathrm{L,R}}$.

(3) If furthermore $(U_n,P,\mathcal{I}_{U_n})$ is stably universal for every $[n] \in \mathbf{\Delta}_s$, then so is $(X,P,\mathcal{I})$.

Proof. By Proposition 2.3.13, we know that

$$\operatorname{colim} \Pi_\infty(U_n,P) \to \Pi_\infty(X,P)$$

is an equivalence. Then, (1) follows from Proposition 8.5.1. Item (2) is an immediate consequence of Corollary 8.5.3. Item (3) follows from Proposition 8.6.7. □

Notation 11.1.8. Let $f\colon (Y,Q,\mathcal{J}) \to (X,P,\mathcal{I})$ be a morphism in **StStrat**. Under the adjunction $f^{*,\mathrm{hyp}} \dashv f_*$, the morphisms $\mathfrak{u}_f^*$ introduced in Construction 11.1.3 determine transformations

$$\mathfrak{f}^*\colon \mathfrak{Fil}_{\mathcal{I},\mathcal{E}} \to f_*(\mathfrak{Fil}_{\mathcal{I}_Y,\mathcal{E}}) \qquad \text{and} \qquad \mathfrak{f}^*\colon \mathfrak{St}_{\mathcal{I},\mathcal{E}} \to f_*(\mathfrak{St}_{\mathcal{I}_Y,\mathcal{E}})$$

in $\mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathbf{CAT}_\infty)$. When f is a cartesian morphism Proposition 11.1.5 shows that we can rewrite the above morphism as

$$\mathfrak{f}^*\colon \mathfrak{Fil}_{\mathcal{I},\mathcal{E}} \to f_*(\mathfrak{Fil}_{\mathcal{J},\mathcal{E}}) \qquad \text{and} \qquad \mathfrak{f}^*\colon \mathfrak{St}_{\mathcal{I},\mathcal{E}} \to f_*(\mathfrak{St}_{\mathcal{J},\mathcal{E}})\ . \tag{11.1.9}$$

When in addition f is a cartesian refinement, the underlying morphism of topological spaces is the identity of X, and therefore $\mathfrak{f}^*$ simply becomes a transformation

$$\mathfrak{f}^* \colon \mathfrak{Fil}_{\mathcal{I},\mathcal{E}} \to \mathfrak{Fil}_{\mathcal{J},\mathcal{E}} \qquad \text{and} \qquad \mathfrak{f}^* \colon \mathfrak{St}_{\mathcal{I},\mathcal{E}} \to \mathfrak{St}_{\mathcal{J},\mathcal{E}} \ .$$

In either case, passing to global sections and committing a slight abuse of notation, we denote by

$$f^* \colon \mathrm{Fil}^{\mathrm{co}}_{\mathcal{I},\mathcal{E}} \to \mathrm{Fil}^{\mathrm{co}}_{\mathcal{J},\mathcal{E}} \qquad \text{and} \qquad f^* \colon \mathrm{St}_{\mathcal{I},\mathcal{E}} \to \mathrm{St}_{\mathcal{J},\mathcal{E}}$$

the induced functors.

Proposition 11.1.10. *Let $f \colon (X, Q, \mathcal{J}) \to (X, P, \mathcal{I})$ be a cartesian refinement in* **StStrat***. Then, the natural transformations*

$$\mathfrak{f}^* \colon \mathfrak{Fil}_{\mathcal{I},\mathcal{E}} \to \mathfrak{Fil}_{\mathcal{J},\mathcal{E}} \qquad \textit{and} \qquad \mathfrak{f}^* \colon \mathfrak{St}_{\mathcal{I},\mathcal{E}} \to \mathfrak{St}_{\mathcal{J},\mathcal{E}}$$

introduced in Notation 11.1.8 are equivalences in $\mathrm{Cons}_Q(X; \mathbf{CAT}_\infty)$.

Proof. It suffices to check on stalks. Fix a point $x \in X$. The functoriality of the exodromy equivalence paired with Corollary 6.1.6 supplies a canonical identification of

$$\mathfrak{f}^*_x \colon (\mathfrak{Fil}_{\mathcal{I},\mathcal{E}})_x \to (\mathfrak{Fil}_{\mathcal{J},\mathcal{E}})_x$$

with

$$f^*_x \colon \mathrm{Fun}(\mathcal{I}_x, \mathcal{E}) \to \mathrm{Fun}(\mathcal{J}_x, \mathcal{E}) \ .$$

Since f is cartesian, $f_x \colon \mathcal{I}_x \to \mathcal{J}_x$ is an equivalence, so the same goes for (f^*_x and hence for) $\mathfrak{f}^*_x$. Using Corollary 11.1.6 as a starting point, the same reasoning shows that

$$\mathfrak{f}^*_x \colon (\mathfrak{St}_{\mathcal{I},\mathcal{E}})_x \to (\mathfrak{St}_{\mathcal{J},\mathcal{E}})_x$$

is an equivalence. □

Observation 11.1.11. Let $f \colon (X, Q, \mathcal{J}) \to (X, P, \mathcal{I})$ be a cartesian refinement in **StStrat**. For every open W in X, write $f_W \colon (W, Q, \mathcal{J}_W) \to (W, P, \mathcal{I}_W)$ for the induced cartesian refinement. Unraveling the definitions and using Proposition 6.1.2-(1), we identify the global sections over W of

$$\mathfrak{f}^*_W \colon \mathfrak{Fil}_{\mathcal{I}_W,\mathcal{E}} \to \mathfrak{Fil}_{\mathcal{J}_W,\mathcal{E}}$$

with

$$u^*_{f_W} \colon \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{I}_W, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{J}_W, \mathcal{E}) \ .$$

Combining Proposition 2.3.8 and Proposition 7.7.3, we see that $u^*_{f_W}$ is an equivalence, and that its inverse is given by the left Kan extension

$$u_{f_W,!} \colon \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{I}_W, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{J}_W, \mathcal{E}) \ .$$

Since each $u_{f_W,!}$ is an equivalence, we deduce that they can all be glued together to define a natural transformation

$$\mathfrak{f}_! \colon \mathfrak{Fil}_{\mathcal{J},\mathcal{E}} \to \mathfrak{Fil}_{\mathcal{I},\mathcal{E}}$$

in $\mathrm{Cons}^{\mathrm{hyp}}_Q(X; \mathbf{Pr}^{\mathrm{L}})$. Passing to global sections and committing a slight abuse of notation, we denote by

$$f_! \colon \mathrm{Fil}^{\mathrm{co}}_{\mathcal{J},\mathcal{E}} \to \mathrm{Fil}^{\mathrm{co}}_{\mathcal{I},\mathcal{E}}$$

the induced functor.

Corollary 11.1.12. *Let $f\colon (X,Q,\mathcal{J}) \to (X,P,\mathcal{I})$ be a cartesian refinement in* **StStrat**. *Then the natural transformation $\mathfrak{f}_! \colon \mathfrak{Fil}_{\mathcal{J},\mathcal{E}} \to \mathfrak{Fil}_{\mathcal{I},\mathcal{E}}$ of Observation 11.1.11 induces a transformation*

$$\mathfrak{f}_! \colon \mathfrak{St}_{\mathcal{J},\mathcal{E}} \to \mathfrak{St}_{\mathcal{I},\mathcal{E}}$$

which is an inverse to $\mathfrak{f}^* \colon \mathfrak{St}_{\mathcal{I},\mathcal{E}} \to \mathfrak{St}_{\mathcal{J},\mathcal{E}}$. In particular, passing to global sections, the adjunction

$$f_! \colon \mathrm{St}_{\mathcal{J},\mathcal{E}} \leftrightarrows \mathrm{St}_{\mathcal{I},\mathcal{E}} \colon f^*$$

is an equivalence of ∞-categories.

Proof. Unraveling the definitions, we see that it is enough to check that for each point $x \in X$, the functor

$$(\mathfrak{f}_!)_x \colon (\mathfrak{Fil}_{\mathcal{J},\mathcal{E}})_x \to (\mathfrak{Fil}_{\mathcal{I},\mathcal{E}})_x$$

takes $(\mathfrak{St}_{\mathcal{J},\mathcal{E}})_x$ to $(\mathfrak{St}_{\mathcal{I},\mathcal{E}})_x$. By construction $(\mathfrak{f}_!)_x$ is an inverse to $\mathfrak{f}_x^*$, which as in the proof of Proposition 11.1.10 is canonically identified with

$$f_x^* \colon \mathrm{Fun}(\mathcal{J}_x, \mathcal{E}) \to \mathrm{Fun}(\mathcal{I}_x, \mathcal{E}) \ ,$$

where f_x is the induced morphism $\mathcal{I}_x \to \mathcal{J}_x$. We therefore find a canonical identification of $(\mathfrak{f}_!)_x$ with the left Kan extension $f_{x,!}$. From, Corollary 11.1.6 we are thus left to check that $f_{x,!}$ takes $\mathrm{St}_{\mathcal{J}_x,\mathcal{E}}$ to $\mathrm{St}_{\mathcal{I}_x,\mathcal{E}}$. Since $f_x \colon \mathcal{J}_x \to \mathcal{I}_x$ is an equivalence, the conclusion follows. □

Corollary 11.1.13. *Let (X,P) be an exodromic stratified space, considered as a Stokes stratified space $(X,P,\Pi_\infty(X,P))$. Then, $\mathfrak{St}_{\Pi_\infty(X,P),\mathcal{E}}$ is canonically equivalent to $\mathfrak{Loc}_{X,\mathcal{E}}$ (see Definition 5.1.6).*

Proof. Observe that $(X,P,\Pi_\infty(X,P)) \to (X,*,\Pi_\infty(X))$ is a cartesian refinement in **StStrat**. From Proposition 11.1.10, we deduce that $\mathfrak{St}_{\Pi_\infty(X,P),\mathcal{E}}$ is canonically equivalent to $\mathfrak{St}_{\Pi_\infty(X),\mathcal{E}}$. The punctually split condition being empty in that case, $\mathfrak{St}_{\Pi_\infty(X),\mathcal{E}}$ is canonically equivalent to $\mathfrak{Fil}_{\Pi_\infty(X),\mathcal{E}}$. Then, the conclusion follows from Proposition 5.1.7. □

Corollary 11.1.14. *Let $(X,P,\mathcal{I})$ be a Stokes stratified space such that $\mathcal{I} \to \Pi_\infty(X,P)$ is locally constant in the sense of Definition 19.1.4. Then, $\mathfrak{St}_{\mathcal{I},\mathcal{E}}$ is locally hyperconstant on X.*

Proof. By definition, the straightening of $\mathcal{I} \to \Pi_\infty(X,P)$ sends every exit path to an isomorphism of posets. From Proposition 2.3.8, we deduce the existence of a cartesian refinement $(X,P,\mathcal{I}) \to (X,*,\mathcal{J})$. Hence, Proposition 11.1.10 ensures that $\mathfrak{St}_{\mathcal{I},\mathcal{E}}$ is canonically equivalent to $\mathfrak{St}_{\mathcal{J},\mathcal{E}}$. By construction, $\mathfrak{St}_{\mathcal{J},\mathcal{E}}$ lies in $\mathrm{Loc}(X;\mathbf{Pr}^{\mathrm{L}})$ so the conclusion follows. □

11.2. **Hyperconstructible hypersheaves and tensor product.** Let $(X,P) \in$ **ExStrat** be an exodromic stratified space. Let $\mathcal{E}$ be a presentable ∞-category. Composition with the colimit-preserving functor

$$(-) \otimes \mathcal{E} \colon \mathbf{Pr}^{\mathrm{L}} \to \mathbf{Pr}^{\mathrm{L}}$$

induces a colimit preserving functor

$$\mathrm{Fun}(\Pi_\infty(X,P), \mathbf{Pr}^{\mathrm{L}}) \to \mathrm{Fun}(\Pi_\infty(X,P), \mathbf{Pr}^{\mathrm{L}}) \ .$$

The exodromy equivalence with coefficients in $\mathbf{Pr}^{\mathrm{L}}$ from Remark 2.3.7 allows therefore to define a functor

$$(-) \otimes^{\mathrm{hyp}} \mathcal{E} \colon \mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathbf{Pr}^{\mathrm{L}}) \to \mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathbf{Pr}^{\mathrm{L}})$$

making the diagram

$$\begin{tikzcd} \mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathbf{Pr}^{\mathrm{L}}) \arrow[r, "\sim"] \arrow[d, "(-)\otimes^{\mathrm{hyp}}\mathcal{E}"] & \mathrm{Fun}(\Pi_\infty(X,P), \mathbf{Pr}^{\mathrm{L}}) \arrow[d, "(-)\otimes\mathcal{E}"] \\ \mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathbf{Pr}^{\mathrm{L}}) \arrow[r, "\sim"] & \mathrm{Fun}(\Pi_\infty(X,P), \mathbf{Pr}^{\mathrm{L}}) \end{tikzcd} \tag{11.2.1}$$

commutative.

Notation 11.2.2. There is a natural forgetful functor $\mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathbf{Pr}^{\mathrm{L}}) \to \mathrm{PSh}(X;\mathbf{Pr}^{\mathrm{L}})$, and $(-)\otimes\mathcal{E}$ induces a well defined functor

$$(-)\otimes\mathcal{E}\colon \mathrm{PSh}(X;\mathbf{Pr}^{\mathrm{L}}) \to \mathrm{PSh}(X;\mathbf{Pr}^{\mathrm{L}}) \ .$$

In other words, given $\mathcal{F}\in\mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathbf{Pr}^{\mathrm{L}})$, $\mathcal{F}\otimes\mathcal{E}$ is the presheaf sending an open U of X to $\mathcal{F}(U)\otimes\mathcal{E}$.

Construction 11.2.3. Let $\mathcal{F}\in\mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathbf{Pr}^{\mathrm{L}})$. Let $\mathcal{E}$ be a presentable ∞-category. Unraveling the definitions, we see that for every point $x\in X$, there is a natural equivalence

$$(\mathcal{F}\otimes^{\mathrm{hyp}}\mathcal{E})_x \simeq \mathcal{F}_x\otimes\mathcal{E} \in \mathbf{Pr}^{\mathrm{L}} \ .$$

Fix an open U in X. Then we have a canonical identification

$$(\mathcal{F}\otimes^{\mathrm{hyp}}\mathcal{E})(U) \simeq \lim_{x\in\Pi_\infty(U,P)} \mathcal{F}_x\otimes\mathcal{E} \ ,$$

and in particular we find a natural comparison map

$$\mathcal{F}(U)\otimes\mathcal{E} \to (\mathcal{F}\otimes^{\mathrm{hyp}}\mathcal{E})(U) \ ,$$

which is a particular case of the Beck-Chevalley transformation considered in Lemma 7.5.6. In other words, we obtain a natural transformation

$$\mathcal{F}\otimes\mathcal{E} \to \mathcal{F}\otimes^{\mathrm{hyp}}\mathcal{E} \tag{11.2.4}$$

Notation 11.2.5. We denote $\mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathbf{Pr}^{\mathrm{L,R}})$ as the full-subcategory of $\mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathbf{Pr}^{\mathrm{L}})$ of hyperconstructible hypersheaves corresponding to objects in $\mathrm{Fun}(\Pi_\infty(X,P),\mathbf{Pr}^{\mathrm{L,R}})$ through the exodromy equivalence (11.2.1).

Lemma 11.2.6. *Let $\mathcal{F}\in\mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathbf{Pr}^{\mathrm{L,R}})$. Let $\mathcal{E}$ be a presentable ∞-category. Then the comparison map* (11.2.4) *is an equivalence, and in particular the presheaf $\mathcal{F}\otimes\mathcal{E}$ is a hypersheaf.*

Proof. This is a particular case of Lemma 7.5.6. We can argue directly as follows: it is enough to show that for every open subset U of X, the canonical map

$$\Big(\lim_{x\in\Pi_\infty(U,P)}\mathcal{F}_x\Big)\otimes\mathcal{E} \to \lim_{x\in\Pi_\infty(U,P)} F_x\otimes\mathcal{E}$$

is an equivalence, and this follows directly from Lemma 7.5.6. ☐

Corollary 11.2.7. *Let $(X,P,\mathcal{I})$ be a Stokes stratified space. Let $\mathcal{E}$ and $\mathcal{E}'$ be presentable ∞-categories.*

(1) The canonical comparison map

$$\mathfrak{Fil}_{\mathcal{I},\mathcal{E}}\otimes^{\mathrm{hyp}}\mathcal{E}' \to \mathfrak{Fil}_{\mathcal{I},\mathcal{E}\otimes\mathcal{E}'}$$

is an equivalence.

(2) Assume that $(X,P,\mathcal{I})$ is a Stokes stratified space in finite posets and that $\mathcal{E}$ and $\mathcal{E}'$ are presentable stable. Then $\mathfrak{Fil}_{\mathcal{I},\mathcal{E}}$ belongs to $\mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathbf{Pr}^{\mathrm{L,R}})$ and the comparison map

$$\mathfrak{Fil}_{\mathcal{I},\mathcal{E}}\otimes\mathcal{E}' \to \mathfrak{Fil}_{\mathcal{I},\mathcal{E}\otimes\mathcal{E}'}$$

is an equivalence.

Proof. The first point follows from Remark 3.3.1. The second point follows combining Lemma 7.3.5 with Lemma 11.2.6. ☐

We conclude by recording the following handy sufficient condition ensuring that a categorical sheaf $\mathcal{F}\in\mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathbf{Pr}^{\mathrm{L}})$ belongs to $\mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathbf{Pr}^{\mathrm{L,R}})$.

Lemma 11.2.8. *Let (X, P) be a subanalytic stratified space. Let $\mathcal{F} \in \mathrm{Cons}_P(X; \mathbf{Pr}^{\mathrm{L}})$ such that for every open subsets $U \subset V$, the functor $\mathcal{F}(V) \to \mathcal{F}(U)$ is a left and right adjoint. Then, $\mathcal{F}$ lies in $\mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathbf{Pr}^{\mathrm{L,R}})$.*

Proof. Let $F\colon \Pi_\infty(X, P) \to \mathbf{Pr}^{\mathrm{L}}$ be the functor corresponding to $\mathcal{F}$ via the exodromy equivalence (2.3.6). Let $\gamma\colon x \to y$ be a morphism in $\Pi_\infty(X, P)$. By Proposition 2.5.6, choose an open neighbourhood V of x such that x is initial in $\Pi_\infty(V, P)$. At the cost of writing γ as the composition of a small enough exit-path followed by an equivalence, we can suppose that γ lies in V. Let $U \subset V$ such that y is initial in $\Pi_\infty(U, P)$. Then, the vertical arrows of the following commutative diagram of $\mathcal{E}$

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r] \arrow[d] & \mathcal{F}(U) \arrow[d] \\ F(x) \arrow[r, "F(\gamma)"] & F(y) \end{tikzcd}$$

are equivalences. Lemma 11.2.8 thus follows. □

11.3. **Elementarity.** We now introduce a fundamental concept in the study of Stokes stratified spaces: the notion of elementarity and its variants. We start discussing the absolute notion:

Definition 11.3.1 (Absolute elementarity)**.** Let $(X, P, \mathcal{I})$ be a Stokes stratified space. We say that $(X, P, \mathcal{I})$ is:

(1) *elementary* if for every presentable stable ∞-category $\mathcal{E}$, the functor

$$i_{\mathcal{I},!}\colon \mathrm{St}_{\mathcal{I}^{\mathrm{set}},\mathcal{E}} \to \mathrm{St}_{\mathcal{I},\mathcal{E}}$$

is an equivalence;

(2) *locally elementary* if X admits a cover by open subsets U such that $(U, P, \mathcal{I}_U)$ is elementary.

Elementarity is a really strong condition, as the following two examples show:

Example 11.3.2. A poset $\mathcal{I}$ seen as a Stokes stratified space $(*, *, \mathcal{I})$ as in Example 4.1.4 is elementary if and only if $\mathcal{I}$ is discrete. Indeed, if $\mathcal{I}$ is discrete then $i_{\mathcal{I}}\colon \mathcal{I}^{\mathrm{set}} \to \mathcal{I}$ is an isomorphism and therefore the three arrows in the commutative triangle

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{I}^{\mathrm{set}}, \mathcal{E}) \arrow[r] \arrow[dr, "i_{\mathcal{I},!}"'] & \mathrm{St}_{\mathcal{I},\mathcal{E}} \arrow[d, hook] \\ & \mathrm{Fun}(\mathcal{I}, \mathcal{E}) \end{tikzcd}$$

are equivalences. Conversely, assume that $\mathcal{I}$ is elementary. Then the top horizontal arrow is an equivalence, and therefore $i_{\mathcal{I},!}$ is forced to be fully faithful. Fix a non-zero object $E \neq 0$ in $\mathcal{E}$ and assume by contradiction that there exists two elements $a, b \in \mathcal{I}$ satisfying $a < b$. Then

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{I}^{\mathrm{set}},\mathcal{E})}(\mathrm{ev}_{b,!}^{\mathcal{I}^{\mathrm{set}}}(E), \mathrm{ev}_{a,!}^{\mathcal{I}^{\mathrm{set}}}(E)) \simeq 0 \ ,$$

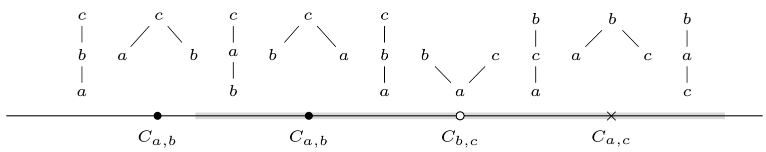
while

$$\begin{aligned} \mathrm{Map}_{\mathrm{Fun}(\mathcal{I},\mathcal{E})}\big(i_{\mathcal{I},!}\mathrm{ev}_{b,!}^{\mathcal{I}^{\mathrm{set}}}(E), i_{\mathcal{I},!}\mathrm{ev}_{a,!}^{\mathcal{I}^{\mathrm{set}}}(E)\big) &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{I},\mathcal{E})}\big(\mathrm{ev}_{b,!}^{\mathcal{I}}(E), \mathrm{ev}_{a,!}^{\mathcal{I}}(E)\big) \\ &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{I},\mathcal{E})}\big(E, \mathrm{ev}_b^{\mathcal{I},*}\mathrm{ev}_{a,!}^{\mathcal{I}}(E)\big) \\ &\simeq \mathrm{Map}_{\mathcal{E}}(E, E) \neq 0 \ , \end{aligned}$$

which contradicts the full faithfulness of $i_{\mathcal{I},!}$.

Example 11.3.3. We consider again the situation of Example 5.2.7. Then the analysis carried out there shows that $(S^1, P, \mathcal{I})$ is not elementary while $(W_1, P, \mathcal{I}_{W_1})$ and $(W_{-1}, P, \mathcal{I}_{W_{-1}})$ are elementary. In other words, $(S^1, P, \mathcal{I})$ is locally elementary.

Example 11.3.4. Take $X = (0,1)$ stratified in four points and take $\mathcal{I}$ the constructible sheaf in posets depicted below:



Here we marked with $C_{\alpha,\beta}$ the Stokes locus for the pair $\{\alpha, \beta\}$. It follows from Theorem 15.2.4 that the shadowed interval is elementary, because it contains exactly one Stokes direction for every possible pair of elements of $\mathcal{I}^{\mathrm{set}} = \{a, b, c\}$. On the other hand, Corollary 15.2.8 shows that the leftmost $C_{a,b}$ cannot have an elementary open neighborhood. In other words, this is an example of a Stokes stratified space which is not locally elementary.

Warning 11.3.5. Let $(X, P, \mathcal{I})$ be a Stokes stratified space. In general, the intersection of two elementary open subsets is no longer elementary: for instance, with the notations of Example 5.2.7, the intersection $W_1 \cap W_{-1}$ is no longer elementary. Also, Example 11.3.2 implies that any point $x \in X$ such that $\mathcal{I}_x$ is not discrete does not have a fundamental system of elementary open neighborhoods. In other words, even when $(X, P, \mathcal{I})$ is locally elementary, the collection of elementary open subsets of X does *not* form a basis for the topology of X.

Let us discuss two variations on Definition 11.3.1. The first one concerns adapting the notion of elementarity to a family of Stokes stratified spaces:

Definition 11.3.6. A morphism $(X, P) \to (Y, Q)$ in **ExStrat** is said to be a *family of exodromic stratified spaces* if for every $y \in Y$ the stratified space (X_y, P) is exodromic.

Notation 11.3.7. Recall from Example 4.1.3 that every exodromic stratified space (Y, Q) gives rise to a Stokes stratified space $(Y, Q, \emptyset)$. We will commit a slight abuse of notation and write (Y, Q) in place of $(Y, Q, \emptyset)$.

Definition 11.3.8. A *family of Stokes stratified spaces* is a morphism

$$f \colon (X, P, \mathcal{I}) \to (Y, Q)$$

in **StStrat** whose underlying morphism $f \colon (X, P) \to (Y, Q)$ is a family of exodromic stratified spaces. We denote the (1-)category of families of Stokes stratified spaces by $\mathbf{FStStrat} \subset \mathbf{StStrat}^{[1]}$.

Example 11.3.9. Let $f \colon (X, P) \to (Y, Q)$ be a morphism of subanalytic stratified spaces. Then for each $y \in Y$, the fiber (X_y, P) is again a subanalytic stratified space, so Remark 2.5.4 guarantees that (X_y, P) is again exodromic. Therefore f is a family of exodromic stratified spaces. In particular, for any Stokes fibration $\mathcal{I}$ on (X, P), the resulting morphism $f \colon (X, P, \mathcal{I}) \to (Y, Q)$ is a family of Stokes analytic stratified spaces.

Definition 11.3.10 (Relative elementarity)**.** Let $f \colon (X, P, \mathcal{I}) \to (Y, Q)$ be a family of Stokes stratified spaces. We say that *f is (locally) elementary at $y \in Y$* if $(X_y, P, \mathcal{I}_y)$ is (locally) elementary. We say that *f is (locally) elementary* if it is (locally) elementary at y for every $y \in Y$.

Remark 11.3.11. It follows from Example 11.3.2 that the empty poset $\mathcal{I} = \emptyset$ is elementary. Thus, a family of Stokes stratified spaces $f \colon (X, P, \mathcal{I}) \to (Y, Q)$ is automatically elementary at every point $y \in Y$ not in the image of f (i.e. for which $X_y = \emptyset$).

Before moving on to the second variation on the notion of elementarity, let us record a couple of important facts concerning relative elementarity. The first is the following easy stability property:

Lemma 11.3.12. *Consider a morphism of families of Stokes stratified spaces*

$$\begin{tikzcd} (Y,Q,\mathcal{J}) \arrow[r, "g"] \arrow[d, "f'"] & (X,P,\mathcal{I}) \arrow[d, "f"] \\ (Y',Q') \arrow[r] & (X',P') \end{tikzcd} \tag{11.3.13}$$

with cartesian horizontal arrows. Consider the following conditions:

(1) The square of stratified spaces underlying (11.3.13) *is a pullback.*

(2) The horizontal arrows are refinements.

Then, in both cases if f is elementary the same goes for f'. In case (2), the converse holds.

Proof. In case (1), the fibers of f' are fibers of f so there is nothing to prove. For (2), let $x \in X'$ and let $\mathcal{E}$ be a presentable stable ∞-category. Then, restricting above x yields a refinement of exodromic stratified spaces $g_x \colon (Y_x, Q) \to (X_x, P)$. Thanks to Corollary 11.1.12 the horizontal arrows in the commutative square

$$\begin{tikzcd} \mathrm{St}_{\mathcal{J}_x^{\mathrm{set}},\mathcal{E}} \arrow[r, "g_{x,!}^{\mathrm{set}}"] \arrow[d, "i_{\mathcal{J}_x,!}"] & \mathrm{St}_{\mathcal{I}_x^{\mathrm{set}},\mathcal{E}} \arrow[d, "i_{\mathcal{I}_x,!}"] \\ \mathrm{St}_{\mathcal{J}_x,\mathcal{E}} \arrow[r, "g_{x,!}"] & \mathrm{St}_{\mathcal{I}_x,\mathcal{E}} \end{tikzcd}$$

are equivalences, so the conclusion follows. $\square$

The second property of relative elementarity is the following important local-to-global principle. An idea that plays a major role subsequent developments is that to establish absolute elementarity of some $(X,P,\mathcal{I})$, it is useful to fiber $(X,P,\mathcal{I})$ over a stratified space (Y,Q), and then establish relative elementarity to apply the following:

Proposition 11.3.14. *Let $f\colon (X,P,\mathcal{I}) \to (Y,Q)$ be an elementary family of Stokes stratified spaces. Assume that the underlying morphism $f\colon X \to Y$ is proper and that at least one of the following conditions hold:*

(1) The induced morphism of ∞-topoi

$$f_* \colon \mathrm{Sh}^{\mathrm{hyp}}(X) \to \mathrm{Sh}^{\mathrm{hyp}}(Y)$$

is proper in the sense of [31, Definition 7.3.1.4].

(2) $f\colon (X,P) \to (Y,Q)$ is a morphism of subanalytic stratified spaces.

Then, $(X,P,\mathcal{I})$ is elementary.

Proof. Let $\mathcal{E}$ be a presentable stable ∞-category. We have to show that

$$i_{\mathcal{I},!} \colon \mathrm{St}_{\mathcal{I}^{\mathrm{set}},\mathcal{E}} \to \mathrm{St}_{\mathcal{I},\mathcal{E}}$$

is an equivalence. To do this, it is enough to show that the morphism

$$f_*(\mathscr{I}_{\mathcal{I},!}) \colon f_*(\mathfrak{St}_{\mathcal{I}^{\mathrm{set}},\mathcal{E}}) \to f_*(\mathfrak{St}_{\mathcal{I},\mathcal{E}})$$

in $\mathrm{Sh}^{\mathrm{hyp}}(Y; \mathbf{Cat}_\infty)$ is an equivalence. This can be done at the level of stalks. Fix therefore $y \in Y$. For every $\mathcal{F} \in \mathrm{Sh}^{\mathrm{hyp}}(X; \mathbf{Cat}_\infty)$, we have a canonical comparison map

$$y^{*,\mathrm{hyp}} f_*(\mathcal{F}) \to \Gamma_{X_y,*}(j_y^{*,\mathrm{hyp}}(\mathcal{F})) \ , \tag{11.3.15}$$

where $j_y \colon X_y \hookrightarrow X$ is the inclusion of the fiber. Notice that Proposition 11.1.5 provides an identifications

$$j_y^{*,\mathrm{hyp}}(\mathfrak{St}_{\mathcal{I},\mathcal{E}}) \simeq \mathfrak{St}_{\mathcal{I}_y,\mathcal{E}} \qquad \text{and} \qquad j_y^{*,\mathrm{hyp}}(\mathfrak{St}_{\mathcal{I}^{\mathrm{set}},\mathcal{E}}) \simeq \mathfrak{St}_{\mathcal{I}_y^{\mathrm{set}},\mathcal{E}} \ ,$$

so the result follows from our elementarity assumption as soon as we know that (11.3.15) is an equivalence for $\mathcal{F} = \mathfrak{St}_{\mathcal{I},\mathcal{E}}$ and for $\mathcal{F} = \mathfrak{St}_{\mathcal{I}^{\mathrm{set}},\mathcal{E}}$. In case (1), since $\mathbf{Cat}_\infty$ is compactly generated, [24, Theorem 0.5] shows that (11.3.15) is an equivalence for every categorical hypersheaf $\mathcal{F} \in \mathrm{Sh}^{\mathrm{hyp}}(X; \mathbf{Cat}_\infty)$. In case (2), Proposition 2.5.10 shows that (11.3.15) is an equivalence for any $\mathcal{F} \in \mathrm{Cons}_P(X; \mathbf{Cat}_\infty)$. So in both cases the conclusion follows. $\square$

Recollection 11.3.16. Let us recall some topological conditions that ensure that assumption (1) in Proposition 11.3.14 are satisfied. Assume that:

(a) X is locally compact and Hausdorff and f is proper;

(b) both X and Y admit an open cover by subsets of finite covering dimensions (see [31, Definition 7.2.3.1]).

Condition (a) ensures via [31, Theorem 7.3.1.16] that the geometric morphism

$$f_* \colon \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$$

is proper. Condition (b) on the other hand guarantees that both $\mathrm{Sh}(X)$ and $\mathrm{Sh}(Y)$ are hypercomplete: combine [31, Theorem 7.2.3.6, Corollary 7.2.1.12 and Remark 6.5.2.22]. Finally, notice that any paracompact and finite dimensional space has finite covering dimension, see for instance [16, Proposition 3.2.2].

We now introduce one final variation on the idea of elementarity in the analytic setting:

Definition 11.3.17 (Absolute piecewise elementarity)**.** Let $(X, P, \mathcal{I})$ be a Stokes analytic stratified space and let $x \in X$ be a point. We say that:

(1) $(X, P, \mathcal{I})$ is *piecewise elementary at x* if there exists a closed subanalytic subset Z containing x such that $(Z, P, \mathcal{I}_Z)$ is elementary;

(2) $(X, P, \mathcal{I})$ is *strongly piecewise elementary at x* if there exists a closed subanalytic neighborhood Z containing x such that $(Z, P, \mathcal{I}_Z)$ is elementary;

We say that $(X, P, \mathcal{I})$ is *(strongly) piecewise elementary* if it is (strongly) piecewise elementary at every point.

Remark 11.3.18. We will see in the next section that piecewise elementarity implies local elementarity: in other words, if one can find a closed subanalytic subset Z containing x such that $(Z, P, \mathcal{I}_Z)$ is elementary, then Z can be spread out to an elementary open neighborhood of x.

Moving to the relative setting, we have:

Definition 11.3.19 (Relative piecewise elementarity)**.** Let $f \colon (X, P, \mathcal{I}) \to (Y, Q)$ be a family of Stokes analytic stratified spaces and let $x \in X$ be a point. We say that:

(1) f is *vertically piecewise elementary at x* if the fiber $(X_{f(x)}, P, \mathcal{I}_{f(x)})$ is piecewise elementary at x;

(2) f is *piecewise elementary at x* if there exists a closed subanalytic subset Z containing x and such that $f|_Z \colon (Z, P, \mathcal{I}_Z) \to (Y, Q)$ is an elementary family of Stokes analytic stratified spaces;

(3) f is *strongly piecewise elementary at x* if there exists a closed subanalytic neighborhood Z of x such that $f|_Z \colon (Z, P, \mathcal{I}_Z) \to (Y, Q)$ is an elementary family of Stokes analytic stratified spaces.

We say that f is *(vertically, strongly) piecewise elementary* if it is (vertically, strongly) piecewise elementary at every point.

Remark 11.3.20. Saying that $f\colon (X, P, \mathcal{I}) \to (Y, Q)$ is vertically piecewise elementary amounts to say that for every $y \in Y$ the fiber $(X_y, P, \mathcal{I}_y)$ is piecewise elementary.

We conclude with a couple of easy facts concerning piecewise elementarity:

Lemma 11.3.21. *Consider a morphism of families of Stokes analytic stratified spaces*

$$\begin{tikzcd} (Y, Q, \mathcal{J}) \arrow[r, "g"] \arrow[d, "f'"] & (X, P, \mathcal{I}) \arrow[d, "f"] \\ (Y', Q') \arrow[r] & (X', P') \end{tikzcd} \tag{11.3.22}$$

with cartesian horizontal arrows. Let $\mathcal{E}$ be a presentable ∞-category. Let $y \in Y$ and put $x = g(y)$. Consider the following conditions :

(1) The square of stratified spaces induced by (11.3.22) is a pull-back.

(2) The horizontal arrows are refinements.

Then, in either case f' is (strongly) piecewise elementary at y if f is (strongly) piecewise elementary at x. In case (2), the converse holds.

Proof. Immediate from Lemma 11.3.12. □

Clearly, if $f\colon (X, P, \mathcal{I}) \to (Y, Q)$ is strongly piecewise elementary it is also piecewise elementary. We also have:

Corollary 11.3.23. *Let $f\colon (X, P, \mathcal{I}) \to (Y, Q)$ be a family of Stokes analytic stratified spaces. If f is piecewise elementary at a point $x \in X$, then it is vertically piecewise elementary at x.*

Proof. Let Z be a closed subanalytic subset containing x such that $(Z, P, \mathcal{I}_Z)$ is elementary. Set $y \coloneqq f(x)$. Then $Z_y \simeq X_y \times_X Z$ is a closed subanalytic subset of X_y, and Lemma 11.3.21 implies that $(Z_y, P, \mathcal{I}_{Z_y})$ is elementary. □

11.4. **Spreading out for Stokes analytic stratified spaces.** The goal is to prove a *spreading out* property for closed subanalytic subset of Stokes analytic stratified spaces that does not change the category of Stokes functors. The proof combines all the functoriality results concerning Stokes functors obtained so far, with the deep results obtained by Thom, Mather, Goresky and Verdier on the local structure of analytic stratified spaces. We will also need terminology and results from the theory of simplicial complexes, for which we refer the reader to Section 2.4.

Theorem 11.4.1 (Spreading out)**.** *Let $(X, P, \mathcal{I})$ be a Stokes analytic stratified space. Let $\mathcal{E}$ be a presentable stable ∞-category. Then any closed subanalytic subset $Z \subset X$ admits a fundamental system of open neighborhoods $i\colon Z \hookrightarrow U$ such that:*

(1) U final at Z (see Definition 2.3.12).

(2) The induction $i_!\colon \mathrm{Fun}(\mathcal{I}_Z, \mathcal{E}) \to \mathrm{Fun}(\mathcal{I}_U, \mathcal{E})$ preserves Stokes functors.

(3) The adjunction $i_! \dashv i^*$ induces an equivalence of ∞-categories between $\mathrm{St}_{\mathcal{I}_Z, \mathcal{E}}$ and $\mathrm{St}_{\mathcal{I}_U, \mathcal{E}}$.

(4) $(Z, P, \mathcal{I}_Z)$ is elementary if and only if $(U, P, \mathcal{I}_U)$ is elementary.

(5) If Z is compact, the open set U can be chosen to be subanalytic.

Proof. Observe that the claim (4) follows from (3) and the commutativity of the following square

$$\begin{tikzcd} \mathrm{St}_{\mathcal{I}_Z^{\mathrm{set}},\mathcal{E}} \arrow[r, "i_!"] \arrow[d, "i_{\mathcal{I}_Z,!}"] & \mathrm{St}_{\mathcal{I}_U^{\mathrm{set}},\mathcal{E}} \arrow[d, "i_{\mathcal{I}_U,!}"] \\ \mathrm{St}_{\mathcal{I}_Z,\mathcal{E}} \arrow[r, "i_!"] & \mathrm{St}_{\mathcal{I}_U,\mathcal{E}} \end{tikzcd} .$$

As a consequence of Proposition 7.6.7 and Lemma 7.6.13, every open subset $U \subset X$ satisfying (1) and (2) automatically satisfies (3). We are thus left to find a fundamental system of open neighborhoods of Z satisfying (1) and (2).

We first observe that to construct such open neighborhoods we can replace $(X, P, \mathcal{I})$ by any cartesian refinement. Indeed, let

$$r \colon (Y, Q, \mathcal{J}) \to (X, P, \mathcal{I})$$

be a cartesian refinement in **ExStrat** and set

$$T \coloneqq Z \times_X Y .$$

Let V be an open neighborhood of T inside Y. Since $r \colon Y \to X$ is a homeomorphism, $U \coloneqq r(V)$ is an open neighborhood of T inside X. We obtain the following commutative diagram in **ExStrat**:

$$\begin{tikzcd} (T, Q) \arrow[r, "r|_T"] \arrow[d, "j"] & (Z, P) \arrow[d, "i"] \\ (V, Q) \arrow[r, "r|_V"] & (U, P) \end{tikzcd} .$$

Passing to the stratified homotopy types, Proposition 2.3.8 shows that the horizontal maps becomes localizations, and hence final maps. Thus, [31, Proposition 4.1.1.3-(2)] implies that if V is final at T the U is final at Z. Besides, Corollary 11.1.12 shows that both

$$(r|_T)_! \colon \mathrm{Fun}(\mathcal{I}_T, \mathcal{E}) \to \mathrm{Fun}(\mathcal{I}_Z, \mathcal{E}) \qquad \text{and} \qquad (r|_T)_! \colon \mathrm{Fun}(\mathcal{I}_T^{\mathrm{set}}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{I}_Z^{\mathrm{set}}, \mathcal{E})$$

preserves the full subcategories of Stokes functors and that the induced morphisms

$$(r|_T)_! \colon \mathrm{St}_{\mathcal{I}_T,\mathcal{E}} \to \mathrm{St}_{\mathcal{I}_Z,\mathcal{E}} \qquad \text{and} \qquad (r_T)_! \colon \mathrm{St}_{\mathcal{I}_T^{\mathrm{set}},\mathcal{E}} \to \mathrm{St}_{\mathcal{I}_Z^{\mathrm{set}},\mathcal{E}}$$

are equivalences of ∞-categories, and similarly for $r|_V$ in place of $r|_T$. It follows that if $j_! \colon \mathrm{Fun}(\mathcal{I}_T, \mathcal{E}) \to \mathrm{Fun}(\mathcal{I}_V, \mathcal{E})$ preserves Stokes functors, then so does $i_!$.

Using [53, Théorème 2.2] we can refine the stratification (X, P) to a Whitney stratification (X, Q) such that Z is union of strata of (X, Q). By [22, Theorem §3], (X, Q) admits a locally finite triangulation. Thus, using the notations from Section 2.4, we can replace (X, Q) by the geometric realization $(|K|, F)$ of a simplicial complex $K = (V, F)$ and we can furthermore assume that (Z, Q) corresponds to the geometric realization $(|S|, F_S)$ a simplicial subcomplex $S = (V_S, F_S)$ of K. At the cost of replacing K by its barycentric subdivision, we can suppose that S is full in K. Fix a real number $0 < \varepsilon \leqslant 1$ and define

$$U_{S,K}^{\varepsilon} \coloneqq \left\{ w \colon V \to [0,1] \mid \mathrm{supp}(w) \cap V_S \neq \emptyset \text{ and } \sum_{v \in V \smallsetminus V_S} w(v) < \varepsilon \right\} .$$

Then $\{U_{S,K}^{\varepsilon}\}_{0<\varepsilon<1}$ is a fundamental system of open neighborhoods of $|S|$ inside $|K|$. We claim that each $U_{S,K}^{\varepsilon}$ satisfies conditions (1) and (2). Notice that for $\varepsilon < \varepsilon'$ the inclusion

$$(U_{S,K}^{\varepsilon}, F) \subset (U_{S,K}^{\varepsilon'}, F)$$

is a stratified homotopy equivalence. It is therefore enough to show that $U_{S,K} \coloneqq U^1_{S,K}$ satisfies conditions (1) and (2). Since S is full in K, Lemma 2.4.3 shows that $U_{S,K}$ is final at $|S|$, i.e. property (1) holds. Concerning (2), observe first that via the equivalence

$$\Pi_\infty(|K|, F) \simeq F$$

supplied by Theorem 2.4.1, $\Pi_\infty(|U_{S,K}|, F)$ corresponds to the subposet $G_S \subset F$ of faces having non-empty intersection with S. Then the inclusion of posets

$$F_S \hookrightarrow G_S$$

satisfies the assumptions of Corollary 7.8.10: indeed, since S is full in K we see that for every $\sigma \in G_S$ the intersection $\sigma \cap S$ is a face of S and therefore provides a final object for $(F_S)_{/\sigma}$. Thus, denoting $i_S \colon |S| \hookrightarrow U$ the canonical inclusion, we deduce from Corollary 7.8.10 that the induction functor

$$i_{S,!} \colon \operatorname{Fun}(\mathcal{I}_{|S|}, \mathcal{E}) \to \operatorname{Fun}(\mathcal{I}_{U_{S,K}}, \mathcal{E})$$

preserves cocartesian functors. At this point, Lemma 8.2.4 guarantees that $i_{S,!}$ also preserves Stokes functor, so property (2) is satisfied as well.

We are left to prove (5). Assume now that Z is compact. In particular, the set G_S is finite. On the other hand, we have

$$U_{S,K} = \bigcup_{\sigma \in G_S} |\overset{\circ}{\sigma}| \ .$$

Furthermore, the triangulation can be constructed so that the interior of each simplex is subanalytic [26, Theorem 2]. See also paragraph 10 and Remark p1585 of [55]. Hence (5) follows from the fact that a finite union of subanalytic subsets is again subanalytic. □

Corollary 11.4.2. *Let $f \colon (X, P, \mathcal{I}) \to (Y, Q)$ be a vertically piecewise elementary family of Stokes analytic stratified spaces. Then:*

(1) $(X, P, \mathcal{I})$ is locally elementary.

(2) If $f \colon X \to Y$ is proper, there exists a cover of X by subanalytic open subsets U such that $(U, P, \mathcal{I}_U)$ is elementary.

Proof. Let x be a point of X and set $y \coloneqq f(x)$. Choose a closed subanalytic subset Z of X_y such that $(Z, P, \mathcal{I}_Z)$ is elementary. Then Theorem 11.4.1-(4) implies the existence of an elementary open neighborhood U of Z, so (1) follows. If furthermore f is proper, then X_y is compact and therefore the same goes for Z, so (2) follows from Theorem 11.4.1-(5). □

11.5. **Level structures.** Local elementarity is a fundamental concept in the theory of Stokes analytic stratified spaces, but it is only rarely satisfied. Level structures provide the key technical tool needed to bypass this difficulty: performing induction on the length of a level structure allows to reduce the complexity of the Stokes analytic stratified space, eventually reducing to the locally elementary case. Conceptually, one takes care of the induction step via the Stokes detection criterion Proposition 10.3.5, developed in Part 2 and that uses the theory of graduation in an essential way. We start importing these ideas in the geometric setting via the following:

Definition 11.5.1. Let (X, P) be an exodromic stratified space. Let $\mathcal{E}$ be a presentable stable ∞-category. Let $p \colon \mathcal{I} \to \mathcal{J}$ be a graduation morphism over $\Pi_\infty(X, P)$ (see Definition 9.1.4). The *graduation relative to* p is the morphism

$$\mathfrak{Gr}_p \colon \mathfrak{Fil}_{\mathcal{I},\mathcal{E}} \to \mathfrak{Fil}_{\mathcal{I}_p,\mathcal{E}}$$

in $\operatorname{Cons}_P^{\mathrm{hyp}}(X; \mathbf{Pr}^{\mathrm{L}})$ corresponding to the exponential graduation functor expGr_p from Definition 9.2.1 under the exodromy equivalence.

Observation 11.5.2. It follows from Corollary 9.3.7 that $\mathfrak{Gr}_p$ induces a well defined morphism

$$\mathfrak{Gr}_p \colon \mathfrak{St}_{\mathcal{I},\mathcal{E}} \to \mathfrak{St}_{\mathcal{I}_p,\mathcal{E}}$$

in $\mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathbf{CAT}_\infty)$.

Definition 11.5.3. Let $(X,P) \to (Y,Q)$ be a family of exodromic stratified spaces and let

$$p \colon \mathcal{I} \to \mathcal{J}$$

be a morphism of Stokes fibrations over (X,P). Fix a full subcategory $\mathcal{C} \subseteq \mathbf{FStStrat}$. We say that p is a *simple* $\mathcal{C}$*-level morphism relative to* (Y,Q) if the following conditions hold:

(1) p is a level morphism in the sense of Definition 10.1.2;

(2) both $\mathcal{I}^{\mathrm{set}}$ and $\mathcal{J}^{\mathrm{set}}$ are pullback of Stokes structures (in sets) over (Y,Q);

(3) for every $q \in Q$, the family of Stokes stratified spaces

$$(X_q, P_q, (\mathcal{I}|_{X_q})_{p|_{X_q}}) \to Y_q$$

belongs to $\mathcal{C}$ (see Construction 9.1.1 for the meaning of the notation $(\mathcal{I}|_{X_q})_{p|_{X_q}}$).

We say that p is a $\mathcal{C}$*-level morphism relative to* (Y,Q) if it can be factored as a finite composition

$$\mathcal{I} = \mathcal{I}^d \xrightarrow{p_d} \mathcal{I}^d \xrightarrow{p_{d-1}} \cdots \xrightarrow{p_2} \mathcal{I}^1 \xrightarrow{p_1} \mathcal{I}^0 = \mathcal{J} \tag{11.5.4}$$

where each $\mathcal{I}^k$ is a Stokes fibration over (X,P) and each $p_k \colon \mathcal{I}^k \to \mathcal{I}^{k-1}$ is a simple $\mathcal{C}$-level morphism relative to (Y,Q). When $\mathcal{C} = \mathbf{FStStrat}$, we simply say that p is a (simple) level morphism relative to (Y,Q).

Remark 11.5.5. Assume that the stratification on Y is trivial. Then Condition (2) ensures that if $p \colon \mathcal{I} \to \mathcal{J}$ is a simple level morphism, then it is also a level graduation morphism above each stratum of Y.

Definition 11.5.6. In the situation of Definition 11.5.3, we refer to a factorization of $p \colon \mathcal{I} \to \mathcal{J}$ of the form (11.5.4) as a $\mathcal{C}$*-level structure for* p and we say that d is its *length.* When $\mathcal{J} = \Pi_\infty(X,P)$, we say that (11.5.4) is a $\mathcal{C}$*-level structure for* $\mathcal{I}$.

Definition 11.5.7. Let $\mathcal{C} \subseteq \mathbf{FStStrat}$ be a full subcategory. We say that a family of Stokes stratified spaces $(X,P,\mathcal{I}) \to (Y,Q)$ *admits a* $\mathcal{C}$*-level structure* if the morphism

$$p \colon \mathcal{I} \to \Pi_\infty(X,P)$$

is a $\mathcal{C}$-level morphism relative to (Y,Q). Similarly, we say that $(X,P,\mathcal{I}) \to (Y,Q)$ *locally admits a* $\mathcal{C}$*-level structure* if Y can be covered by open subset U such that each $(X_U,P,\mathcal{I}_U) \to (U,Q)$ admits a $\mathcal{C}$-level structure.

Example 11.5.8. Let $(X,P,\mathcal{I})$ be a Stokes stratified space. Then the canonical morphism $\mathcal{I} \to \Pi_\infty(X,P)$ is a level morphism, so $(X,P,\mathcal{I})$ always admits a level structure of length 1.

Example 11.5.9. Let $\mathcal{C} \subseteq \mathbf{FStStrat}$ be the full subcategory spanned by locally elementary families. Then a Stokes stratified space $(X,P,\mathcal{I})$ admits a level structure of length 1 if and only if the canonical morphism $p \colon \mathcal{I} \to \Pi_\infty(X,P)$ is a simple locally elementary level morphism. Since in this case $\mathcal{I}_p = \mathcal{I}$, this happens if and only if $(X,P,\mathcal{I})$ is locally elementary.

As a consequence of Theorem 11.4.1, we obtain:

Corollary 11.5.10. *Let* $f \colon (X,P,\mathcal{I}) \to (Y,Q)$ *be a family of Stokes analytic stratified spaces. Then:*

(1) If f *has a vertically piecewise elementary level structure then it has a locally elementary level structure;*

(2) if f has a piecewise elementary level structure, then it has a vertically piecewise elementary level structure;

(3) if f has a strongly piecewise elementary level structure, then it has a piecewise elementary level structure.

Proof. (1) follows from Corollary 11.4.2, (2) follows from Corollary 11.3.23 and (3) is automatic unraveling the definitions. □

In the classical theory of Stokes structures, level structures exist only after some suitable ramified cover. The following definition axiomatizes this phenomenon.

Definition 11.5.11. A morphism in **FStStrat**

$$\begin{tikzcd} (X', P', \mathcal{J}) \arrow[r] \arrow[d] & (X, P, \mathcal{I}) \arrow[d] \\ (Y', Q') \arrow[r] & (Y, Q) \end{tikzcd}$$

is a *finite Galois stratified cover* if the upper arrow is cartesian in **StStrat** and if for ever $q \in Q$, the horizontal arrows of the induced diagram

$$\begin{tikzcd} X'_q \arrow[r] \arrow[d] & X_q \arrow[d] \\ Y'_q \arrow[r] & Y_q \end{tikzcd}$$

are finite étale Galois covers.

Definition 11.5.12. Let $\mathcal{C} \subseteq$ **FStStrat** be a full subcategory. We say that a family of Stokes stratified spaces $(X, P, \mathcal{I}) \to (Y, Q)$ *admits a ramified* $\mathcal{C}$*-level structure* if there exists a finite Galois stratified cover as in Definition 11.5.11 such that $(X', P', \mathcal{J}) \to (Y', Q')$ admits a $\mathcal{C}$-level structure. We say that $(X, P, \mathcal{I}) \to (Y, Q)$ *locally admits a ramified* $\mathcal{C}$*-level structure* if Y can be covered by opens U such that each $(X_U, P_U, \mathcal{I}_U) \to (U, Q)$ admits a ramified $\mathcal{C}$-level structure.

11.6. **Hybrid descent for Stokes functors.** As observed in Warning 11.3.5 that even when they exist, elementary open subsets do not form a basis of the topology. For this reason, we need to discuss a hybrid descent property for the ∞-category of Stokes functors that combines $\mathfrak{St}$ on elementary opens and $\mathfrak{Fil}$ on their further intersections. This is achieved via the following:

Construction 11.6.1. Let $(X, P, \mathcal{I})$ be a Stokes stratified space and let $\mathcal{E}$ be a presentable ∞-category. Let $\mathcal{U} = \{\mathbf{U}_\bullet\}$ be a hypercover of X. We define the semi-simplicial diagram

$$\mathrm{StFil}^{\mathcal{U}}_{\mathcal{I},\mathcal{E}} \colon \mathbf{\Delta}_s^{\mathrm{op}} \to \mathbf{CAT}_\infty$$

as the subfunctor of

$$\mathfrak{Fil}_{\mathcal{I},\mathcal{E}} \circ \mathbf{U}_\bullet \colon \mathbf{\Delta}_s^{\mathrm{op}} \to \mathbf{CAT}_\infty$$

defined by

$$\mathrm{StFil}^{\mathcal{U}}_{\mathcal{I},\mathcal{E}}([n]) \coloneqq \begin{cases} \mathfrak{St}_{\mathcal{I},\mathcal{E}}(\mathbf{U}_0) & \text{if } n = 0 \\ \mathfrak{Fil}_{\mathcal{I},\mathcal{E}}(\mathbf{U}_n) & \text{if } n > 0 \ . \end{cases}$$

Notice that it is well defined thanks to the commutativity of (11.1.4).

Proposition 11.6.2. *Let $(X, P, \mathcal{I})$ be a Stokes stratified space and let $\mathcal{E}$ be a presentable ∞-category. Let $\mathcal{U} = \{\mathbf{U}_\bullet\}$ be a hypercover of X. Then the canonical functor*

$$\mathrm{St}_{\mathcal{I},\mathcal{E}} \to \lim_{\mathbf{\Delta}_s^{\mathrm{op}}} \mathrm{StFil}^{\mathcal{U}}_{\mathcal{I},\mathcal{E}}$$

is an equivalence of ∞-categories.

Proof. For every $n \geq 0$, the functors

$$\mathfrak{St}_{\mathcal{I},\mathcal{E}}(\mathbf{U}_n) \to \mathfrak{Fil}_{\mathcal{I},\mathcal{E}}(\mathbf{U}_n)$$

are fully-faithful. Since $\mathfrak{St}_{\mathcal{I},\mathcal{E}}$ and $\mathfrak{Fil}_{\mathcal{I},\mathcal{E}}$ are hypersheaves, passing to the limit thus yields fully-faithful functors

$$\mathrm{St}_{\mathcal{I},\mathcal{E}} \hookrightarrow \lim_{[n] \in \mathbf{\Delta}_s^{\mathrm{op}}} \mathrm{StFil}^{\mathcal{U}}_{\mathcal{I},\mathcal{E}}([n]) \hookrightarrow \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{I}, \mathcal{E}) \ .$$

By definition, an object of the middle term is a cocartesian functor $F \colon \mathcal{I} \to \mathcal{E}$ such that $F|_{U_0}$ is a Stokes functor. In particular, F is punctually split at every point of X. Hence, F is a Stokes functor. This concludes the proof of Proposition 11.6.2. $\square$

Remark 11.6.3. If $\mathbf{U}_\bullet$ is the hypercover induced by a finite cover $U_1, \dots, U_n$ of X, then the limit appearing in Proposition 11.6.2 can be performed over the *finite* subcategory $\Delta^{\mathrm{op}}_{\leq n,s}$ of Δ^{op}_s.

Under some suitable finiteness and stability conditions, the diagram $\mathrm{StFil}_{\mathcal{I},\mathcal{E}}$ takes value in $\mathbf{Pr}^{\mathrm{L,R}}$ (Definition 7.5.2):

Corollary 11.6.4. *Let $(X, P, \mathcal{I})$ be a Stokes stratified space in finite posets. Let $\mathcal{E}$ be a presentable stable ∞-category. Let $\mathcal{U} = \{\mathbf{U}_\bullet\}$ be a hypercover of X such that $(\mathbf{U}_0, P, \mathcal{I}_{\mathbf{U}_0})$ is elementary. Then the semi-simplicial diagram of Construction 11.6.1 lifts to a functor*

$$\mathrm{StFil}^{\mathcal{U}}_{\mathcal{I},\mathcal{E}} \colon \mathbf{\Delta}_s^{\mathrm{op}} \to \mathbf{Pr}^{\mathrm{L,R}} \ .$$

In particular, the equivalence

$$\mathrm{St}_{\mathcal{I},\mathcal{E}} \simeq \lim_{\mathbf{\Delta}_s^{\mathrm{op}}} \mathrm{StFil}^{\mathcal{U}}_{\mathcal{I},\mathcal{E}}$$

supplied by Proposition 11.6.2 is an equivalence in $\mathbf{Pr}^{\mathrm{L}}$, where the limit is computed in $\mathbf{Pr}^{\mathrm{L}}$.

Proof. Since U_0 is elementary, the definition of $\mathrm{StFil}^{\mathcal{U}}_{\mathcal{I},\mathcal{E}}$ yields:

$$\mathrm{StFil}^{\mathcal{U}}_{\mathcal{I},\mathcal{E}}(\mathbf{U}_n) \simeq \begin{cases} \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{I}^{\mathrm{set}}_{\mathbf{U}_0}, \mathcal{E}) & \text{if } n = 0 \\ \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{I}_{\mathbf{U}_n}, \mathcal{E}) & \text{if } n > 0 \ . \end{cases}$$

In both cases, $\mathrm{StFil}_{\mathcal{I},\mathcal{E}}$ takes values in $\mathbf{Pr}^{\mathrm{L}}$ by Corollary 7.2.4. Let $f \colon [n] \to [m]$ be a morphism in $\mathbf{\Delta}^{\mathrm{op}}$ and let $i_f \colon \mathbf{U}_n \to \mathbf{U}_m$ be the associated morphism. When $m > 0$ the corresponding transition functor for $\mathrm{StFil}^{\mathcal{U}}_{\mathcal{I},\mathcal{E}}$ is just

$$i_f^* \colon \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{I}_{\mathbf{U}_m}, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{I}_{\mathbf{U}_n}, \mathcal{E}) \ ,$$

while for $m = 0$ Corollary 8.3.4-(1) identifies it with

$$i_f^* \circ i_{\mathcal{I}_{\mathbf{U}_0},!} \colon \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{I}^{\mathrm{set}}_{\mathbf{U}_0}, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{I}_{\mathbf{U}_n}, \mathcal{E}) \ .$$

In both cases, Lemma 7.3.5 shows they are both left and right adjoints. To conclude the proof of Corollary 11.6.4, use Proposition 11.6.2 and the fact that $\mathbf{Pr}^{\mathrm{L}} \to \mathbf{Cat}_\infty$ commutes with limits. $\square$

12. Stokes structures as a smooth non-commutative space

Let $(X, P, \mathcal{I})$ be a Stokes stratified space and fix an animated ring k. We consider the ∞-category

$$\mathrm{St}_{\mathcal{I},k} := \mathrm{St}_{\mathcal{I},\mathrm{Mod}_k} \ .$$

We saw in Example 5.2.6 that in general $\mathrm{St}_{\mathcal{I},k}$ does not inherit any of the good properties of Mod_k: for instance, it is neither presentable nor stable. The goal of this section is to prove that on the other hand $\mathrm{St}_{\mathcal{I},k}$ is well behaved when $(X, P, \mathcal{I})$ admits a *locally ramified piecewise elementary level structure*. This is a strong condition forcing a highly non-trivial interaction between the geometry of X and $\mathcal{I}$.

12.1. Stability.

The goal of this section is to prove one of the key results of this work, namely:

Theorem 12.1.1. *Let $f\colon (X, P, \mathcal{I}) \to (Y, Q)$ be a family of Stokes stratified spaces in finite posets. Assume that f locally admits a ramified locally elementary level structure. Then for every presentable stable ∞-category $\mathcal{E}$, the ∞-category $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ is presentable and stable.*

Theorem 12.1.1 will follow from a more precise statement (see Corollary 12.1.4 below) exhibiting $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ as a localization of $\mathrm{Fun}(\mathcal{I}, \mathcal{E})$. With this goal in mind, we start setting up the stage with a couple of preliminaries lemmas.

Lemma 12.1.2. *Let $(X, P, \mathcal{I})$ be a locally elementary Stokes stratified space and let $\mathcal{E}$ be a presentable ∞-category. Then:*

(1) $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ is closed under colimits in $\mathrm{Fun}(\mathcal{I}, \mathcal{E})$;

(2) if in addition the fibers of $\mathcal{I}$ are finite and $\mathcal{E}$ is stable, then $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ is closed under limits in $\mathrm{Fun}(\mathcal{I}, \mathcal{E})$. In other words, $(X, P, \mathcal{I})$ is stably bireflexive.

Proof. Thanks to Proposition 7.2.9 we see that $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{I}, \mathcal{E})$ is closed under colimits in $\mathrm{Fun}(\mathcal{I}, \mathcal{E})$. Similarly, when the fibers of $\mathcal{I}$ are finite posets, Remark 17.2.4 and Proposition 7.2.14 imply that $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{I}, \mathcal{E})$ is stable under limits in $\mathrm{Fun}(\mathcal{I}, \mathcal{E})$. Let now $F_\bullet\colon I \to \mathrm{Fun}(\mathcal{I}, \mathcal{E})$ be a diagram such that for every $i \in I$, the functor $F_i\colon \mathcal{I} \to \mathcal{E}$ is Stokes and set

$$F_{\lhd} := \lim_{i \in I} F_i \ , \qquad F_{\rhd} := \operatorname*{colim}_{i \in I} F_i \ ,$$

where the limit and the colimit are computed in $\mathrm{Fun}(\mathcal{I}, \mathcal{E})$. To check that $F_{\lhd}$ and $F_{\rhd}$ are Stokes, we are left to check that they are pointwise split. This question is local on X and since $(X, P, \mathcal{I})$ is locally elementary, we can therefore assume that it is elementary to begin with. In this case, the top horizontal arrow in the commutative triangle

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{I}^{\mathrm{set}}, \mathcal{E}) \arrow[rr] \arrow[dr, "i_{\mathcal{I},!}"'] & & \mathrm{St}_{\mathcal{I},\mathcal{E}} \arrow[dl, hook'] \\ & \mathrm{Fun}(\mathcal{I}, \mathcal{E}) & \end{tikzcd}$$

is an equivalence. Thus, we deduce that $F_{\rhd}$ is Stokes from the fact that $i_{\mathcal{I},!}$ commutes with colimits. Similarly, when the fibers of $\mathcal{I}$ are finite posets, we deduce that $F_{\lhd}$ is Stokes from Proposition 17.2.3 and Remark 17.2.4. □

Theorem 12.1.3. *Let $f\colon (X, P, \mathcal{I}) \to (Y, Q)$ be a family of Stokes stratified spaces in finite posets locally admitting a ramified locally elementary level structure. Then $(X, P, \mathcal{I})$ is stably bireflexive.*

Proof. Let $\mathcal{E}$ be a presentable stable ∞-category. Let $F_\bullet\colon I \to \mathrm{Fun}(\mathcal{I}, \mathcal{E})$ be a diagram such that for every $i \in I$ the functor $F_i\colon \mathcal{I} \to \mathcal{E}$ is Stokes and set

$$F_{\lhd} := \lim_{i \in I} F_i \ , \qquad F_{\rhd} := \operatorname*{colim}_{i \in I} F_i \ ,$$

where the limit and the colimit are computed in $\mathrm{Fun}(\mathcal{I}, \mathcal{E})$. By Proposition 7.2.9 and Proposition 7.2.14, the functors $F_{\lhd}$ and $F_{\rhd}$ are cocartesian. We are thus left to show that they are punctually split. Hence, we can suppose that Y is a point and that $(X, P, \mathcal{I})$ admits a ramified locally elementary level structure. Since we are checking a punctual condition on X, we can further suppose that $(X, P, \mathcal{I})$ admits a locally elementary level structure. We argue by induction on the length d of the locally elementary level structure. If $d = 0$, then $\mathcal{I} = \Pi_\infty(X, P)$ is a fibration in sets, so the result follows from Proposition 8.4.1. Otherwise, our assumption guarantees the existence of a level morphism $p \colon \mathcal{I} \to \mathcal{J}$ such that:

(1) $\mathcal{J}$ admits a locally elementary level structure of length $< d$;

(2) $(X, P, \mathcal{I}_p)$ is locally elementary.

Notice that since level morphisms are surjective, the fibers of $\mathcal{J}$ are again finite posets, so the inductive hypothesis applies to the Stokes stratified space $(X, P, \mathcal{J})$. Consider the pullback square

$$\begin{tikzcd} \mathrm{St}_{\mathcal{I},\mathcal{E}} \arrow[r, "p_!"] \arrow[d, "\mathrm{Gr}_p"] & \mathrm{St}_{\mathcal{J},\mathcal{E}} \arrow[d, "\mathrm{Gr}"] \\ \mathrm{St}_{\mathcal{I}_p,\mathcal{E}} \arrow[r, "\pi_!"] & \mathrm{St}_{\mathcal{J}^{\mathrm{set}},\mathcal{E}} \end{tikzcd}$$

supplied by Theorem 10.2.1. The Stokes detection criterion of Proposition 10.3.5 implies that $F_{\lhd}$ is Stokes if and only if both $\mathrm{Gr}_p(F_{\lhd})$ and $p_!(F_{\lhd})$ are Stokes, and similarly for $F_{\rhd}$ in place of $F_{\lhd}$. Propositions 9.1.12 and 9.1.15 guarantee that Gr_p commutes with both limits and colimits. Similarly, $p_!$ commutes with colimits because it is a left adjoint; since the fibers of $\mathcal{I}$ are finite posets Proposition 17.2.3 and Remark 17.2.4 imply that $p_!$ commutes with limits as well. Thus, we are reduced to check that

$$\mathrm{Gr}_p(F_{\lhd}) \simeq \lim_i \mathrm{Gr}_p(F_i) \in \mathrm{Fun}(\mathcal{I}_p, \mathcal{E}) \qquad \text{and} \qquad p_!(F_{\lhd}) \simeq \lim_i p_!(F_i) \in \mathrm{Fun}(\mathcal{J}, \mathcal{E})$$

are Stokes functors, and similarly for the colimit in place of the limit and $F_{\rhd}$ in place of $F_{\lhd}$. Proposition 9.4.9 ensures that $\mathrm{Gr}_p(F_i)$ is Stokes for every $i \in I$, while Corollary 8.3.4-(2) guarantees that $p_!(F_i)$ is Stokes for every $i \in I$. Thus, the induction hypothesis implies that $p_!(F_{\lhd})$ and $p_!(F_{\rhd})$ are Stokes. On the other hand, since $\mathcal{I}_p$ is locally elementary, Lemma 12.1.2 implies that $\mathrm{Gr}_p(F_{\lhd})$ and $\mathrm{Gr}_p(F_{\rhd})$ are Stokes as well, and the conclusion follows. □

At this point, Theorem 12.1.1 follows from the following more precise statement:

Corollary 12.1.4. *Let $f \colon (X, P, \mathcal{I}) \to (Y, Q)$ be a family of Stokes stratified spaces in finite posets locally admitting a ramified locally elementary level structure. Let $\mathcal{E}$ be a presentable stable ∞-category. Then $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ is a localization of $\mathrm{Fun}(\mathcal{I}, \mathcal{E})$, and in particular it is presentable and stable.*

Proof. Combine Lemma 8.4.6 with Theorem 12.1.3. □

Corollary 12.1.5. *Let $f \colon (X, P, \mathcal{I}) \to (Y, Q)$ be a family of Stokes stratified spaces in finite posets locally admitting a ramified locally elementary level structure. Let $\mathcal{E}$ be a presentable stable compactly generated ∞-category. Let $\{E_\alpha\}_{\alpha \in I}$ be a set of compact generators for $\mathcal{E}$. Then $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ is presentable stable compactly generated by the $\{\mathrm{LSt}_{\mathcal{I},\mathcal{E}}(\mathrm{ev}_{a,!}(E_\alpha))\}_{\alpha \in I, a \in \mathcal{I}}$ where the $\mathrm{ev}_a : \{a\} \to \mathcal{I}$ are the canonical inclusions.*

Proof. Combine Lemma 8.4.8 with Theorem 12.1.3. □

Thanks to the results of Section 8.7, we obtain the following:

Corollary 12.1.6. *Let $f\colon (X,P,\mathcal{I}) \to (Y,Q)$ be a family of Stokes stratified spaces in finite posets locally admitting a ramified locally elementary level structure. Let $\mathcal{A}$ be a Grothendieck abelian category. Then $\mathrm{St}_{\mathcal{I},\mathcal{A}}$ is a Grothendieck abelian category.*

Proof. Combine Corollary 8.7.14 with Theorem 12.1.3. □

Remark 12.1.7. Thanks to Corollary 11.5.10, all the results stated so far hold for families of Stokes analytic stratified spaces in finite posets $f\colon (X,P,\mathcal{I}) \to (Y,Q)$ locally admitting a ramified vertical piecewise elementary level structure.

The following lemma is an amplification of Corollary 12.1.4 in the analytic setting:

Proposition 12.1.8. *Let $f\colon (X,P,\mathcal{I}) \to (Y,Q)$ be a proper family of Stokes analytic stratified spaces in finite posets locally admitting a ramified locally elementary level structure. Let $\mathcal{E}$ be a presentable stable ∞-category. Then, the following hold:*

1. *For every open subsets $U \subset V$, the functor $f_*(\mathfrak{St}_{\mathcal{I},\mathcal{E}})(V) \to f_*(\mathfrak{St}_{\mathcal{I},\mathcal{E}})(U)$ is a left and right adjoint.*
2. *There exists a subanalytic refinement $R \to Q$ such that $f_*(\mathfrak{St}_{\mathcal{I},\mathcal{E}}) \in \mathrm{Cons}_R^{\mathrm{hyp}}(X;\mathbf{Pr}^{\mathrm{L}})$.*
3. *For every subanalytic refinement $R \to Q$ such that $f_*(\mathfrak{St}_{\mathcal{I},\mathcal{E}}) \in \mathrm{Cons}_R^{\mathrm{hyp}}(X;\mathbf{Pr}^{\mathrm{L}})$, the hypersheaf $f_*(\mathfrak{St}_{\mathcal{I},\mathcal{E}})$ is an object of $\mathrm{Cons}_R^{\mathrm{hyp}}(X;\mathbf{Pr}^{\mathrm{L,R}})$.*

Proof. Item (1) is an immediate consequence of Theorem 12.1.3 and Corollary 12.1.4. The existence of an analytic refinement as in (2) is a consequence of Proposition 2.5.10. Then (3) follows from (1) and Lemma 11.2.8. □

12.2. **Stokes functors and tensor product.** In this section, we analyze more thoroughly the interaction between the category of Stokes functor and the tensor product in $\mathbf{Pr}^{\mathrm{L}}$.

Lemma 12.2.1. *Let $(X,P,\mathcal{I})$ be a locally elementary Stokes stratified space in finite posets. Then $(X,P,\mathcal{I})$ is stably universal.*

Proof. Note that $(X,P,\mathcal{I})$ is stably bireflexive by Lemma 12.1.2. Let $\mathcal{E},\mathcal{E}'$ be presentable stable ∞-categories. Since $(X,P,\mathcal{I})$ is locally elementary, we can find a cover $\{U_i\}$ such that $(U_i,P,\mathcal{I}_{U_i})$ is elementary. Let $\mathcal{U} = \{\mathbf{U}_\bullet\}$ be its Čech nerve. Recall from Construction 11.6.1 the semi-simplicial diagram

$$\mathrm{StFil}^{\mathcal{U}}_{\mathcal{I},\mathcal{E}}\colon \mathbf{\Delta}_s^{\mathrm{op}} \to \mathbf{Cat}_\infty \ .$$

By Corollary 11.6.4, this functor takes values in $\mathbf{Pr}^{\mathrm{L,R}}$. Therefore, we can tensor it with $\mathcal{E}'$, finding:

$$\begin{aligned}
\mathrm{St}_{\mathcal{I},\mathcal{E}} \otimes \mathcal{E}' &\simeq \big(\lim_{\mathbf{\Delta}_s^{\mathrm{op}}} \mathrm{StFil}_{\mathcal{I},\mathcal{E}}\big) \otimes \mathcal{E}' && \text{By Cor. 11.6.4} \\
&\simeq \lim_{\mathbf{\Delta}_s^{\mathrm{op}}} \big(\mathrm{StFil}^{\mathcal{U}}_{\mathcal{I},\mathcal{E}} \otimes \mathcal{E}'\big) && \text{By Lem. 7.5.5} \\
&\simeq \lim_{\mathbf{\Delta}_s^{\mathrm{op}}} \mathrm{StFil}^{\mathcal{U}}_{\mathcal{I},\mathcal{E}\otimes\mathcal{E}'} && \text{By Cor. 7.5.7 \& E.g. 8.2.3} \\
&\simeq \mathrm{St}_{\mathcal{I},\mathcal{E}\otimes\mathcal{E}'} && \text{By Cor. 11.6.4}
\end{aligned}$$

The conclusion follows. □

Proposition 12.2.2. *Let $(X,P,\mathcal{I})$ be a Stokes stratified space in finite posets admitting a locally elementary level structure. Then $(X,P,\mathcal{I})$ is stably universal.*

Proof. Note that $(X, P, \mathcal{I})$ is bireflexive by Theorem 12.1.3. Let $\mathcal{E}, \mathcal{E}'$ be presentable stable ∞-categories. We proceed by induction on the length d of the locally elementary level structure. When $d = 0$, $\mathcal{I} = \Pi_\infty(X, P)$ and $(X, P, \mathcal{I})$ is elementary, so the conclusion follows from Lemma 12.2.1. Otherwise, our assumption guarantees the existence of a level morphism $p \colon \mathcal{I} \to \mathcal{J}$ such that:

(1) $\mathcal{J}$ admits a locally elementary level structure of length $< d$;

(2) $(X, P, \mathcal{I}_p)$ is locally elementary.

Notice that since level morphisms are surjective, the fibers of $\mathcal{J}$ are again finite posets, so the inductive hypothesis applies to the Stokes stratified space $(X, P, \mathcal{J})$. Consider the following commutative cube:

$$
\begin{tikzcd}
\mathrm{St}_{\mathcal{I},\mathcal{E}} \otimes \mathcal{E}' \arrow[rr, "p_! \otimes \mathcal{E}'"] \arrow[dr] \arrow[dd, "\mathrm{Gr}_p \otimes \mathcal{E}'"'] & & \mathrm{St}_{\mathcal{J},\mathcal{E}} \otimes \mathcal{E}' \arrow[dr] \arrow[dd, "\mathrm{Gr} \otimes \mathcal{E}'"] & \\
& \mathrm{St}_{\mathcal{I},\mathcal{E}\otimes\mathcal{E}'} \arrow[rr, "p_!"] \arrow[dd, "\mathrm{Gr}_p"] & & \mathrm{St}_{\mathcal{J},\mathcal{E}\otimes\mathcal{E}'} \arrow[dd, "\mathrm{Gr}"] \\
\mathrm{St}_{\mathcal{I}_p,\mathcal{E}} \otimes \mathcal{E}' \arrow[rr, "\pi_! \otimes \mathcal{E}'"] \arrow[dr] & & \mathrm{St}_{\mathcal{J}^{\mathrm{set}},\mathcal{E}} \otimes \mathcal{E}' \arrow[dr] & \\
& \mathrm{St}_{\mathcal{I}_p,\mathcal{E}\otimes\mathcal{E}'} \arrow[rr, "\pi_!"] & & \mathrm{St}_{\mathcal{J}^{\mathrm{set}},\mathcal{E}\otimes\mathcal{E}'} .
\end{tikzcd}
\tag{12.2.3}
$$

whose front face is a pull-back in virtue of Theorem 10.2.1. Combining Theorem 12.1.3, Corollary 8.4.11 and Lemma 7.5.5 we deduce that the back face is a pullback in $\mathbf{Pr}^{\mathrm{L,R}}$. Lemma 12.2.1 shows that the bottom diagonal arrows are equivalences while the upper right diagonal arrow is an equivalence by the inductive hypothesis. Hence, so is the top left diagonal arrow. $\square$

Working in the analytic setting, we can formulate a stronger version of the above result. To begin with, let us improve the construction of the comparison functor of Construction 8.6.3:

Construction 12.2.4. Let $f \colon (X, P, \mathcal{I}) \to (Y, Q)$ be a family of Stokes analytic stratified spaces in finite posets admitting a vertically piecewise elementary level structure. Fix stable presentable ∞-categories $\mathcal{E}$ and $\mathcal{E}'$. For every open subset $V \subset Y$, the induced family $(X_V, P, \mathcal{I}_V) \to (Y, Q)$ admits again a vertically piecewise elementary level structure. Thus, Theorem 12.1.3 shows that $\mathrm{St}_{\mathcal{I}_V,\mathcal{E}}$ and $\mathrm{St}_{\mathcal{I}_V,\mathcal{E}\otimes\mathcal{E}'}$ is closed under limits and colimits in $\mathrm{Fun}(\mathcal{I}_V, \mathcal{E})$ and $\mathrm{Fun}(\mathcal{I}_V, \mathcal{E} \otimes \mathcal{E}')$, respectively. The assumptions of Construction 8.6.3 are therefore satisfied, and they yield a comparison map

$$\mathrm{St}_{\mathcal{I}_V,\mathcal{E}} \otimes \mathcal{E}' \to \mathrm{St}_{\mathcal{I}_V,\mathcal{E}\otimes\mathcal{E}'} .$$

Furthermore, Proposition 8.6.5 shows that it is fully faithful. Since this comparison map depends functorially on V, we deduce the existence of a commutative diagram

$$
\begin{tikzcd}
f_*(\mathfrak{St}_{\mathcal{I},\mathcal{E}}) \otimes \mathcal{E}' \arrow[r, hook] \arrow[d, hook] & f_*(\mathfrak{St}_{\mathcal{I},\mathcal{E}\otimes\mathcal{E}'}) \arrow[d, hook] \\
f_*(\mathfrak{Fil}_{\mathcal{I},\mathcal{E}}) \otimes \mathcal{E}' \arrow[r, "\sim"] & f_*(\mathfrak{Fil}_{\mathcal{I},\mathcal{E}\otimes\mathcal{E}'})
\end{tikzcd}
$$

in $\mathrm{PSh}(Y; \mathbf{Pr}^{\mathrm{L}})$.

Lemma 12.2.5. *Let $f \colon (Y, P, \mathcal{J}) \to (X, P, \mathcal{I})$ be a cartesian finite Galois cover in* **StStrat** *where (X, P) is conically refineable and where $(Y, P, \mathcal{J})$ is stably universal. Then $(X, P, \mathcal{I})$ is stably universal and for every presentable stable ∞-category $\mathcal{E}$, the canonical functor*

$$\mathrm{Loc}(Y; \mathbf{Sp}) \otimes_{\mathrm{Loc}(X;\mathbf{Sp})} \mathrm{St}_{\mathcal{I},\mathcal{E}} \to \mathrm{St}_{\mathcal{J},\mathcal{E}}$$

is an equivalence.

Proof. Let $Y_\bullet \colon \mathbf{\Delta}_s^{\mathrm{op}} \to \mathcal{T}\mathrm{op}_{/X}$ be the Cech complex of $f \colon Y \to X$ and put

$$\mathcal{I}_\bullet \coloneqq \Pi_\infty(Y_\bullet, P) \times_{\Pi_\infty(X,P)} \mathcal{I} \ .$$

Since $f \colon Y \to X$ is Galois, Y_n is a finite coproduct of copies of Y over X. Hence, (Y_n, P) is conically refineable for every $[n] \in \mathbf{\Delta}_s$ and $(Y_n, P, \mathcal{I}_n)$ is stably universal for every $[n] \in \mathbf{\Delta}_s$. Then $(X, P, \mathcal{I})$ is stably universal in virtue of Proposition 8.6.7. Since the $Y \to X$ is a finite étale cover, Lemma 19.2.9 implies that

$$\Pi_\infty(Y, P) \to \Pi_\infty(X, P)$$

is a finite étale fibration in the sense of Definition 19.2.1. We deduce from Corollary 8.8.5 that the canonical functor

$$\mathrm{Loc}(Y; \mathbf{Sp}) \otimes_{\mathrm{Loc}(X;\mathbf{Sp})} \mathrm{St}_{\mathcal{I},\mathbf{Sp}} \to \mathrm{St}_{\mathcal{J},\mathbf{Sp}}$$

is an equivalence. Tensoring the above equivalence with $\mathcal{E}$ and using the stable universality thus concludes the proof of Lemma 12.2.5. ☐

Theorem 12.2.6. *Let $f \colon (X, P, \mathcal{I}) \to (Y, Q)$ be a proper family of Stokes analytic stratified spaces in finite posets locally admitting a ramified vertically piecewise elementary level structure. Let $\mathcal{E}$ and $\mathcal{E}'$ be stable presentable ∞-categories. Then the canonical functor*

$$f_*(\mathfrak{St}_{\mathcal{I},\mathcal{E}}) \otimes \mathcal{E}' \to f_*(\mathfrak{St}_{\mathcal{I},\mathcal{E}\otimes\mathcal{E}'}) \tag{12.2.7}$$

is an equivalence. In particular, $(X, P, \mathcal{I})$ is stably universal.

Proof. The second half follows from the first because $f_*(\mathfrak{St}_{\mathcal{I},\mathcal{E}}) \otimes \mathcal{E}'$ is by definition the tensor product computed in $\mathrm{PSh}(Y; \mathbf{CAT}_\infty)$. To prove the first half, observe that in fact both sides of (12.2.7) are hyperconstructible hypersheaves, thanks to Proposition 12.1.8 and Lemma 11.2.6. Hence, the equivalence can be checked at the level of stalks. Since f is proper, Propositions 2.5.10 and 11.1.5 allow to reduce ourselves to the case where Y is a point. That is, we are left to show that $(X, P, \mathcal{I})$ is stably universal. In that case, there exists a cartesian finite Galois cover $(Y, P, \mathcal{J}) \to (X, P, \mathcal{I})$ such that $(Y, P, \mathcal{J})$ admits a vertically piecewise elementary level structure. Recall that (X, P) is conically refineable in virtue of Remark 2.5.4. By Lemma 12.2.5, it is thus enough to show that $(Y, P, \mathcal{J})$ is stably universal. Hence, we can suppose that $(X, P, \mathcal{I})$ admits a vertically piecewise elementary level structure. In this case, Corollary 11.5.10 guarantees that $(X, P, \mathcal{I})$ admits a locally elementary level structure, so the conclusion follows from Proposition 12.2.2. ☐

12.3. **Finite type property for Stokes structures.** We proved Theorem 12.1.1 under two key assumptions on the Stokes stratified space $(X, P, \mathcal{I})$: the local existence of a ramified locally elementary level structure and the fibers of $\mathcal{I}$ are finite posets. Under these same assumptions, we also proved that $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ is compactly generated provided that $\mathcal{E}$ is. We now analyze the categorical finiteness properties of $\mathrm{St}_{\mathcal{I},\mathcal{E}}$: under some stricter geometrical assumptions on $(X, P, \mathcal{I})$ and working in the *analytic setting* we establish that it is of finite type and hence smooth in the non-commutative sense (see e.g. [33, Definition 11.3.1.1]).

Definition 12.3.1. Let $f \colon (M, X) \to (N, Y)$ be a subanalytic morphism. We say that f is *strongly proper* if it is proper and for every finite subanalytic stratifications $X \to P$ and $Y \to Q$ such that $f \colon (X, P) \to (Y, Q)$ is a subanalytic stratified map, there exists a categorically *finite* subanalytic refinement $R \to Q$ such that for every $\mathcal{F} \in \mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathbf{CAT}_\infty)$, we have $f_*(\mathcal{F}) \in \mathrm{Cons}_R^{\mathrm{hyp}}(Y; \mathbf{CAT}_\infty)$.

Example 12.3.2. By Proposition 2.5.10 and Proposition 2.5.7, every proper subanalytic map $f : (M, X) \to (N, Y)$ with Y compact is strongly proper.

The following lemma is our main source of strongly proper morphisms.

Lemma 12.3.3. *Let $f\colon (M,X) \to (N,Y)$ be a proper subanalytic morphism. Assume the existence of a commutative diagram*

$$\begin{tikzcd} (M,X) \arrow[r, hook, "j"] \arrow[d, "f"] & (\overline{M},\overline{X}) \arrow[d, "g"] \\ (N,Y) \arrow[r, hook, "i"] & (\overline{N},\overline{Y}) \end{tikzcd}$$

such that g is proper, $\overline{Y}$ is compact and the horizontal arrows are open immersions with closed subanalytic complements. Then $f\colon (M,X) \to (N,Y)$ is strongly proper.

Proof. Let $X \to P$ and $Y \to Q$ be finite subanalytic stratifications such that $f\colon (X,P) \to (Y,Q)$ is a subanalytic stratified map. Extend $X \to P$ to a subanalytic stratification $\overline{X} \to P^{\lhd}$ by sending $\overline{X} \setminus X$ to the initial object of $P^{\lhd}$. Extend $Y \to Q$ to a subanalytic stratification $\overline{Y} \to Q^{\lhd}$ by sending $\overline{Y} \setminus Y$ to the initial object of $Q^{\lhd}$. By Proposition 2.5.10 applied to the proper map $g\colon (\overline{M},\overline{X}) \to (\overline{N},\overline{Y})$, there is a finite subanalytic refinement $S \to Q^{\lhd}$ such that for every $F \in \mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathbf{Cat}_\infty)$, we have $g_*(j_!(F)) \in \mathrm{Cons}_S^{\mathrm{hyp}}(\overline{Y};\mathbf{Cat}_\infty)$. Let $R \subset S$ be the (finite) open subset of elements not mapped to the initial object of $Q^{\lhd}$ by $S \to Q^{\lhd}$. Then, $f_*(\mathcal{F}) \in \mathrm{Cons}_R^{\mathrm{hyp}}(Y;\mathbf{Cat}_\infty)$ with (Y,R) categorically finite by Proposition 2.5.7. $\square$

From now on, we fix an animated commutative ring k and a compactly generated k-linear stable ∞-category $\mathcal{E}$.

Observation 12.3.4. For every Stokes stratified space $(X,P,\mathcal{I})$, we see that $\mathrm{Fun}(\mathcal{I},\mathcal{E})$ is again compactly generated and k-linear. When the fibers of $\mathcal{I}$ are finite, Proposition 7.2.14 implies that $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{I},\mathcal{E})$ is a localization of $\mathrm{Fun}(\mathcal{I},\mathcal{E})$ and therefore inherits a k-linear structure. Finally, when $(X,P,\mathcal{I})$ admits a locally elementary level structure Corollary 12.1.4 implies that $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ inherits an k-linear structure as well.

Theorem 12.3.5. *Let $f\colon (X,P,\mathcal{I}) \to (Y,Q)$ be a strongly proper family of Stokes analytic stratified spaces in finite posets locally admitting a ramified piecewise elementary level structure. Let k be an animated ring and let $\mathcal{E}$ be a compactly generated k-linear stable ∞-category of finite type (Definition 17.3.1). Then $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ is of finite type relative to k as well.*

Remark 12.3.6. It is possible to formulate an analogous statement without analyticity assumption assuming that f is strongly proper in the topological sense: for every finite exodromic stratifications $X \to P$ and $Y \to Q$ such that $(X,P) \to (Y,Q)$ is a morphism of stratified spaces, there exists a categorically finite exodromic refinement $R \to Q$ such that for every $\mathcal{F} \in \mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathbf{Cat}_\infty)$, $f_*(\mathcal{F})$ belongs to $\mathrm{Cons}_R^{\mathrm{hyp}}(Y;\mathbf{Cat}_\infty)$. However, to ensure the existence and the abundance of strongly proper maps, analyticity is extremely handy, as it enables to use techniques of [23] that are not available in the purely topological setting.

Corollary 12.3.7. *In the setting of Theorem 12.3.5,*

$$\mathrm{St}_{\mathcal{I},k} \coloneqq \mathrm{St}_{\mathcal{I},\mathrm{Mod}_k}$$

is a smooth k-linear presentable stable ∞-category.

Proof. This simply follows because finite type k-linear categories are smooth over k in the non-commutative sense, see e.g. [51, Proposition 2.14]. $\square$

Lemma 12.3.8. *Let $(X,P,\mathcal{I})$ be a compact piecewise elementary Stokes analytic stratified space in finite posets. Let k be an animated ring and let $\mathcal{E}$ be a compactly generated k-linear stable ∞-category of finite type. If $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ is of finite type relative to k.*

Proof. Thanks to Theorem 11.4.1, X admits a finite cover by relatively compact subanalytic open subsets $U_1, \dots, U_n$ such that $(U_i, P, \mathcal{I}_{U_i})$ is elementary for every $i = 1, \dots, n$. In particular, each term of the associated hypercover $\mathcal{U} = \{\mathbf{U}_\bullet\}$ is a relatively compact *subanalytic* open subset. From Proposition 11.6.2 and Remark 11.6.3, we have a canonical equivalence

$$\mathrm{St}_{\mathcal{I},\mathcal{E}} \simeq \lim_{\mathbf{\Delta}^{\mathrm{op}}_{\leq n,s}} \mathrm{StFil}^{\mathcal{U}}_{\mathcal{I},\mathcal{E}} |_{\Delta^{\mathrm{op}}_{\leq n,s}} .$$

Since $\Delta^{\mathrm{op}}_{\leq n,s}$ is a finite category, Lemma 17.3.3 reduces us to show that the transition maps in the diagram

$$\mathrm{StFil}^{\mathcal{U}}_{\mathcal{I},\mathcal{E}} |_{\Delta^{\mathrm{op}}_{\leq n,s}} : \mathbf{\Delta}^{\mathrm{op}}_{\leq n,s} \to \mathbf{C}\mathrm{\mathbf{AT}}_\infty$$

are both left and right adjoints and that $\mathrm{StFil}^{\mathcal{U}}_{\mathcal{I},\mathcal{E}}([m])$ is of finite type for every $m \leqslant n$. The first point follows from Corollary 11.6.4, while the second one follows from Corollary 17.3.4 and Proposition 2.5.7 stating that for every relatively compact open subanalytic subset $U \subset X$, the stratified space (U, P) is categorically compact. $\square$

Lemma 12.3.9. *Let $f \colon (Y, P, \mathcal{J}) \to (X, P, \mathcal{I})$ be a cartesian finite Galois cover in* **StStrat** *where (X, P) is conically refineable with $\Pi_\infty(X)$ compact and where $(Y, P, \mathcal{J})$ is stably universal. Let $Y_\bullet \colon \mathbf{\Delta}^{\mathrm{op}}_s \to \mathcal{T}\mathrm{op}_{/X}$ be the Cech complex of $f \colon Y \to X$ and put*

$$\mathcal{I}_\bullet := \Pi_\infty(Y_\bullet, P) \times_{\Pi_\infty(X,P)} \mathcal{I} .$$

Then $(X, P, \mathcal{I})$ is stably universal and there exists an integer $m \geq 1$ such that for every $\mathcal{E}$ presentable stable ∞-category, $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ is a retract of

$$\lim_{[n] \in \mathbf{\Delta}_{s,\leq m}} \mathrm{St}_{\mathcal{I}_n,\mathcal{E}}$$

in $\mathbf{Pr}^{\mathrm{L,R}}$.

Proof. The Stokes stratified space $(X, P, \mathcal{I})$ is stably universal in virtue of Lemma 12.2.5. Since $f \colon Y \to X$ is Galois, Y_n is a finite coproduct of copies of Y over X, so that $(Y_n, P, \mathcal{I}_n)$ is stably universal for every $[n] \in \mathbf{\Delta}_s$. Since the $Y_n \to X$ is a finite étale cover for every $[n] \in \mathbf{\Delta}_s$, Lemma 19.2.9 implies that

$$\Pi_\infty(Y_n, P) \to \Pi_\infty(X, P)$$

is a finite étale fibration in the sense of Definition 19.2.1. Corollary 8.8.6 thus implies the existence of an integer $m \geq 1$ such that there exists a retract

$$\mathrm{St}_{\mathcal{I},\mathbf{Sp}} \to \lim_{[n] \in \mathbf{\Delta}_{s,\leq m}} \mathrm{St}_{\mathcal{I}_n,\mathbf{Sp}} \to \mathrm{St}_{\mathcal{I},\mathbf{Sp}} .$$

in $\mathbf{Pr}^{\mathrm{L,R}}$. Lemma 12.3.9 follows from Lemma 7.5.5 by tensoring the above retract with $\mathcal{E}$. $\square$

We are now ready for:

Proof of Theorem 12.3.5. Since f is strongly proper, we can choose a categorically finite subanalytic refinement $R \to Q$ such that $f_*(\mathfrak{St}_{\mathcal{I},\mathcal{E}})$ is R-hyperconstructible. Let $F \colon \Pi_\infty(Y, R) \to \mathbf{C}\mathrm{\mathbf{AT}}_\infty$ be the functor corresponding to $f_*(\mathfrak{St}_{\mathcal{I},\mathcal{E}})$ via the exodromy equivalence (2.3.6). By Recollection 2.3.5, we have

$$\mathrm{St}_{\mathcal{I},\mathcal{E}} \simeq f_*(\mathfrak{St}_{\mathcal{I},\mathcal{E}})(Y) \simeq \lim_{\Pi_\infty(Y,R)} F(y) .$$

Recall from Proposition 12.1.8 that $f_*(\mathfrak{St}_{\mathcal{I},\mathcal{E}})$ belongs to $\mathrm{Cons}^{\mathrm{hyp}}_R(Y; \mathbf{Pr}^{\mathrm{L,R}}_k)$, and therefore that F factors through $\mathbf{Pr}^{\mathrm{L,R}}_k$ as well. By Lemma 7.5.5, the above limit can thus equally be computed in $\mathbf{Pr}^{\mathrm{L}}$. Since (Y, R) is categorically finite, Lemma 17.3.3 reduces us to check that for each $y \in Y$,

$F(y)$ is compactly generated and of finite type relative to k. By Proposition 2.5.6, we can choose an open neighborhood U of y such that y is initial in $\Pi_\infty(U, R)$. Then

$$F(y) \simeq (f_*(\mathfrak{St}_{\mathcal{I},\mathcal{E}}))_y \simeq (f_*(\mathfrak{St}_{\mathcal{I},\mathcal{E}}))(U) \ ,$$

so compact generation of $F(y)$ follows from Corollary 12.1.5. To check that $F(y)$ is of finite type relative to k, we first observe that the base-change results Propositions 11.1.5 and 2.5.10 and Lemma 11.3.12 allow to reduce to the case where Y is a point and X is compact. In that case, there exists a cartesian finite Galois cover $(Y, P, \mathcal{J}) \to (X, P, \mathcal{I})$ such that $(Y, P, \mathcal{J})$ admits a piecewise elementary level structure. Recall that (X, P) is conically refineable in virtue of Remark 2.5.4 and that $\Pi_\infty(X)$ is finite by Proposition 2.5.7. Hence, Lemma 12.3.9 implies the existence of an integer $m \geq 1$ such that $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ is a retract of

$$\lim_{[n]\in \mathbf{\Delta}_{s,\leq m}} \mathrm{St}_{\mathcal{I}_n,\mathcal{E}} \tag{12.3.10}$$

in $\mathbf{Pr}^{\mathrm{L,R}}$, where $Y_\bullet \colon \mathbf{\Delta}_s^{\mathrm{op}} \to \mathcal{T}\mathrm{op}_{/X}$ is the Cech complex of $f \colon Y \to X$ and where

$$\mathcal{I}_\bullet \coloneqq \Pi_\infty(Y_\bullet, P) \times_{\Pi_\infty(X,P)} \mathcal{I} \ .$$

Hence, it is enough to show that (14.2.7) is of finite type relative to k. Since $Y \to X$ is a finite Galois cover, each Y_n is a finite coproduct of copies of Y. By Lemma 17.3.3, it is thus enough to show that $\mathrm{St}_{\mathcal{J},\mathcal{E}}$ is of finite type relative to k. Hence, we can suppose that $(X, P, \mathcal{I})$ admits a piecewise elementary level structure. We now argue by induction on the length d of the piecewise elementary level structure of $(X, P, \mathcal{I})$. When $d = 0$, $\mathcal{I} = \Pi_\infty(X, P)$ is a fibration in sets, so $(X, P, \mathcal{I})$ is (globally) elementary and the conclusion follows from Lemma 12.3.8. Otherwise, we can assume the existence of a level morphism $p \colon \mathcal{I} \to \mathcal{J}$ such that:

(1) $\mathcal{J}$ admits a piecewise elementary level structure of length $< d$;

(2) $(X, P, \mathcal{I}_p)$ is piecewise elementary.

Notice that since level morphisms are surjective, the fibers of $\mathcal{J}$ are again finite posets, so the inductive hypothesis applies to the Stokes stratified space $(X, P, \mathcal{J})$. Consider the pullback square

$$\begin{tikzcd} \mathrm{St}_{\mathcal{I},\mathcal{E}} \arrow[r, "p_!"] \arrow[d, "\mathrm{Gr}_p"] & \mathrm{St}_{\mathcal{J},\mathcal{E}} \arrow[d, "\mathrm{Gr}"] \\ \mathrm{St}_{\mathcal{I}_p,\mathcal{E}} \arrow[r, "\pi_!"] & \mathrm{St}_{\mathcal{J}^{\mathrm{set}},\mathcal{E}} \end{tikzcd}$$

supplied by Theorem 10.2.1. Both $\mathrm{St}_{\mathcal{I}_p,\mathcal{E}}$ and $\mathrm{St}_{\mathcal{J}^{\mathrm{set}},\mathcal{E}}$ are of finite type thanks to Lemma 12.3.8, while the inductive hypothesis guarantees that $\mathrm{St}_{\mathcal{J},\mathcal{E}}$ are of finite type. Finally, Theorem 12.1.3 implies that the assumptions of Corollary 8.4.11 are satisfied, so that the above square is a pullback in $\mathbf{Pr}^{\mathrm{L,R}}$. Thus, it follows from Lemma 17.3.3 that $\mathrm{St}_{\mathcal{I},\mathcal{E}}$ is of finite type. □

13. Geometricity

We now turn to the main theorem of this paper, namely the construction of a derived Artin stack parametrizing Stokes functors. Similarly to Theorems 12.1.1 and 12.3.5 we prove this result in the analytic setting and assuming the existence of a locally elementary level structure. The geometricity is essentially a consequence of Theorem 12.3.5, but we need to run more time the level induction to provide an alternative description of the functor of points.

13.1. **Description of the moduli functor.** We fix an animated commutative ring k. For every animated commutative k-algebra A, we let Mod_A denote the associated stable ∞-category of A-modules and by Perf_A the full subcategory of perfect A-modules (see e.g. [32, Definition 7.2.4.1]).

Recollection 13.1.1. When A is an ordinary commutative k-algebra, Mod_A is the derived ∞-category of the abelian category of A-modules. In particular, its homotopy category coincides with the ordinary derived category of A, seen as a triangulated category.

Fix now a Stokes stratified space $(X, P, \mathcal{I})$.

Notation 13.1.2. Let $\mathcal{E}$ be a compactly generated stable ∞-category. We set

$$\mathrm{St}_{\mathcal{I},\mathcal{E},\omega} \coloneqq \mathrm{St}_{\mathcal{I},\mathcal{E}} \times_{\mathrm{Fun}(\mathcal{I},\mathcal{E})} \mathrm{Fun}(\mathcal{I}, \mathcal{E}^\omega) .$$

When $\mathcal{E} = \mathrm{Mod}_A$, we write

$$\mathrm{St}_{\mathcal{I},A} \coloneqq \mathrm{St}_{\mathcal{I},\mathrm{Mod}_A} \qquad \text{and} \qquad \mathrm{St}_{\mathcal{I},A,\omega} \coloneqq \mathrm{St}_{\mathcal{I},\mathrm{Mod}_A,\omega} .$$

Let $f\colon \mathcal{E} \to \mathcal{E}'$ be a functor of stable presentable ∞-categories. Via Proposition 8.6.1 we see that f functorially induces a morphism

$$f\colon \mathrm{St}_{\mathcal{I},\mathcal{E}} \to \mathrm{St}_{\mathcal{I},\mathcal{E}'} .$$

When in addition both $\mathcal{E}$ and $\mathcal{E}'$ are compactly generated and f preserves compact objects, this further descends to a morphism

$$f\colon \mathrm{St}_{\mathcal{I},\mathcal{E},\omega} \to \mathrm{St}_{\mathcal{I},\mathcal{E}',\omega} .$$

This gives rise to a well defined functor

$$\mathbf{St}^{\mathrm{cat}}_{\mathcal{I},k}\colon \mathrm{dAff}_k^{\mathrm{op}} \to \mathbf{C}\mathbf{AT}_\infty$$

that sends the spectrum of an animated commutative k-algebra $\mathrm{Spec}(A)$ to $\mathrm{St}_{\mathcal{I},A,\omega}$. Passing to the maximal ∞-groupoid, we obtain a presheaf

$$\mathbf{St}_{\mathcal{I},k}\colon \mathrm{dAff}_k^{\mathrm{op}} \to \mathbf{Spc}$$

that sends $\mathrm{Spec}(A)$ to

$$\mathbf{St}_{\mathcal{I},k}(\mathrm{Spec}(A)) \coloneqq (\mathrm{St}_{\mathcal{I},A,\omega})^{\simeq} \in \mathbf{Spc} .$$

When k is clear out of the context, we write $\mathbf{St}_{\mathcal{I}}$ instead of $\mathbf{St}_{\mathcal{I},k}$.

Example 13.1.3. When $\mathcal{I}$ is the trivial fibration, Corollary 11.1.13 shows that $\mathbf{St}_{\mathcal{I}}$ coincide with the derived stack of perfect local systems.

With these notations, we can state the main theorem of this section as follows:

Theorem 13.1.4. *Let $f\colon (X, P, \mathcal{I}) \to (Y, Q)$ be a strongly proper family of Stokes analytic stratified spaces in finite posets locally admitting a ramified piecewise elementary level structure. Let k be an animated commutative ring. Then, $\mathbf{St}_{\mathcal{I}}$ is locally geometric locally of finite presentation over k. Moreover, for every animated commutative k-algebra A and every morphism*

$$x\colon \mathrm{Spec}(A) \to \mathbf{St}_{\mathcal{I}}$$

classifying a Stokes functor $F\colon \mathcal{I} \to \mathrm{Perf}_A$, there is a canonical equivalence

$$x^*\mathbb{T}_{\mathbf{St}_{\mathcal{I}}} \simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{I},\mathrm{Mod}_A)}(F, F)[1] ,$$

where $\mathbb{T}_{\mathbf{St}_{\mathcal{I}}}$ denotes the tangent complex of $\mathbf{St}_{\mathcal{I}}$ and the right hand side denotes the Mod_A*-enriched* Hom *of* $\mathrm{Fun}(\mathcal{I}, \mathrm{Mod}_A)$.

Remark 13.1.5. Theorem 13.1.4 implies in particular that $\mathbf{St}_{\mathcal{I}}$ satisfies étale hyperdescent and that it is infinitesimally cohesive and nilcomplete. Using the locally elementary level structure and proceeding by induction on the length of this level structure, it is possible to verify these three properties by hand, but they typically fail in absence of such a structure. See Theorem 14.2.2 for a more general statement.

Remark 13.1.6. The cotangent complexes of $\mathbf{St}_{\mathcal{I}}$ can be given a cohomological interpretation at the cost of constructing a monoidal structure on Stokes functors. This is delicate since the source cocartesian fibration for $F \otimes G$ is bigger than that of F and G, which in turn forces the base stratification to be finer. This specific issue will be addressed in a later work.

We will deduce Theorem 13.1.4 from Theorem 12.3.5 and of the work of Toën-Vaquié on the moduli of objects of a stable k-linear ∞-category [51]. To do so, we need a brief digression on the behavior of Stokes functors and the tensor product of presentable ∞-categories.

13.2. **Stokes moduli functor as a moduli of objects.** Throughout this section we fix an animated commutative ring k.

Recollection 13.2.1. Let $\mathcal{C}$ be a compactly generated presentable stable k-linear category. Its *moduli of objects* is the derived stack

$$\mathcal{M}_{\mathcal{C}} \colon \mathrm{dAff}_k^{\mathrm{op}} \to \mathbf{Spc}$$

given by the rule

$$\mathcal{M}_{\mathcal{C}}(\mathrm{Spec}(A)) \coloneqq \mathrm{Fun}_k^{\mathrm{st}}((\mathcal{C}^{\omega})^{\mathrm{op}}, \mathrm{Perf}_A)^{\simeq}$$

where $\mathrm{Fun}_k^{\mathrm{st}}((\mathcal{C}^{\omega})^{\mathrm{op}}, \mathrm{Perf}(A))$ denotes the fully subcategory of $\mathrm{Fun}((\mathcal{C}^{\omega})^{\mathrm{op}}, \mathrm{Perf}(A))$ spanned by exact k-linear functors. When $\mathcal{C}$ is of finite type relative to k in the sense of Definition 17.3.1, [51, Theorem 0.2] states that $\mathcal{M}_{\mathcal{C}}$ is a locally geometric derived stack which is furthermore locally of finite presentation.

Let $(X, P, \mathcal{I})$ be a stably bireflexive Stokes stratified space. Then Lemma 8.4.8 implies that the ∞-category $\mathrm{St}_{\mathcal{I},k}$ is stable presentable and compactly generated. In particular, its moduli of objects is well defined. We have:

Proposition 13.2.2. *Let $(X, P, \mathcal{I})$ be a stably universal Stokes stratified space. Then the derived prestacks $\mathbf{St}_{\mathcal{I}}$ and $\mathcal{M}_{\mathrm{St}_{\mathcal{I},k}}$ are canonically equivalent.*

Proof. Fix a derived affine $\mathrm{Spec}(A) \in \mathrm{dAff}_k$ and consider the following chain of canonical equivalences:

$$\begin{aligned}
\mathrm{Fun}_k^{\mathrm{st}}\left((\mathrm{St}_{\mathcal{I},k})^{\omega})^{\mathrm{op}}, \mathrm{Mod}_A\right) &\simeq \mathrm{Fun}_k^{\mathrm{st}}\left((\mathrm{St}_{\mathcal{I},k})^{\omega}, \mathrm{Mod}_A^{\mathrm{op}}\right)^{\mathrm{op}} \\
&\simeq \mathrm{Fun}_k^{\mathrm{L}}(\mathrm{St}_{\mathcal{I},k}, \mathrm{Mod}_A^{\mathrm{op}})^{\mathrm{op}} && \text{By [1, §3.1]} \\
&\simeq \mathrm{Fun}_k^{\mathrm{R}}(\mathrm{St}_{\mathcal{I},k}^{\mathrm{op}}, \mathrm{Mod}_A) \\
&\simeq \mathrm{St}_{\mathcal{I},k} \otimes_k \mathrm{Mod}_A && \text{By [32, 4.8.1.7]} \\
&\simeq \mathrm{St}_{\mathcal{I},A}
\end{aligned}$$

Let $\mathrm{LSt}_{\mathcal{I},\mathcal{E}} \colon \mathrm{Fun}(\mathcal{I}, \mathcal{E}) \to \mathrm{St}_{\mathcal{I},\mathcal{E}}$ be the left adjoint to the canonical inclusion $\mathrm{St}_{\mathcal{I},\mathcal{E}} \hookrightarrow \mathrm{Fun}(\mathcal{I}, \mathcal{E})$. By Lemma 8.4.8, a system of compact generators of $\mathrm{St}_{\mathcal{I},\mathrm{Mod}_A}$ is given by $\{\mathrm{LSt}_{\mathcal{I},\mathcal{E}}(\mathrm{ev}_{a,!}(A))\}_{a \in \mathcal{I}}$, where the $\mathrm{ev}_a \colon \{a\} \to \mathcal{I}$ are the canonical inclusions. Then via the embedding

$$\mathrm{Fun}_k^{\mathrm{st}}\left((\mathrm{St}_{\mathcal{I},k})^{\omega})^{\mathrm{op}}, \mathrm{Perf}(A)\right) \hookrightarrow \mathrm{Fun}_k^{\mathrm{st}}\left((\mathrm{St}_{\mathcal{I},k})^{\omega})^{\mathrm{op}}, \mathrm{Mod}_A\right)$$

induced by $\mathrm{Perf}(A) \hookrightarrow \mathrm{Mod}_A$, the above chain of equivalences exhibits

$$\mathrm{Fun}_k^{\mathrm{st}}\left((\mathrm{St}_{\mathcal{I},k})^{\omega})^{\mathrm{op}}, \mathrm{Perf}(A)\right)$$

as the full-subcategory of $\mathrm{St}_{\mathcal{I},A}$ spanned by Stokes functors $F\colon \mathcal{I} \to \mathrm{Mod}_A$ such that

$$\mathrm{Hom}_{\mathrm{St}_{\mathcal{I},A}}(\mathrm{LSt}_{\mathcal{I},\mathcal{E}}(\mathrm{ev}_{a,!}(A)), F) \in \mathrm{Perf}(A)$$

for every $a \in \mathcal{I}$. Hence for every $F \in \mathrm{St}_{\mathcal{I},A}$, we have

$$\begin{aligned} F \in \mathcal{M}_{\mathrm{St}_{\mathcal{I},k}}(\mathrm{Spec}\, A) &\Leftrightarrow \mathrm{Hom}_{\mathrm{St}_{\mathcal{I},A}}(\mathrm{LSt}_{\mathcal{I},\mathcal{E}}(\mathrm{ev}_{a,!}(A)), F) \in \mathrm{Perf}(A) && \forall a \in \mathcal{I} \\ &\Leftrightarrow \mathrm{Hom}_{\mathrm{St}_{\mathcal{I},A}}(\mathrm{ev}_{a,!}(A), F) \in \mathrm{Perf}(A) && \forall a \in \mathcal{I} \\ &\Leftrightarrow F(a) \in \mathrm{Perf}(A) && \forall a \in \mathcal{I} \\ &\Leftrightarrow F \in \mathbf{St}_{\mathcal{I},k}(\mathrm{Spec}(A)) \end{aligned}$$

This concludes the proof of Proposition 13.2.2. □

We are now ready for:

Proof of Theorem 13.1.4. By Corollary 12.1.5 and Proposition 13.2.2, the prestack $\mathcal{M}_{\mathrm{St}_{\mathcal{I},k}}$ and $\mathbf{St}_{\mathcal{I},k}$ are canonically equivalent. By Theorem 12.3.5, the ∞-category $\mathrm{St}_{\mathcal{I},k}$ is stable presentable and of finite type relative to k. The conclusion thus follows from [51, Theorem 0.2]. □

13.3. **The moduli of Stokes vector bundles.** We fix once more an animated commutative ring k. A k-point of $\mathbf{St}_{\mathcal{I},k}$ is a Stokes functor $F\colon \mathcal{I} \to \mathrm{Perf}_k$. In particular, even when k is a field the stack $\mathbf{St}_{\mathcal{I},k}$ classifies $\mathcal{I}$-Stokes structures on perfect complexes, rather than vector bundles. Thus, when the Stokes stratified space is of dimension 1, $\mathbf{St}_{\mathcal{I},k}$ provides an extension of [3]. We are going to see how to extract from $\mathbf{St}_{\mathcal{I},k}$ a more classical substack.

Let $(X, P, \mathcal{I})$ be a Stokes stratified space. For every animated commutative k-algebra A, consider the standard t-structure $\tau = ((\mathrm{Mod}_A)_{\geqslant 0}, (\mathrm{Mod}_A)_{\leqslant 0})$ on the stable derived ∞-category Mod_A. It is accessible and compatible with filtered colimits, and $\mathrm{Fun}(\mathcal{I}, \mathrm{Mod}_A)$ inherits an induced t-structure defined objectwise and satisfying the same properties. Besides, $\mathrm{Fun}(\mathcal{I}, \mathrm{Mod}_A)$ has a canonical A-linear structure, with underlying tensor product

$$(-) \otimes_A (-)\colon \mathrm{Mod}_A \otimes \mathrm{Fun}(\mathcal{I}, \mathrm{Mod}_A) \to \mathrm{Fun}(\mathcal{I}, \mathrm{Mod}_A) \ ,$$

that sends (M, F) to the functor $M \otimes_A F(-)\colon \mathcal{I} \to \mathrm{Mod}_A$. Using Proposition 8.6.1, we deduce that if F is a Stokes functor, then the same goes for $M \otimes_A F$. Following [15], we introduce the following:

Definition 13.3.1. Let A be an animated commutative k-algebra and let $F\colon \mathcal{I} \to \mathrm{Mod}_A$ be a filtered functor. We say that F is *flat relative to A* (or *A-flat*) if for every $M \in \mathrm{Mod}_A^{\heartsuit}$, the functor $M \otimes_k F\colon \mathcal{I} \to \mathrm{Mod}_A$ belongs to $\mathrm{Fun}(\mathcal{I}, \mathrm{Mod}_A)^{\heartsuit}$.

Remark 13.3.2. Since $\mathrm{Fun}(\mathcal{I}, \mathrm{Mod}_A)^{\heartsuit} \simeq \mathrm{Fun}(\mathcal{I}, \mathrm{Mod}_A^{\heartsuit})$, we see that a filtered functor F is A-flat if and only if it takes values in $\mathrm{Mod}_A^{\heartsuit}$.

Example 13.3.3. Assume that A is an discrete commutative algebra. If a Stokes functor $F\colon \mathcal{I} \to \mathrm{Mod}_A$ is flat relative to A, then automatically $F \in \mathrm{St}_{\mathcal{I},A}^{\heartsuit}$. The vice-versa holds provided that A is a field.

Sending $\mathrm{Spec}(A) \in \mathrm{dAff}_k^{\mathrm{op}}$ to the full subgroupoid of $\mathbf{St}_{\mathcal{I},k}(\mathrm{Spec}(A))$ spanned by flat Stokes functors defines a full sub-prestack $\mathbf{St}_{\mathcal{I},k}^{\mathrm{flat}}$ of $\mathbf{St}_{\mathcal{I},k}$. The goal is to prove the following:

Theorem 13.3.4. *Let k be an animated commutative algebra and let $f\colon (X, P, \mathcal{I}) \to (Y, Q)$ be a strongly proper family of Stokes analytic stratified spaces in finite posets admitting a ramified piecewise elementary level structure. Then the morphism*

$$\mathbf{St}_{\mathcal{I},k}^{\mathrm{flat}} \to \mathbf{St}_{\mathcal{I},k}$$

is representable by open immersions. In particular, $\mathbf{St}^{\mathrm{flat}}_{\mathcal{I},k}$ *is a derived* 1*-Artin stack locally of finite type.*

We start discussing some preliminaries.

Lemma 13.3.5. *Let* $(X,P,\mathcal{I})$ *be a Stokes stratified space and let* A *be an animated commutative* k*-algebra. Assume that* $\Pi_\infty(X,P)$ *has an initial object* x*. Then a Stokes functor* $F\colon \mathcal{I}\to \mathrm{Mod}_A$ *is* A*-flat if and only if* $j_x^*(F)$ *is* A*-flat.*

Proof. Notice that for every $M\in\mathrm{Mod}_A$, the canonical comparison map

$$M\otimes_A j_x^*(F)\to j_x^*(M\otimes_A F)$$

is an equivalence. Then the lemma follows directly from Corollary 8.7.17. □

Notation 13.3.6. Let $(X,P,\mathcal{I})$ be a Stokes stratified space. For every $a\in\mathcal{I}$, Proposition 6.5.2 shows that $\mathrm{ev}_a\colon\{a\}\to\mathcal{I}$ induces a morphism of derived prestacks

$$\mathbf{ev}_a\colon \mathbf{St}_{\mathcal{I},k}\to\mathbf{Perf}_k\ .$$

Proposition 13.3.7. *Let* $(X,P,\mathcal{I})$ *be a compact Stokes stratified space. Then the map*

$$\mathbf{St}^{\mathrm{flat}}_{\mathcal{I},k}\to\mathbf{St}_{\mathcal{I},k}$$

is representable by open immersions.

Proof. Thanks to Proposition 2.5.6 and since X is compact we can find an open cover of X by finitely many open subsets $U_1,U_2,\dots,U_n$ such that each $\Pi_\infty(U_i,P)$ has an initial object x_i. Let

$$e\colon \mathbf{St}_{\mathcal{I},k}\to\prod_{i=1}^n\prod_{a\in\mathcal{I}_{x_i}}\mathbf{Perf}_k$$

be the product of the evaluation maps of Notation 13.3.6. Notice that both products are finite, so the map

$$\prod_{i=1}^n\prod_{a\in\mathcal{I}_{x_i}}\mathrm{BGL}\to\prod_{i=1}^n\prod_{a\in\mathcal{I}_{x_i}}\mathbf{Perf}_k$$

is representable by open immersions (see e.g. [33, Proposition 6.1.4.5]). Besides, Lemma 13.3.5 implies that the square

$$\begin{tikzcd}
\mathbf{St}^{\mathrm{flat}}_{\mathcal{I},k} \arrow[r] \arrow[d] & \mathbf{St}_{\mathcal{I},k} \arrow[d] \\
\prod_{i=1}^n\prod_{a\in\mathcal{I}_{x_i}}\mathrm{BGL} \arrow[r] & \prod_{i=1}^n\prod_{a\in\mathcal{I}_{x_i}}\mathbf{Perf}_k
\end{tikzcd}$$

is a fiber product. The conclusion follows. □

This proves Theorem 13.3.4 when the base is reduced to a single point. To prove the general case, we need a couple of extra preliminaries.

Lemma 13.3.8. *Let* $(X,P,\mathcal{I})$ *be a Stokes stratified space. Then the derived prestack* $\mathbf{St}^{\mathrm{flat}}_{\mathcal{I},k}$ *is* 1*-truncated.*

Proof. We have to prove that for every discrete commutative k-algebra A, $\mathbf{St}^{\mathrm{flat}}_{\mathcal{I},k}(\mathrm{Spec}(A))$ is a 1-groupoid. Since $\mathrm{St}_{\mathcal{I},A}$ is fully faithful inside $\mathrm{Fun}(\mathcal{I},A)$, using [31, Proposition 2.3.4.18] we see that it is enough to show that for every pair of A-flat Stokes functors $F,G\colon\mathcal{I}\to\mathrm{Mod}_A$, the mapping space $\mathrm{Map}_{\mathrm{Fun}(\mathcal{I},\mathrm{Mod}_A)}(F,G)$ is discrete. Since A is discrete, both F and G belongs to

$\mathrm{St}^{\heartsuit}_{\mathcal{I},A}$, as already observed in Example 13.3.3. Thus, Corollary 8.7.12 implies that both F and G take values in the 1-category $\mathrm{Mod}^{\heartsuit}_A$. Then the conclusion follows from [31, Corollary 2.3.4.8]. □

Lemma 13.3.9. *Let I be a finite ∞-category and let*

$$f_\bullet \colon F_\bullet \to G_\bullet$$

be a natural transformation between diagrams $I \to \mathrm{dSt}_k$. Let

$$F \coloneqq \lim_{i\in I} F_i \qquad \text{and} \qquad G \coloneqq \lim_{i \in I} G_i$$

be the limits computed in dSt_k. Assume that:

(1) for every $i \in I$, F_i is geometric and locally of finite type and G_i is locally geometric and locally of finite type;

(2) for every $i \in I$, $f_i \colon F_i \to G_i$ is representable by open immersions;

(3) G is locally geometric and locally of finite presentation.

Then F is a geometric derived stack and the induced morphism $f \colon F \to G$ is an open immersion.

Proof. It follows from [52, Proposition 1.3.3.3 and Lemma 1.4.1.12] that geometric stacks locally of finite type are closed under finite limits. Thus, F is geometric and locally of finite type. We are left to check that f is an open immersion. Since both F and G are locally geometric and locally of finite type, it follows that f is an open immersion if and only if it is étale and the diagonal

$$\delta_f \colon F \to F \times_G F$$

is an equivalence. Besides, since f is automatically locally of finite presentation, [52, Corollary 2.2.5.6] shows that f is étale if and only if it is formally étale, i.e. the relative cotangent complex $\mathbb{L}_f$ vanishes. Since limits commutes with limits, we see that δ_f is the limit of the diagonal maps

$$\delta_{f_i} \colon F_i \to F_i \times_{G_i} F_i \ ,$$

and since each f_i is an open immersion, it automatically follows that each δ_{f_i} is an equivalence. Therefore, the same goes for f. Similarly, the property of being formally étale is clearly closed under retracts. On the other hand, [52, Lemma 1.4.1.12] implies that formally étale maps are closed under pullbacks and hence under finite limits. The conclusion follows. □

We are now ready for:

Proof of Theorem 13.3.4. Since f is strongly proper, we can choose a categorically finite subanalytic refinement $R \to Q$ such that $f_*(\mathfrak{St}_{\mathcal{I},k})$ is R-hyperconstructible. Let $F \colon \Pi_\infty(Y,R) \to \mathbf{Cat}_\infty$ be the functor corresponding to $f_*(\mathfrak{St}_{\mathcal{I},k})$ via the exodromy equivalence. As we argued in Theorem 12.3.5, we obtain a canonical equivalence

$$\mathrm{St}_{\mathcal{I},k} \simeq \lim_{y \in \Pi_\infty(Y,R)} F_y \ ,$$

the limit being computed in $\mathbf{Pr}^{\mathrm{L,R}}$. Besides, the base-change results of Propositions 11.1.5 and 2.5.10 and Lemma 11.3.12 provide a canonical identification $F_y \simeq \mathrm{St}_{\mathcal{I}_y,k}$. Passing to the moduli of objects and applying Proposition 13.2.2, we deduce that

$$\mathbf{St}_{\mathcal{I},k} \simeq \lim_{y \in \Pi_\infty(Y,R)} \mathbf{St}_{\mathcal{I}_y,k} \ .$$

Using Proposition 8.7.11, we deduce from here that the induced morphism

$$\mathbf{St}^{\mathrm{flat}}_{\mathcal{I},k} \to \lim_{y \in \Pi_\infty(Y,R)} \mathbf{St}^{\mathrm{flat}}_{\mathcal{I}_y,k}$$

is an equivalence as well. Besides, $\mathbf{St}^{\mathrm{flat}}_{\mathcal{I},k}$ and $\mathbf{St}^{\mathrm{flat}}_{\mathcal{I}_y,k}$ are 1-truncated for every $y \in \Pi_\infty(Y,R)$ thanks to Lemma 13.3.8. Thus, Lemma 13.3.9 reduces us to the case where Y is reduced to a single point. Since in this case X is compact, the conclusion follows from Proposition 13.3.7. $\square$

14. Permanence theorem

Theorems 12.1.1, 12.3.5 and 13.1.4 were all proven via the level induction. We now provide an axiomatization of this technique, that takes the form of a meta-theorem that can be used to prove general statements about Stokes structures.

14.1. **The general setup.** To maximize the range of applicability, we fix throughout this entire section an ∞-category of *test objects* $\mathcal{C}$. Recall from Section 6.5 that we have an ∞-functor

$$\exp\colon \mathbf{CoCart} \times \mathbf{Pr}^{\mathrm{L}} \to \mathbf{PrFib}^{\mathrm{L}} \ .$$

Composing with the evaluation

$$\mathrm{ev}\colon \mathcal{C} \times \mathrm{Fun}(\mathcal{C}, \mathbf{Pr}^{\mathrm{L}}) \to \mathbf{Pr}^{\mathrm{L}} \ ,$$

we obtain a functor

$$\exp \circ \mathrm{ev}\colon \mathbf{CoCart} \times \mathcal{C} \times \mathrm{Fun}(\mathcal{C}, \mathbf{Pr}^{\mathrm{L}}) \to \mathbf{PrFib}^{\mathrm{L}} \ .$$

By adjunction, it determines a functor

$$\exp\colon \mathbf{CoCart} \times \mathrm{Fun}(\mathcal{C}, \mathbf{Pr}^{\mathrm{L}}) \to \mathrm{Fun}(\mathcal{C}, \mathbf{PrFib}^{\mathrm{L}}) \ .$$

Given $\mathcal{F} \in \mathrm{Fun}(\mathcal{C}, \mathbf{Pr}^{\mathrm{L}})$, we denote by

$$\exp_{\mathcal{F}}\colon \mathbf{CoCart} \to \mathrm{Fun}(\mathcal{C}, \mathbf{PrFib}^{\mathrm{L}})$$

the resulting functor, and we refer to it as the *exponential construction with coefficients in* $\mathcal{F}$.

Notice that $\mathrm{Fun}(\mathcal{C}, \mathbf{CAT}_\infty)$ is compactly generated. Therefore, any exodromic stratified space (X,P) gives rise to an equivalence

$$\mathrm{Cons}^{\mathrm{hyp}}_P(X; \mathrm{Fun}(\mathcal{C}, \mathbf{CAT}_\infty)) \simeq \mathrm{Fun}(\Pi_\infty(X,P), \mathrm{Fun}(\mathcal{C}, \mathbf{CAT}_\infty)) \simeq \mathbf{COCART}_{\Pi_\infty(X,P)\times\mathcal{C}} \ .$$

It is straightforward to adapt the definition of $\mathrm{Cons}^{\mathrm{hyp}}_P(X; \mathbf{Pr}^{\mathrm{L}})$ given in [25, Definition 4.2.4] to this more general setting. Then, the same argument given in [25, Proposition 4.2.5] yields an equivalence

$$\mathrm{Cons}^{\mathrm{hyp}}_P(X; \mathrm{Fun}(\mathcal{C}, \mathbf{Pr}^{\mathrm{L}})) \simeq \mathrm{Fun}(\Pi_\infty(X,P), \mathrm{Fun}(\mathcal{C}, \mathbf{Pr}^{\mathrm{L}})) \simeq \mathbf{PrFib}^{\mathrm{L}}_{\Pi_\infty(X,P)\times\mathcal{C}} \ ,$$

making the diagram

$$\begin{tikzcd} \mathrm{Cons}^{\mathrm{hyp}}_P(X; \mathrm{Fun}(\mathcal{C}, \mathbf{Pr}^{\mathrm{L}})) \arrow[rr, "\sim"] \arrow[dr, "\Gamma_{X,*}"'] & & \mathrm{Fun}(\Pi_\infty(X,P), \mathrm{Fun}(\mathcal{C}, \mathbf{Pr}^{\mathrm{L}})) \arrow[dl, "\lim"] \\ & \mathrm{Fun}(\mathcal{C}, \mathbf{Pr}^{\mathrm{L}}) & \end{tikzcd}$$

commutative.

Fix now a Stokes stratified space $(X,P,\mathcal{I})$ and a coefficient functor $\mathcal{F}\colon \mathcal{C} \to \mathbf{Pr}^{\mathrm{L}}$. The exponential construction gives rise to

$$\exp_{\mathcal{F}}(\mathcal{I}/\Pi_\infty(X,P)) \in \mathbf{PrFib}^{\mathrm{L}}_{\Pi_\infty(X,P)\times\mathcal{C}} \ .$$

Following Definition 5.1.2, we let

$$\mathfrak{Fil}_{\mathcal{I},\mathcal{F}} \in \mathrm{Cons}^{\mathrm{hyp}}_P(X; \mathrm{Fun}(\mathcal{C}, \mathbf{Pr}^{\mathrm{L}}))$$

be the $\mathrm{Fun}(\mathcal{C}, \mathbf{Pr}^{\mathrm{L}})$-valued P-hyperconstructible hypersheaf associated to $\exp_{\mathcal{F}}(\mathcal{I}/\Pi_\infty(X,P))$. We denote its global sections by

$$\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{I}, \mathcal{F}) \in \mathrm{Fun}(\mathcal{C}, \mathbf{Pr}^{\mathrm{L}}) \ ,$$

and we refer to them as the $\mathcal{F}$-valued cocartesian functors on $(X, P, \mathcal{I})$. The morphism $i_{\mathcal{I}} \colon \mathcal{I}^{\mathrm{set}} \to \mathcal{I}$ gives rise to a transformation

$$\mathcal{F}_!^{i_{\mathcal{I}}} \colon \exp_{\mathcal{F}}(\mathcal{I}^{\mathrm{set}}/\Pi_\infty(X,P)) \to \exp_{\mathcal{F}}(\mathcal{I}/\Pi_\infty(X,P)) \ .$$

Applying Lemma 5.2.1 we obtain an object

$$\mathrm{EssIm}(\mathcal{F}_!^{i_{\mathcal{I}}}) \in \mathbf{CoCart}_{\Pi_\infty(X,P)\times\mathcal{C}} \ .$$

Following Definition 5.2.3, we let

$$\mathfrak{St}_{\mathcal{I},\mathcal{F}} \in \mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathrm{Fun}(\mathcal{C}; \mathbf{Cat}_\infty))$$

be the $\mathrm{Fun}(\mathcal{C}, \mathbf{Cat}_\infty)$-valued P-hyperconstructible hypersheaf associated to $\mathrm{EssIm}(\mathcal{F}_!^{i_{\mathcal{I}}})$. We denote its global sections by

$$\mathrm{St}_{\mathcal{I},\mathcal{F}} \in \mathrm{Fun}(\mathcal{C}, \mathbf{Cat}_\infty) \ ,$$

and we refer to them as the $\mathcal{F}$-valued Stokes functors on $(X, P, \mathcal{I})$.

Example 14.1.1. When $\mathcal{C}$ is reduced to a single point, the given of $\mathcal{F}$ amounts to fix a presentable ∞-category $\mathcal{E}$. In this case, $\mathrm{St}_{\mathcal{I},\mathcal{F}} = \mathrm{St}_{\mathcal{I},\mathcal{E}}$ coincides with the ∞-category of $\mathcal{E}$-valued Stokes functors previously considered.

Example 14.1.2. Fix an animated commutative ring k and take $\mathcal{C} = \mathrm{dAff}_k^{\mathrm{op}}$. Consider the functor

$$\mathbf{QCoh} \colon \mathrm{dAff}_k^{\mathrm{op}} \to \mathbf{Cat}_\infty \ ,$$

that sends $\mathrm{Spec}(A)$ to $\mathbf{QCoh}(\mathrm{Spec}(A)) \coloneqq \mathrm{Mod}_A$ (whose existence is guaranteed by [32, Theorem 4.5.3.1]). Then

$$\mathrm{St}_{\mathcal{I},\mathbf{QCoh}} \colon \mathrm{dAff}_k^{\mathrm{op}} \to \mathbf{Cat}_\infty$$

is the categorical derived prestack sending $\mathrm{Spec}(A)$ to $\mathrm{St}_{\mathcal{I},A}$.

14.2. **The abstract permanence theorem.** Before stating the permanence theorem, we need to set the following:

Notation 14.2.1. Let $\mathcal{C}$ be an ∞-category and let $\mathcal{F} \colon \mathcal{C} \to \mathbf{Pr}^{\mathrm{L}}$. Given any other small ∞-category $\mathcal{A}$, we let $\mathcal{F}^{\mathcal{A}}$ be the functor

$$\mathcal{F}^{\mathcal{A}} \colon \mathcal{C} \to \mathbf{Pr}^{\mathrm{L}}$$

defined by

$$\mathcal{F}^{\mathcal{A}}(c) \coloneqq \mathrm{Fun}(\mathcal{A}, \mathcal{F}(c)) \ .$$

Theorem 14.2.2. *Let $f \colon (X, P, \mathcal{I}) \to (Y, Q)$ be a strongly proper family of Stokes analytic stratified spaces in finite posets locally admitting a ramified piecewise elementary level structure. Let $\mathcal{C}$ be an ∞-category and let $\mathcal{D} \subset \mathrm{Fun}(\mathcal{C}, \mathbf{Cat}_\infty)$ be a subcategory stable under finite limits and retracts. Let $\mathcal{F} \colon \mathcal{C} \to \mathbf{Pr}^{\mathrm{L}}_{\mathrm{st}}$. Assume that:*

(1) either f is strongly proper;

(2) both $\mathcal{F}$ and $\mathcal{F}^{\Delta^1}$ belong to $\mathcal{D}$.

Then, $\mathrm{St}_{\mathcal{I},\mathcal{F}} \in \mathcal{D}$.

We need several preliminaries. In what follows, we fix an ∞-category of test objects.

Lemma 14.2.3. *Let $\mathcal{F} \colon \mathcal{C} \to \mathbf{Pr}^{\mathrm{L}}_{\mathrm{st}}$ and let $\mathcal{D} \subset \mathrm{Fun}(\mathcal{C}, \mathbf{Cat}_\infty)$ be a subcategory closed under finite limits and containing $\mathcal{F}$ and $\mathcal{F}^{\Delta^1}$. Then:*

(1) for every finite ∞-category $\mathcal{A}$, $\mathcal{F}^{\mathcal{A}} \in \mathcal{D}$;

(2) if in addition $\mathcal{D}$ is closed under retracts, then for every compact ∞-category $\mathcal{A}$, $\mathcal{F}^{\mathcal{A}} \in \mathcal{D}$.

Proof. First notice that if I is a finite set, then

$$\mathcal{F}^I \simeq \prod_I \mathcal{F} .$$

Since $\mathcal{D}$ is closed under finite limits and $\mathcal{F} \in \mathcal{D}$, it follows that $\mathcal{F}^I \in \mathcal{D}$ as well. Second, observe that if we are given a pushout

$$\begin{tikzcd} \mathcal{B} \arrow[r] \arrow[d] & \mathcal{B}' \arrow[d] \\ \mathcal{A} \arrow[r] & \mathcal{A}' \end{tikzcd}$$

in $\mathbf{Cat}_\infty$, then one has

$$\mathcal{F}^{\mathcal{A}'} \simeq \mathcal{F}^{\mathcal{A}} \times_{\mathcal{F}^{\mathcal{B}}} \mathcal{F}^{\mathcal{B}'}$$

in $\mathrm{Fun}(\mathcal{C}, \mathbf{Pr}^{\mathrm{L}})$. Thus, if $\mathcal{F}^{\mathcal{B}}$, $\mathcal{F}^{\mathcal{B}'}$ and $\mathcal{F}^{\mathcal{A}}$ belong to $\mathcal{D}$, it follows that $\mathcal{F}^{\mathcal{A}'}$ belong to $\mathcal{D}$ as well. This immediately implies (1), and (2) follows from the fact that every compact ∞-category is a retract of a finite one. $\square$

Lemma 14.2.4. *Let $(X, P, \mathcal{I})$ be a Stokes stratified space in finite posets where (X, P) is categorically finite. Let $\mathcal{F} \colon \mathcal{C} \to \mathbf{Pr}^{\mathrm{L}}_{\mathrm{st}}$ and let $\mathcal{D} \subset \mathrm{Fun}(\mathcal{C}, \mathbf{CAT}_\infty)$ be a subcategory stable under finite limits and containing both $\mathcal{F}$ and $\mathcal{F}^{\Delta^1}$. Then, $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{I}, \mathcal{F})$ belongs to $\mathcal{D}$ as well. If $\mathcal{D}$ is furthermore closed under retracts, (X, P) can be taken to be categorically compact.*

Proof. Since limits in $\mathrm{Fun}(\mathcal{C}, \mathbf{Pr}^{\mathrm{L}})$ are computed objectwise, applying Corollary 7.2.4 we find

$$\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{I}, \mathcal{F}) \simeq \lim_{x \in \Pi_\infty(X,P)} \mathcal{F}^{\mathcal{I}_x} .$$

Since $\mathcal{I}_x$ is a finite category, Lemma 14.2.3 implies that $\mathcal{F}^{\mathcal{I}_x} \in \mathcal{D}$, and the conclusion follows from our assumptions. $\square$

The following is a restatement of Lemma 12.3.8 with essentially the same proof (we leave to the reader to adapt Construction 11.6.1 and Proposition 11.6.2 to the setting of a test ∞-category $\mathcal{C}$):

Lemma 14.2.5. *Let $(X, P, \mathcal{I})$ be a compact piecewise elementary analytic Stokes stratified space in finite posets. Let $\mathcal{F} \colon \mathcal{C} \to \mathbf{Pr}^{\mathrm{L}}_{\mathrm{st}}$ and let $\mathcal{D} \subset \mathrm{Fun}(\mathcal{C}, \mathbf{CAT}_\infty)$ be a subcategory stable under finite limits and containing both $\mathcal{F}$ and $\mathcal{F}^{\Delta^1}$. Then, $\mathrm{St}_{\mathcal{I},\mathcal{F}} \in \mathcal{D}$.*

Proof. Thanks to Theorem 11.4.1, X admits a finite cover by relatively compact subanalytic open subsets $U_1, \dots, U_n$ such that $(U_i, P, \mathcal{I}_{U_i})$ is elementary for every $i = 1, \dots, n$. Let $\mathcal{U} = \{\mathbf{U}_\bullet\}$ be the associated hypercover of X. From Proposition 11.6.2 and Remark 11.6.3, we have a canonical equivalence in $\mathrm{Fun}(\mathcal{C}, \mathbf{CAT}_\infty)$:

$$\mathrm{St}_{\mathcal{I},\mathcal{F}} \simeq \lim_{\mathbf{\Delta}^{\mathrm{op}}_{\leq n,s}} \mathrm{StFil}^{\mathcal{U}}_{\mathcal{I},\mathcal{F}}|_{\Delta^{\mathrm{op}}_{\leq n,s}} .$$

Since $\Delta^{\mathrm{op}}_{\leq n,s}$ is a finite category, it is enough to check that $\mathrm{StFil}^{\mathcal{U}}_{\mathcal{I},\mathcal{E}}([m])$ belongs to $\mathcal{D}$. Since each $(U_i, P, \mathcal{I}_{U_i})$ is elementary,

$$\mathrm{StFil}_{\mathcal{I},\mathcal{F}}([0]) \simeq \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{I}_{\mathbf{U}_0}, \mathcal{F}) ,$$

while for $m > 0$,

$$\mathrm{StFil}_{\mathcal{I},\mathcal{F}}([m]) \simeq \mathrm{Fun}^{\mathrm{cocart}}(\mathcal{I}_{\mathbf{U}_m}, \mathcal{F})$$

by definition. Thus, the conclusion follows from Lemma 14.2.4. $\square$

The following is the analogue of Lemma 12.3.9:

Lemma 14.2.6. *Let $f\colon (Y,P,\mathcal{J}) \to (X,P,\mathcal{I})$ be a cartesian finite Galois cover in* **StStrat** *where (X,P) is conically refineable with $\Pi_\infty(X)$ compact and where $(Y,P,\mathcal{J})$ is stably universal. Let $Y_\bullet\colon \mathbf{\Delta}_s^{\mathrm{op}} \to \mathcal{T}\mathrm{op}_{/X}$ be the Cech complex of $f\colon Y \to X$ and put*

$$\mathcal{I}_\bullet := \Pi_\infty(Y_\bullet, P) \times_{\Pi_\infty(X,P)} \mathcal{I} \ .$$

Then there exists an integer $m \geq 1$ such that for every $\mathcal{F}\colon \mathcal{C} \to \mathbf{Pr}^{\mathrm{L}}_{\mathrm{st}}$, $\mathrm{St}_{\mathcal{I},\mathcal{F}}$ is a retract of

$$\lim_{[n]\in \mathbf{\Delta}_{s,\leq m}} \mathrm{St}_{\mathcal{I}_n,\mathcal{F}}$$

in $\mathrm{Fun}(\mathcal{C}, \mathbf{Pr}^{\mathrm{L,R}})$.

Proof. The Stokes stratified space $(X,P,\mathcal{I})$ is stably universal in virtue of Lemma 12.2.5. Since $f\colon Y \to X$ is Galois, Y_n is a finite coproduct of copies of Y over X, so that $(Y_n, P, \mathcal{I}_n)$ is stably universal for every $[n] \in \mathbf{\Delta}_s$. Since the $Y_n \to X$ is a finite étale cover for every $[n] \in \mathbf{\Delta}_s$, Lemma 19.2.9 implies that

$$\Pi_\infty(Y_n, P) \to \Pi_\infty(X,P)$$

is a finite étale fibration in the sense of Definition 19.2.1. Corollary 8.8.6 thus implies the existence of an integer $m \geq 1$ such that there exists a retract

$$\mathrm{St}_{\mathcal{I},\mathbf{Sp}} \to \lim_{[n]\in \mathbf{\Delta}_{s,\leq m}} \mathrm{St}_{\mathcal{I}_n,\mathbf{Sp}} \to \mathrm{St}_{\mathcal{I},\mathbf{Sp}} \ .$$

in $\mathbf{Pr}^{\mathrm{L,R}}$. Thus, we conclude from Lemma 7.5.5 by tensoring the above retract with $\mathcal{F}$. $\square$

We are now ready for:

Proof of Theorem 14.2.2. Since f is strongly proper, we can choose a categorically compact subanalytic refinement $R \to Q$ such that for every $c \in \mathcal{C}$, $f_*(\mathfrak{St}_{\mathcal{I},\mathcal{F}}(c))$ is R-hyperconstructible. Thus, $f_*(\mathfrak{St}_{\mathcal{I},\mathcal{F}})$ is itself R-hyperconstructible. Let $F\colon \Pi_\infty(Y,R) \to \mathrm{Fun}(\mathcal{C}, \mathbf{CAT}_\infty)$ be the functor corresponding to $f_*(\mathfrak{St}_{\mathcal{I},\mathcal{F}})$ via the exodromy equivalence (2.3.6). By Recollection 2.3.5, we have

$$\mathrm{St}_{\mathcal{I},\mathcal{F}} \simeq f_*(\mathfrak{St}_{\mathcal{I},\mathcal{F}})(Y) \simeq \lim_{y\in \Pi_\infty(Y,R)} F(y) \ .$$

Since $\Pi_\infty(Y,R)$ is categorically compact and $\mathcal{D}$ is closed under finite limits and retracts, we are reduced to show that $F(y) \in \mathcal{D}$ for every $y \in Y$. The base-change results Propositions 11.1.5 and 2.5.10 and Lemma 11.3.12 allow to reduce to the case where Y is a point and X is compact.

In this case, there exists a cartesian finite Galois cover $(Y,P,\mathcal{J}) \to (X,P,\mathcal{I})$ such that $(Y,P,\mathcal{J})$ admits a piecewise elementary level structure. Recall that (X,P) is conically refineable in virtue of Remark 2.5.4 and that $\Pi_\infty(X)$ is finite by Proposition 2.5.7. Hence, Lemma 14.2.6 implies the existence of an integer $m \geq 1$ such that $\mathrm{St}_{\mathcal{I},\mathcal{F}}$ is a retract of

$$\lim_{[n]\in \mathbf{\Delta}_{s,\leq m}} \mathrm{St}_{\mathcal{I}_n,\mathcal{F}} \tag{14.2.7}$$

in $\mathrm{Fun}(\mathcal{C}, \mathbf{Pr}^{\mathrm{L,R}})$, where $Y_\bullet\colon \mathbf{\Delta}_s^{\mathrm{op}} \to \mathcal{T}\mathrm{op}_{/X}$ is the Cech complex of $f\colon Y \to X$ and where

$$\mathcal{I}_\bullet := \Pi_\infty(Y_\bullet, P) \times_{\Pi_\infty(X,P)} \mathcal{I} \ .$$

Since $\mathcal{D}$ is closed under finite limits and retracts, it is enough to show that for every integer n, $\mathrm{St}_{\mathcal{I}_n,\mathcal{F}}$ belongs to $\mathcal{D}$. Since $Y \to X$ is a finite Galois cover, each Y_n is a finite coproduct of copies of Y. So it is enough to treat the case of Y. Equivalently, we can suppose that $(X,P,\mathcal{I})$ admits a piecewise elementary level structure.

We now argue by induction on the length d of the locally elementary level structure of $(X,P,\mathcal{I})$. When $d = 0$, $\mathcal{I} = \Pi_\infty(X,P)$ is a fibration in sets, so $(X,P,\mathcal{I})$ is (globally) elementary and the conclusion follows from Lemma 14.2.5 (or from Lemma 14.2.4). Otherwise, we can assume the existence of a level morphism $p\colon \mathcal{I} \to \mathcal{J}$ such that:

(1) $\mathcal{J}$ admits a piecewise elementary level structure of length $< d$;

(2) $(X, P, \mathcal{I}_p)$ is piecewise elementary.

Notice that since level morphisms are surjective, the fibers of $\mathcal{J}$ are again finite posets, so the inductive hypothesis applies to the Stokes stratified space $(X, P, \mathcal{J})$. Consider the pullback square

$$\begin{tikzcd} \mathrm{St}_{\mathcal{I},\mathcal{F}} \arrow[r, "p_!"] \arrow[d, "\mathrm{Gr}_p"] & \mathrm{St}_{\mathcal{J},\mathcal{F}} \arrow[d, "\mathrm{Gr}"] \\ \mathrm{St}_{\mathcal{I}_p,\mathcal{F}} \arrow[r, "\pi_!"] & \mathrm{St}_{\mathcal{J}^{\mathrm{set}},\mathcal{F}} \end{tikzcd}$$

supplied by an objectwise application of Theorem 10.2.1. Both $\mathrm{St}_{\mathcal{I}_p,\mathcal{F}}$ and $\mathrm{St}_{\mathcal{J}^{\mathrm{set}},\mathcal{F}}$ belong to $\mathcal{D}$ thanks to Lemma 14.2.5, while the inductive hypothesis guarantees that $\mathrm{St}_{\mathcal{J},\mathcal{E}}$ are of finite type. Thus, the conclusion follows from the closure of $\mathcal{D}$ under finite limits. $\square$

15. Elementarity and polyhedral Stokes stratified spaces

The goal of this section is to prove an elementarity criterion for a specific class of Stokes stratified spaces that we now introduce.

15.1. Polyhedral Stokes stratified spaces.

Recollection 15.1.1. For $n \geq 0$, recall that a *polyhedron* of $\mathbb{R}^n$ is a non empty subset obtained as the intersection of a finite number of closed half spaces.

In what follows, $\{-, 0, +\}$ will denote the span poset where 0 is declared to be the initial object. Let $n \geq 0$. For a non zero affine form $\varphi\colon \mathbb{R}^n \to \mathbb{R}$, we denote by H_φ the zero locus of φ.

Definition 15.1.2. Let $n \geq 0$ and let $C \subset \mathbb{R}^n$ be a polyhedron. Let Φ be a finite set of non zero affine forms on $\mathbb{R}^n$. Let $(\mathbb{R}^n, \Phi)$ be the stratified space given by the continuous function $C \to \{-, 0, +\}^\Phi$ sending $x \in C$ to the function sending φ to the sign of $\varphi(x)$ if $x \notin H_\varphi$, and to 0 otherwise.

Remark 15.1.3. The stratified space (C, Φ) is conical and the induced functor $\Pi_\infty(C, \Phi) \to \{-, 0, +\}^\Phi$ is an equivalence of ∞-categories.

Definition 15.1.4. A *polyhedral Stokes stratified space* is a Stokes stratified space in finite posets of the form $(C, \Phi, \mathcal{I})$ where (C, Φ) is as in Definition 15.1.2 and such that $\mathcal{I}^{\mathrm{set}} \to \Pi_\infty(C, \Phi)$ is locally constant (Definition 19.1.4).

15.2. Elementarity criterion: statements and counter-examples.

The main result of this section is the following theorem whose statement is inspired from [36, Proposition 3.16].

Theorem 15.2.1. *Let $(C, \Phi, \mathcal{I})$ be a polyhedral Stokes stratified space. Suppose that for every distinct $a, b \in \mathscr{I}(C)$, there exists $\varphi \in \Phi$ such that*

(1) The Stokes locus of $\{a, b\}$ is $C \cap H_\varphi$ (Definition 4.2.2).

(2) $C \setminus H_\varphi$ admits exactly two connected components C_1 and C_2.

(3) $a <_x b$ for every $x \in C_1$ and $b <_x a$ for every $x \in C_2$.

Then $(C, \Phi, \mathcal{I})$ is elementary (Definition 11.3.10).

Remark 15.2.2. In the setting of Theorem 15.2.1, the order of $\mathcal{I}_x$ is total for every x lying in an open stratum of (C, Φ).

Remark 15.2.3. Fully-faithfulness in Theorem 15.2.1 will not require any extra technology that the one developed so far and will be proved in Proposition 15.5.1. On the other hand, essential surjectivity will require more work and will be ultimately proved in Proposition 15.5.3.

Theorem 15.2.1 will be used via the following:

Theorem 15.2.4. *Let $(C, P, \mathcal{I})$ be a Stokes analytic stratified space in finite posets where $C \subset \mathbb{R}^n$ is a polyhedron and $\mathcal{I}^{\mathrm{set}} \to \Pi_\infty(C, P)$ is locally constant. Assume that for every distinct $a, b \in \mathscr{I}(C)$, there exists a non zero affine form $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ such that*

(1) The Stokes locus of $\{a, b\}$ is $C \cap H_\varphi$.

(2) $C \setminus H_\varphi$ admits exactly two connected components C_1 and C_2.

(3) $a <_x b$ for every $x \in C_1$ and $b <_x a$ for every $x \in C_2$.

Then $(C, P, \mathcal{I})$ is elementary.

Proof. Let Φ be a finite set of non zero affine forms such that for every distinct $a, b \in \mathscr{I}(C)$, there is $\varphi \in \Phi$ satisfying (1),(2),(3) for a, b. By Lemma 11.3.12, the conclusion of Theorem 15.2.4 is insensitive to subanalytic refinements. Hence, at the cost of refining (C, P), we can suppose that there exists a refinement $(C, P) \to (C, \Phi)$. By Proposition 2.3.8, the induced functor

$$\Pi_\infty(C, P) \to \Pi_\infty(C, \Phi)$$

exhibits $\Pi_\infty(C, \Phi)$ as the localization of $\Pi_\infty(C, P)$ at the set of arrows sent to equivalences by $\Pi_\infty(C, P) \to P \to \Phi$. On the other hands, conditions (1) and (3) say that for every morphism $\gamma\colon x \to y$ in $\Pi_\infty(C, P)$ sent to an equivalence by $\Pi_\infty(C, P) \to P \to \Phi$, the induced morphism of posets $\mathcal{I}_x \to \mathcal{I}_y$ is an isomorphism. Hence, there is a cocartesian fibration in finite posets $\mathcal{J} \to \Pi_\infty(C, \Phi)$ and a cartesian morphism

$$(C, P, \mathcal{I}) \to (C, \Phi, \mathcal{J}) \ .$$

Hence, we are left to show that $(C, \Phi, \mathcal{J})$ is elementary, which is a consequence of Theorem 15.2.1. $\square$

Polyhedral Stokes stratified spaces are rarely elementary, as the following Lemmas below show:

Lemma 15.2.5. *Let $(C, \Phi, \mathcal{I})$ be a polyhedral Stokes stratified space of $\mathbb{R}$. Let $\mathcal{E}$ be a presentable stable ∞-category. Assume that $i_{\mathcal{I},!}\colon \mathrm{St}_{\mathcal{I}^{\mathrm{set}},\mathcal{E}} \to \mathrm{St}_{\mathcal{I},\mathcal{E}}$ is fully faithful. Then for every $a, b \in \mathscr{I}(C)$ distinct, the Stokes locus $C_{a,b}$ is not empty.*

Proof. Assume the existence of $a, b \in \mathscr{I}(C)$ distinct such that $C_{a,b}$ is empty. We show that $i_{\mathcal{I},!}\colon \mathrm{St}_{\mathcal{I}^{\mathrm{set}},\mathcal{E}} \to \mathrm{St}_{\mathcal{I},\mathcal{E}}$ is not fully faithful. Let $i\colon \mathcal{J} \hookrightarrow \mathcal{I}$ be the cocartesian subfibration defined by a and b. By Corollary 9.5.3, it is enough to show that $i_{\mathcal{J},!}\colon \mathrm{St}_{\mathcal{J}^{\mathrm{set}},\mathcal{E}} \to \mathrm{St}_{\mathcal{J},\mathcal{E}}$ is not fully faithful. Since $C_{a,b}$ is empty, the cocartesian fibration $\mathcal{J} \to \Pi_\infty(C, \Phi)$ is locally constant. By Proposition 2.3.8, we deduce the existence of a cartesian refinement

$$(C, \Phi, \mathcal{J}) \to (C, *, \mathcal{K})$$

in PosFib. From Corollary 11.1.12, we have $\mathrm{St}_{\mathcal{J},\mathcal{E}} \simeq \mathrm{St}_{\mathcal{K},\mathcal{E}}$. On the other hand, $\Pi_\infty(C)$ is contractible. Hence, if we pick $x \in C$, we deduce $\mathrm{St}_{\mathcal{J},\mathcal{E}} \simeq \mathrm{St}_{\mathcal{J}_x,\mathcal{E}}$ and $\mathrm{St}_{\mathcal{J}^{\mathrm{set}},\mathcal{E}} \simeq \mathrm{St}_{\mathcal{J}^{\mathrm{set}}_x,\mathcal{E}}$. Hence, we are left to show that

$$i_{\Delta^1,!}\colon \mathrm{Fun}(\{0,1\}, \mathcal{E}) \to \mathrm{Fun}(\Delta^1, \mathcal{E})$$

is not fully faithful, which is obvious. $\square$

Lemma 15.2.6. *Let $(C, \Phi, \mathcal{I})$ be a polyhedral Stokes stratified space of $\mathbb{R}$. Let $\mathcal{E}$ be a presentable stable ∞-category. Assume that $i_{\mathcal{I},!}\colon \mathrm{St}_{\mathcal{I}^{\mathrm{set}},\mathcal{E}} \to \mathrm{St}_{\mathcal{I},\mathcal{E}}$ is essentially surjective. Then for every $a, b \in \mathscr{I}(C)$ distinct, the Stokes locus $C_{a,b}$ admits at most one point.*

Proof. Assume the existence of $a, b \in \mathscr{I}(C)$ distinct such that $C_{a,b}$ has at least two points. We show that $i_{\mathcal{I},!} \colon \mathrm{St}_{\mathcal{I}^{\mathrm{set}},\mathcal{E}} \to \mathrm{St}_{\mathcal{I},\mathcal{E}}$ is not essentially surjective. Let $i \colon \mathcal{J} \hookrightarrow \mathcal{I}$ be the cocartesian subfibration defined by a and b. By Corollary 9.5.3, it is enough to show that $i_{\mathcal{J},!} \colon \mathrm{St}_{\mathcal{J}^{\mathrm{set}},\mathcal{E}} \to \mathrm{St}_{\mathcal{J},\mathcal{E}}$ is not essentially surjective. Hence we are reduced to the case where $\mathscr{I}(C)$ has exactly two elements a and b. Let $n \geq 2$ and put $C_{a,b} = \{x_1 < \cdots < x_n\}$. Put $D = C \cap \{x \leq (x_{n-1} + x_n)/2\}$ and consider the commutative square

$$\begin{tikzcd}
\mathrm{St}_{\mathcal{I}^{\mathrm{set}},\mathcal{E}} \arrow[r] \arrow[d, "i_{\mathcal{I},!}"] & \mathrm{St}_{\mathcal{I}_D^{\mathrm{set}},\mathcal{E}} \arrow[d, "i_{\mathcal{I}_D,!}"] \\
\mathrm{St}_{\mathcal{I},\mathcal{E}} \arrow[r] & \mathrm{St}_{\mathcal{I}_D,\mathcal{E}} .
\end{tikzcd}$$

Since D contains all Stokes points but the last one, one readily checks that the bottom arrow of the above square is essentially surjective. Hence, to show that the left vertical arrow is not essentially surjective, it is enough to show that the right vertical arrow is not essentially surjective. Arguing by descending recursion on n, we can thus suppose that $n = 2$. In that case, $\mathcal{I} \to \Pi_\infty(C, \Phi)$ is locally constant away from x_1 and x_2. Arguing as in the proof of Lemma 15.2.5, we can thus suppose that x_1 and x_2 are the only closed strata of (C, Φ). Put $U \coloneqq C \cap \{x < x_2\}$ and $V \coloneqq C \cap \{x > x_1\}$. Then, x_1 is an initial object in $\Pi_\infty(U, \Phi)$ and x_2 is an initial object in $\Pi_\infty(V, \Phi)$. By Proposition 8.2.5, we deduce

$$\mathrm{St}_{\mathcal{I}_U,\mathcal{E}} \simeq \mathrm{St}_{\mathcal{I}_{x_1},\mathcal{E}} \simeq \mathrm{Fun}(\{0,1\}, \mathcal{E})$$

and

$$\mathrm{St}_{\mathcal{I}_V,\mathcal{E}} \simeq \mathrm{St}_{\mathcal{I}_{x_2},\mathcal{E}} \simeq \mathrm{Fun}(\{0,1\}, \mathcal{E}) .$$

Furthermore, since $U \cap V$ is contractible, we have

$$\mathrm{St}_{\mathcal{I}_{U\cap V},\mathcal{E}} \simeq \mathrm{Fun}(\Delta^1 \mathcal{E}) .$$

Since $\mathfrak{St}_{\mathcal{I},\mathcal{E}}$ is a hypersheaf, we thus deduce the existence of a pull-back square

$$\begin{tikzcd}
\mathrm{St}_{\mathcal{I},\mathcal{E}} \arrow[r] \arrow[d] & \mathrm{Fun}(\{0,1\}, \mathcal{E}) \arrow[d, "i_{\Delta^1,!}"] \\
\mathrm{Fun}(\{0,1\}, \mathcal{E}) \arrow[r, "i_{\Delta^1,!}"] & \mathrm{Fun}(\Delta^1, \mathcal{E}) .
\end{tikzcd}$$

We are thus left to show that the diagonal functor

$$\mathrm{Fun}(\{0,1\}, \mathcal{E}) \to \mathrm{Fun}(\{0,1\}, \mathcal{E}) \times_{\mathrm{Fun}(\Delta^1,\mathcal{E})} \mathrm{Fun}(\{0,1\}, \mathcal{E}) \tag{15.2.7}$$

is not essentially surjective. This comes from the observation that any object (V, V, f) with $f_1 \colon V_0 \oplus V_1 \to V_0 \oplus V_1$ not diagonal does not lie in the essential image of (15.2.7). This concludes the proof of Lemma 15.2.6. □

Corollary 15.2.8. *Let $(C, \Phi, \mathcal{I})$ be a polyhedral Stokes stratified space of $\mathbb{R}$. If $(C, \Phi, \mathcal{I})$ is elementary, then for every $a, b \in \mathscr{I}(C)$ distinct, the Stokes locus $C_{a,b}$ is reduced to a point.*

Proof. Combine Lemma 15.2.5 and Lemma 15.2.6. □

15.3. **Distance on the set of open strata.**

Definition 15.3.1. Let (C, Φ) be a stratified polyhedron. For $A, B \subset C$, we say that *A and B are separated by* $\varphi \in \Phi$ if they lie in distinct connected components of $C \setminus H_\varphi$. We let $\Phi(A, B) \subset \Phi$ be the set of forms separating A and B and denote by $d(A, B)$ its cardinality.

Remark 15.3.2. If U, V, W are open strata of (C, Φ), then

$$\Phi(U,V) \subset \Phi(U,W) \cup \Phi(W,V) \ .$$

In particular, d induces a distance on the set of open strata of (C, Φ).

Lemma 15.3.3. *Let (C, Φ) be a stratified polyhedron and let U, V, W be open strata of (C, Φ). Suppose that V and W are distinct and adjacent along a face lying in H_φ for some $\varphi \in \Phi$. Then $\Phi(U,V)$ and $\Phi(U,W)$ differ exactly by φ.*

Proof. Let $\psi \in \Phi(U,V)$. If ψ does not appear in $\Phi(U,W)$, then ψ separates V and W. Hence, $\overline{V} \cap \overline{W} \subset H_\varphi \cap H_\psi$. Since V and W are assumed to be adjacent, $\overline{V} \cap \overline{W}$ has codimension 1. Hence, so does $H_\varphi \cap H_\psi$. Thus $\psi = \varphi$. □

Definition 15.3.4. Let (C, Φ) be a stratified polyhedron and let U be an open stratum. For $k = -1$, put $U_{\leq -1} = U$. For $k \geq 0$, put

$$U_{\leq k} \coloneqq \bigcup_{V, d(U,V) \leq k} \overline{V}$$

where the union runs over the open strata V of (C, Φ) satisfying $d(U,V) \leq k$.

Remark 15.3.5. Let V be an open stratum of (C, Φ) mapping to $f \in \{-,+\}^\Phi$. Then, $\overline{V}$ is the set of points of C lying above the closed subset $S(V) \coloneqq (\{-,0,+\}^\Phi)_{\leq f}$. In particular $U_{\leq k}$ is the set of points of C lying above the closed subset

$$S(U,k) \coloneqq \bigcup_{V, d(U,V) \leq k} S(V) \ .$$

Lemma 15.3.6. *Let (C, Φ) be a stratified polyhedron and let U, V be distinct open strata. Put $k \coloneqq d(U,V) - 1$. Let F be a face of $\overline{V}$. Let $\varphi \in \Phi$ be the unique form such that $F = \overline{V} \cap H_\varphi$. Then, $F \subset U_{\leq k}$ if and only if φ separates U and V. In particular,*

$$\overline{V} \cap U_{\leq k} = \bigcup_{\varphi \in \Phi(U,V)} \overline{V} \cap H_\varphi \ .$$

Proof. Suppose that φ separates U and V. Hence, there is an open stratum $W \neq V$ adjacent to V along F. From Lemma 15.3.3, we have $d(U,W) = k$. Hence, $F \subset \overline{W} \subset U_{\leq k}$. On the other hand, suppose that $F \subset U_{\leq k}$. By definition, there is an open stratum W with $d(U,W) \leq k$ such that F is a face of $\overline{W}$. In particular, $W \neq V$. Thus, Lemma 15.3.3 ensures that $\Phi(U,V)$ and $\Phi(U,W)$ differ exactly by φ. Since $d(U,V) > d(U,W)$, we necessarily have $\varphi \in \Phi(U,V)$ and Lemma 15.3.6 is proved. □

Lemma 15.3.7. *Let (C, Φ) be a stratified polyhedron and let U, V be distinct open strata. Put $k \coloneqq d(U,V) - 1$. Then, $\overline{V} \cap U_{\leq k} \to \overline{V}$ admits a deformation retract. In particular, $\overline{V} \cap U_{\leq k}$ is contractible.*

Proof. Fix $x \in U$. At the cost of replacing some forms in Φ by their opposite, we can suppose that $\overline{V}$ is the set of points $x \in C$ such that $\varphi(x) \geq 0$ for every $\varphi \in \Phi$. For $y \in \overline{V}$, define the following degree $k+1$ polynomial

$$P_V(y) \colon t \mapsto \prod_{\varphi \in \Phi(U,V)} \varphi((1-t) \cdot x + t \cdot y) \ .$$

Then, $P_V(y)$ has exactly $k+1$ roots in $(0,1]$ counted with multiplicities. Let $t_V(y) \in (0,1]$ be the biggest root of $P_V(y)$ and put

$$p_V(y) \coloneqq (1 - t_V(y)) \cdot x + t_V(y) \cdot y \ .$$

Since the coefficients of $P_V(y)$ depend continuously on y, so does $t_V(y)$. Hence, p_V varies continuously in y. Let $y \in \overline{V}$. We want to show that $[y, p_V(y)] \subset \overline{V}$. If $y = p_V(y)$, there is nothing to prove. Suppose that $y \neq p_V(y)$ and pick $z \in (y, p_V(y))$. If φ separates U and V, the non zero real numbers $\varphi(y)$ and $\varphi(z)$ have the same sign by construction. Hence $\varphi(z) > 0$. If φ does not separate U and V, we have $\varphi(x) > 0$. Since $\varphi(y) \geq 0$, we deduce $\varphi(z) \geq 0$. Hence, $(y, p_V(y)) \subset \overline{V}$, so that $[y, p_V(y)] \subset \overline{V}$. By Lemma 15.3.6, we deduce that $p_V(y) \in \overline{V} \cap U_{\leq k}$. Note that if $y \in \overline{V} \cap U_{\leq k}$, then y lies on a face of $\overline{V}$ separating U and V by Lemma 15.3.6. Hence, $P_V(y)$ vanishes at $t = 1$, so that $p_V(y) = y$. Thus, the continuous function $[0,1] \times \overline{V} \to \overline{V}$ defined as

$$(u, y) \mapsto u \cdot p_V(y) + (1-u) \cdot y$$

provides the sought-after deformation retract. □

Construction 15.3.8. Let (C, Φ) be a stratified polyhedron and let U be an open strata. Let $k \geq 0$ and put $S(U, k+1)^\circ \coloneqq S(U, k+1) \setminus S(U, k)$. Observe that $S(U, k+1)^\circ$ is open in $S(U, k+1)$. Consider the following pushout of posets

$$\begin{tikzcd} S(U,k+1)^\circ \arrow[r] \arrow[d] & S(U,k+1) \arrow[d] \\ * \arrow[r] & P(U,k+1) \end{tikzcd} .$$

Since $S(U, k+1)^\circ$ is open in $S(U, k+1)$, the stratified space $(U_{k+1}, P(U, k+1))$ is conically stratified and admits $U_{\leq k+1} \setminus U_{\leq k}$ as open stratum.

Lemma 15.3.9. *Let (C, Φ) be a stratified polyhedron and let U be an open stratum. Let $k \geq 0$. Then, the induced functor*

$$\Pi_\infty(U_{\leq k}, S(U,k)) \to \Pi_\infty(U_{\leq k+1}, P(U, k+1)) \tag{15.3.10}$$

is final.

Proof. To prove Lemma 15.3.9, it is enough to prove that for $x \in U_{\leq k+1}$, the ∞-category

$$\mathcal{X} \coloneqq \Pi_\infty(U_{\leq k}, S(U,k)) \times_{\Pi_\infty(U_{\leq k+1}, P(U,k+1))} \Pi_\infty(U_{\leq k+1}, P(U, k+1))_{/x}$$

is weakly contractible. Since $S(U, k)$ is closed in $P(U, k+1)$, the functor (15.3.10) is fully-faithful. Hence, we can suppose that $x \in U_{\leq k+1} \setminus U_{\leq k}$. In that case, let $V \subset U_{\leq k+1}$ be an open stratum at distance $k+1$ from U such that $x \in \overline{V}$. By Remark 15.1.3, the ∞-category $\mathcal{X}$ is equivalent to the full subcategory of $\Pi_\infty(U_{\leq k}, S(U,k))$ spanned by points y at the source of some exit-path $\gamma\colon y \to x$ in $\Pi_\infty(U_{\leq k+1}, P(U, k+1))$. In particular $\gamma((0,1]) \subset U_{\leq k+1} \setminus U_{\leq k}$. Note that $\gamma((0,1]) \subset \overline{V}$. Indeed if this was not the case, there would exist an open stratum $W \neq V$ adjacent to V with $d(U, W) = k+1$. This is impossible by Lemma 15.3.3. Hence $y \in \overline{V} \cap U_{\leq k}$. On the other hand, for $y \in \overline{V} \cap U_{\leq k}$, the line joining y to x is a morphism in $\Pi_\infty(U_{\leq k+1}, P(U, k+1))$. Hence, $\mathcal{X}$ is equivalent to the full subcategory of $\Pi_\infty(U_{\leq k}, S(U,k))$ spanned by points $y \in \overline{V} \cap U_{\leq k}$, that is

$$\mathcal{X} \simeq \Pi_\infty(\overline{V} \cap U_{\leq k}, S(V) \cap S(U, k)) .$$

Hence,

$$\mathrm{Env}(\mathcal{X}) \simeq \Pi_\infty(\overline{V} \cap U_{\leq k}) \simeq *$$

where the last equivalence follows from Lemma 15.3.7. □

15.4. **Splitting propagation.**

Definition 15.4.1. Let (C,Φ) be a stratified polyhedron and let U be an open stratum. Let $W(U)$ be the class of morphisms $\gamma\colon x\to y$ in $\Pi_\infty(C,\Phi)$ such that for every $\varphi\in\Phi$ with $x\in H_\varphi$, one of the following condition is satisfied:

(i) We have $y\in H_\varphi$.

(ii) The point y and U are not separated by H_φ.

In particular, $W(U)$ contains every equivalence of $\Pi_\infty(C,\Phi)$.

Here are some examples of arrows in the class $W(U)$.

Lemma 15.4.2. *Let (C,Φ) be a stratified polyhedron and let U be an open stratum. Let $k\geq 0$. Then, every exit path of $(U_{\leq k+1}\setminus U_{\leq k},\Phi)$ lies in $W(U)$.*

Proof. Let $\gamma\colon x\to y$ be an exit path of $(U_{\leq k+1}\setminus U_{\leq k},\Phi)$. Let V be a stratum at distance $k+1$ from U with $x\in\overline{V}$. Let $\varphi\in\Phi$ with $x\in H_\varphi$ and assume that $y\notin H_\varphi$. Since $x\notin U_{\leq k}$, Lemma 15.3.6 ensures that φ does not separate U and V. Since $\gamma\colon x\to y$ lies in $U_{\leq k+1}$ we deduce that φ does not separate y and U. □

The class of maps from Definition 15.4.1 is useful because of the following

Lemma 15.4.3. *Let (C,Φ) be a stratified polyhedron and let U be an open stratum. Let $\mathcal{E}$ be a presentable ∞-category. Let $F\colon\Pi_\infty(C,\Phi)\to\mathcal{E}$ be a functor inverting every arrow in $W(U)$. Then, the canonical morphism*

$$\lim_{\Pi_\infty(C,\Phi)} F\to\lim_{\Pi_\infty(U,\Phi)} F|_U$$

is an equivalence.

Proof. To prove Lemma 15.4.3, it is enough to prove that

$$\lim_{\Pi_\infty(U_{\leq k},S(U,k))} F|_{U_{\leq k}}\to\lim_{\Pi_\infty(U_{\leq k-1},S(U,k-1))} F|_{U_{\leq k-1}} \tag{15.4.4}$$

is an equivalence for every $k\geq 0$, where we used the notations of Construction 15.3.8. Assume that $k\geq 1$. Since

$$(U_{\leq k},S(U,k))\to(U_{\leq k},P(U,k))$$

is a refinement, we know by Proposition 2.3.8 that the functor

$$\Pi_\infty(U_{\leq k},S(U,k))\to\Pi_\infty(U_{\leq k},P(U,k)) \tag{15.4.5}$$

exhibits the target as the localization of the source at the exit paths in $U_{\leq k}\setminus U_{\leq k-1}$. By Lemma 15.4.2, the functor (15.4.5) is thus a localization functor at some arrows in $W(U)$. Hence, the functor

$$F|_{U_{\leq k}}\colon\Pi_\infty(U_{\leq k},S(U,k))\to\mathcal{E}$$

factors uniquely through $\Pi_\infty(U_{\leq k},P(U,k))$. Since a localization functor is final, to prove that (15.4.4) is an equivalence thus amounts to prove that the functor

$$\Pi_\infty(U_{\leq k-1},S(U,k-1))\to\Pi_\infty(U_{\leq k},P(U,k))$$

is final, which follows from Lemma 15.3.9. The case where $k=0$ is treated similarly. □

The following lemma provides two examples of functors to which Lemma 15.4.3 applies :

Lemma 15.4.6. *Let $(C,\Phi,\mathcal{I})$ be a polyhedral Stokes stratified space satisfying the conditions of Theorem 15.2.1. Let U be an open stratum. Let $a\in\mathscr{I}(C)$ minimal on U. Let $\mathcal{E}$ be a presentable stable ∞-category. Let $F\colon\mathcal{I}\to\mathcal{E}$ be a Stokes functor. Then $F_{<a}$ and F_a invert arrows in $W(U)$.*

Proof. Consider the fibre sequence

$$F_{<a} \to F_a \to \operatorname{Gr}_a F \ .$$

By Corollary 18.1.2, the functor $\operatorname{Gr}_a F$ inverts every arrow of $\Pi_\infty(C, \Phi)$. Hence, we are left to show that F_a invert arrows in $W(U)$. Let $\gamma \in W(U)$. At the cost of writing γ as the composition of a smaller path followed by an equivalence, we can suppose that γ lies in an open subset V such that x is initial in $\Pi_\infty(V, \Phi)$. From Proposition 8.2.5, we have $F|_V = i_{\mathcal{I}!}(V)$ where $V \colon \mathcal{I}^{\mathrm{set}} \to \mathcal{E}$. Then, $F_a(\gamma)$ reads as

$$\bigoplus_{\substack{b \in \mathscr{I}(C) \\ b \leq_x a}} V_b \to \bigoplus_{\substack{b \in \mathscr{I}(C) \\ b \leq_y a}} V_b \ .$$

Let $b \in \mathscr{I}(C)$ with $b \neq a$. To prove Lemma 15.4.6, we are left to show that $b <_x a$ if and only if $b <_y a$. The direct implication is obvious. We thus suppose that $b <_y a$. Let $\varphi \in \Phi$ such that the Stokes locus of $\{a, b\}$ is $C \cap H_\varphi$. Since a is minimal on U, the assumption (1) from Theorem 15.2.1 implies that φ separates y and U. If $x \in H_\varphi$, then the definition of $W(U)$ yields $y \in H_\varphi$, which contradicts $b <_y a$. Hence, $x \notin H_\varphi$. In particular, $b <_x a$ or $a <_x b$. Note that the inequality $a <_x b$ contradicts $b <_y a$. Hence, $b <_x a$ and the proof of Lemma 15.4.6 is complete. □

Lemma 15.4.7. *Let $(C, \Phi, \mathcal{I})$ be a polyhedral Stokes stratified space satisfying the conditions of Theorem 15.2.1. Let U be an open stratum. Let $a \in \mathscr{I}(C)$ minimal element on U. Let $\mathcal{E}$ be a presentable stable ∞-category. Let $F \colon \mathcal{I} \to \mathcal{E}$ be a Stokes functor. Then, the fiber sequence*

$$F_{<a} \to F_a \to \operatorname{Gr}_a F \tag{15.4.8}$$

admits a splitting.

Proof. Since a is minimal on U, the restriction of $F_{<a}$ to U is the zero functor. Hence, (15.4.8) admits a canonical splitting on U. By Corollary 9.4.6, the functor $\operatorname{Gr} F \colon \mathcal{I}^{\mathrm{set}} \to \mathcal{E}$ is cocartesian. By Corollary 18.1.2, we deduce that $\operatorname{Gr}_a F \colon \Pi_\infty(C, \Phi) \to \mathcal{E}$ inverts every arrows. Since

$$\operatorname{Env}(\Pi_\infty(C, \Phi)) \simeq \Pi_\infty(C) \simeq * \ ,$$

we deduce that $\operatorname{Gr}_a F \colon \Pi_\infty(C, \Phi) \to \mathcal{E}$ is a constant functor. Hence, it is enough to show that

$$\operatorname{Map}(\operatorname{Gr}_a F, F_a) \to \operatorname{Map}(\operatorname{Gr}_a F|_U, F_a|_U)$$

is an equivalence This amounts to show that

$$\lim_{\Pi_\infty(C, \Phi)} F_a \to \lim_{\Pi_\infty(U, \Phi)} F_a|_U$$

is an equivalence. By Lemma 15.4.3, we are thus left to show that F_a inverts every arrow in $W(U)$. This in turn holds by Lemma 15.4.6. □

15.5. **Proof of Theorem 15.2.1.** The proof will be the consequence of the following propositions below.

Proposition 15.5.1. *Let $(C, \Phi, \mathcal{I})$ be a polyhedral Stokes stratified space satisfying the conditions of Theorem 15.2.1. Let $\mathcal{E}$ be a presentable stable ∞-category. Then, the induction functor*

$$i_{\mathcal{I},!} \colon \mathrm{St}_{\mathcal{I}^{\mathrm{set}}, \mathcal{E}} \to \mathrm{St}_{\mathcal{I}, \mathcal{E}}$$

is fully faithful.

Proof. Let $V, W \colon \mathcal{I}^{\mathrm{set}} \to \mathcal{E}$ be Stokes functors. We have to show that the canonical map

$$\operatorname{Map}(V, W) \to \operatorname{Map}_{\mathrm{St}_{\mathcal{I}, \mathcal{E}}}(i_{\mathcal{I},!}(V), i_{\mathcal{I},!}(W)) \simeq \operatorname{Map}(V, i_{\mathcal{I}}^* i_{\mathcal{I},!}(W))$$

is an equivalence. This is equivalent to show that for every $a \in \mathscr{I}(C)$, the map

$$\operatorname{Map}(V_a, W_a) \to \operatorname{Map}(V_a, (i_{\mathcal{I},!}(W))_a)$$

is an equivalence. By Corollary 18.1.2, the cocartesian functor $V_a\colon \Pi_\infty(C,\Phi) \to \mathcal{E}$ inverts every arrow in $\Pi_\infty(C,\Phi)$. Since C is contractible, we deduce that $V_a\colon \Pi_\infty(C,\Phi) \to \mathcal{E}$ is a constant functor. Thus, we are left to show that for every $a \in \mathscr{I}(C)$, the map

$$\lim_{\Pi_\infty(C,\Phi)} W_a \to \lim_{\Pi_\infty(C,\Phi)} (i_{\mathcal{I},!}(W))_a \tag{15.5.2}$$

is an equivalence. At the cost of writing $W\colon \mathcal{I}^{\mathrm{set}} \to \mathcal{E}$ as a finite direct sum over $\mathscr{I}(C)$, we can suppose the existence of $b \in \mathscr{I}(C)$ such that $W_a \simeq 0$ for $a \neq b$. In that case, let $i_b\colon \mathcal{I}_b \hookrightarrow \mathcal{I}$ be the cocartesian fibration constant to b, so that $W \simeq i^{\mathrm{set}}_{b,!}(W_b)$ with $W_b\colon \Pi_\infty(C,\Phi) \to \mathcal{E}$ constant to an object $e \in \mathcal{E}$. Thus, $i_{\mathcal{I},!}(W) \simeq i_{b,!}(W_b)$. In particular,

$$(i_{\mathcal{I},!}(W))_b \simeq i_b^* i_{b,!}(W_b) \simeq W_b \ .$$

Hence, we are left to prove that (15.5.2) is an equivalence for $a \in \mathscr{I}(C)$ with $a \neq b$. Let $\varphi \in \Phi$ such that the Stokes locus of $\{a,b\}$ is H_φ. Let C_1 and C_2 be the two connected components of $C \setminus H_\varphi$ such that $a <_x b$ for every $x \in C_1$ and $b <_x a$ for every $x \in C_2$. Then

$$\begin{aligned} (i_{\mathcal{I},!}(W))_a(x) \simeq (i_{b,!}(W_b))_a(x) \quad &\simeq 0 \quad \text{if } x \in H_\varphi \text{ or } x \in C_1, \\ &\simeq e \quad \text{if } x \in C_2 \ . \end{aligned}$$

Hence both functors in (15.5.2) invert every exit-path in C_1, in C_2 and in H_φ. Consider the map

$$\mathrm{ev}_\varphi\colon \{-,0,+\}^\Phi \to \{-,0,+\}$$

given by evaluation at φ. By Proposition 2.3.8, the refinement

$$(C,\Phi) \to (C,\{-,0,+\})$$

induces a functor

$$\Pi_\infty(C,\Phi) \to \Pi_\infty(C,\{\varphi\})$$

exhibiting the target as the localization of the source at the exit paths in C_1, in C_2 and in H_φ. Since localization functors are final, we are left to prove that (15.5.2) is an equivalence when $\Phi = \{\varphi\}$ and $W = W \simeq i^{\mathrm{set}}_{b,!}(W_b)$. In that case, $\Pi_\infty(C,\Phi) \to \{-,0,+\}$ is an equivalence. Thus, any point x of H_φ is initial in $\Pi_\infty(C,\Phi)$. Hence, the map (15.5.2) identifies canonically with

$$(i^{\mathrm{set}}_{b,!}(W_b))_a(x) \to (i_{\mathcal{I},!}(W))_a(x) \ .$$

Since both terms are 0, Proposition 15.5.1 follows. □

Proposition 15.5.3. *Let $(C,\Phi,\mathcal{I})$ be a polyhedral Stokes stratified space satisfying the conditions of Theorem 15.2.1. Let $\mathcal{E}$ be a presentable stable ∞-category. Then, the induction functor*

$$i_{\mathcal{I},!}\colon \mathrm{St}_{\mathcal{I}^{\mathrm{set}},\mathcal{E}} \to \mathrm{St}_{\mathcal{I},\mathcal{E}}$$

is essentially surjective.

Proof. The proof follows the method from [36, Proposition 3.16]. Let $F\colon \mathcal{I} \to \mathcal{E}$ be a Stokes functor. By Corollary 9.4.10, it is enough to show that F splits. We argue by recursion on the cardinality of $\mathscr{I}(C)$. If $\mathscr{I}(C)$ has one element, there is nothing to prove. Suppose that $\mathscr{I}(C)$ has at least two elements. Then, there exist open strata U and V and $a,b \in \mathscr{I}(C)$ distinct such that a is minimal on U and b is minimal on V. Let $i_a\colon \mathcal{I}_a \hookrightarrow \mathcal{I}$ (resp. $i_b\colon \mathcal{I}_b \hookrightarrow \mathcal{I}$) be the cocartesian fibration constant to a (resp. b) and let $i\colon \mathcal{M} \hookrightarrow \mathcal{I}$ be the full subcategory spanned by objects not in $\mathcal{I}_a$ nor $\mathcal{I}_b$. In particular, we have $\mathcal{I}^{\mathrm{set}} = \mathcal{I}^{\mathrm{set}}_a \sqcup \mathcal{I}^{\mathrm{set}}_b \sqcup \mathcal{M}^{\mathrm{set}}$. By Lemma 15.4.7, the fiber sequences

$$F_{<a} \to F_a \to \mathrm{Gr}_a F \quad \text{and} \quad F_{<b} \to F_b \to \mathrm{Gr}_b F$$

admit some splittings. Let us choose some and let $F^{\backslash \mathcal{I}_a}\colon \mathcal{I} \to \mathcal{E}$ and $F^{\backslash \mathcal{I}_b}\colon \mathcal{I} \to \mathcal{E}$ be the corresponding functors as constructed in Section 9.7. By Corollary 9.7.17, we have to show that $F^{\backslash \mathcal{I}_a}$ and $F^{\backslash \mathcal{I}_b}$ split. We are going to show that $F^{\backslash \mathcal{I}_a}$ splits as the argument is the same for $F^{\backslash \mathcal{I}_b}$.

Let $i\colon \mathcal{I}_b \cup \mathcal{M} \hookrightarrow \mathcal{I}$ be the subcategory spanned by the objects of $\mathcal{I}$ not in $\mathcal{I}_a$. Since F is a Stokes functor, Lemma 9.7.7 implies that $F^{\setminus \mathcal{I}_a}$ is a Stokes functor as well. By Lemma 9.7.6, we have $(\mathrm{Gr}\, F^{\setminus \mathcal{I}_a})(c) \simeq 0$ for every c not in $\mathcal{I}_b \cup \mathcal{M}$. By Proposition 9.5.2, we deduce that $F^{\setminus \mathcal{I}_a}$ lies in the essential image of $i_!\colon \mathrm{St}_{\mathcal{I}_b \cup \mathcal{M}, \mathcal{E}} \to \mathrm{St}_{\mathcal{I},\mathcal{E}}$. By recursion assumption applied to $(C, \Phi, \mathcal{I}_b \cup \mathcal{M})$, we deduce that $F^{\setminus \mathcal{I}_a}$ splits. This concludes the proof of Proposition 15.5.3. □

16. Stokes structures and flat bundles

16.1. Real blow-up.

Definition 16.1.1. A *strict normal crossing pair* is the data of (X, D) where X is a complex manifold and D is a strict normal crossing divisor in X.

Notation 16.1.2. Let (X, D) be a strict normal crossing pair and put $U \coloneqq X \setminus D$. Let $D_1, \dots, D_l$ be the irreducible components of D. For $I \subset \{1, \dots, l\}$, we put

$$D_I \coloneqq \bigcap_{i \in I} D_i \quad \text{ and } \quad D_I^\circ \coloneqq \bigcap_{I \subsetneq J} D_I \setminus D_J \ .$$

We denote by $i_I\colon D_I \hookrightarrow X$ and $i_I^\circ\colon D_I \hookrightarrow X$ the canonical inclusions. We note (X, D) for the stratification $X \to \mathrm{Fun}(\{D_1, \dots, D_l\}, \Delta^1)$ induced by the irreducible components of D (see Remark 2.2.5).

Remark 16.1.3. The canonical functor $\Pi_\infty(X, D) \to \mathrm{Fun}(\{1, \dots, l\}, \Delta^1)$ is an equivalence of ∞-categories.

Construction 16.1.4 ([44, §8.b]). Let (X, D) be a strict normal crossing pair. Let $D_1, \dots, D_l$ be the irreducible components of D. For $i = 1, \dots, l$, let $L(D_i)$ be the line bundle over X corresponding to the sheaf $\mathcal{O}_X(D_i)$ and let $S^1 L(D_i)$ be the associated circle bundle. Put

$$S^1 L(D) \coloneqq \bigoplus_{i=1}^{l} S^1 L(D_i) \ .$$

Let $U \subset X$ be an open polydisc with coordinates $(z_1, \dots, z_n)$ and let $z_i = 0$ be an equation of D_i in U. Let $\widetilde{X}_U \subset S^1 L(D)|_U$ be the closure of the image of $(z_i/|z_i|)_{1 \leq i \leq l}\colon U \setminus D \to S^1 L(D)$. Then, the $\widetilde{X}_U$ are independent of the choices made and thus glue as a closed subspace $\widetilde{X} \subset S^1 L(D)$ called the *real-blow up of* X *along* D. We denote by $\pi\colon \widetilde{X} \to X$ the induced proper morphism and by $j\colon X \setminus D \to \widetilde{X}$ the canonical open immersion. For $I \subset \{1, \dots, l\}$ of cardinal $1 \leq k \leq l$, we put $\widetilde{D}_I \coloneqq \pi^{-1}(D_I)$ and $\widetilde{D}_I^\circ \coloneqq \pi^{-1}(D_I^\circ)$ and observe that the restriction

$$\pi|_{D_I^\circ}\colon \widetilde{D}_I^\circ \to D_I^\circ$$

is a S^k-bundle.

Example 16.1.5. Let $\Delta \subset \mathbb{C}^l$ be a polydisc with coordinates $(z_1, \dots, z_l)$, let Y be a complex manifold and put $X = \Delta \times Y$. Let D be the divisor defined by $z_1 \cdots z_l = 0$. Then, $S^1 L(D) = \Delta \times (S^1)^l \times Y$ and

$$\widetilde{X} = \{(z, y, u) \in S^1 L(D) \text{ such that } z_k = |z_k| u_k, 1 \leq k \leq l\} \ .$$

In particular,

$$\widetilde{X} \simeq (\mathbb{R}_{\geq 0} \times S^1)^l \times \mathbb{C}^{n-l}$$

and via the above identification, the inclusion $\widetilde{X} \hookrightarrow S^1 L(D)$ reads

$$(r, u, y) \to (r_1 u_1, \dots, r_l u_l, y, u) \ .$$

Remark 16.1.6. In the situation from Example 16.1.5, let $1 \leq k \leq l$. Then, the map

$$z_k/|z_k| \colon X \setminus D_k \to S^1$$

extends as a map $S^1L(D) \to S^1$ given by $(z, y, u) \to u_k$.

Example 16.1.5 implies the following

Lemma 16.1.7. *Let (X, D) be a strict normal crossing pair. Then, $\widetilde{X}$ is a closed subanalytic subset of $S^1L(D)$ and $\pi \colon \widetilde{X} \to X$ is a subanalytic map.*

Lemma 16.1.8. *Let (X, D) be a strict normal crossing pair such that X admits a smooth compactification. Then, $\pi \colon \widetilde{X} \to X$ is strongly proper (Definition 12.3.1).*

Proof. Let $X \hookrightarrow Y$ be a smooth compactification of X. At the cost of applying resolution of singularities, we can suppose that $Z \coloneqq Y \setminus X$ is a divisor such that $E \coloneqq Z + D$ is a strict normal crossing divisor. In particular, there is a pull-back square

$$\begin{tikzcd} (S^1L(D), \widetilde{X}) \arrow[r, hook] \arrow[d] & (S^1L(E), \widetilde{Y}) \arrow[d] \\ X \arrow[r, hook] & Y \end{tikzcd} .$$

Then Lemma 16.1.8 follows from Lemma 12.3.3. □

Recollection 16.1.9 ([44, §8.c])**.** Let (X, D) be a strict normal crossing pair and put $U \coloneqq X \setminus D$. Let $\pi \colon \widetilde{X} \to X$ be the real blow-up along D and let $j \colon U \hookrightarrow \widetilde{X}$ be the canonical inclusion. We denote by $\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}} \subset j_*\mathcal{O}_U$ the sheaf of analytic functions with moderate growth along D. By definition for every open subset $V \subset \widetilde{X}$, a section of $\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}}$ on V is an analytic function $f \colon V \cap U \to \mathbb{C}$ such that for every open subset $W \subset V$ with D defined by $h = 0$ in a neighbourhood of $\pi(W)$, for every compact subset $K \subset W$, there exist $C_K > 0$ and $N_K \in \mathbb{N}$ such that for every $z \in K \cap U$, we have

$$|f(z)| \leq C_K \cdot |h(z)|^{-N_K} .$$

The following lemma is obvious:

Lemma 16.1.10. *In the setting of Recollection 16.1.9, let $(j_*\mathcal{O}_U)^{\mathrm{lb}} \subset j_*\mathcal{O}_U$ be the subsheaf of locally bounded functions. Then $\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}}$ is a unitary sub $(j_*\mathcal{O}_U)^{\mathrm{lb}}$-algebra of $j_*\mathcal{O}_U$ such that*

$$\mathcal{A}^{\mathrm{mod},\times}_{\widetilde{X}} \subset (j_*\mathcal{O}_U)^{\mathrm{lb}} .$$

Recollection 16.1.11 ([44, Definition 9.2])**.** Let (X, D) be a strict normal crossing pair and put $U \coloneqq X \setminus D$. Let $\pi \colon \widetilde{X} \to X$ be the real blow-up along D and let $j \colon U \hookrightarrow \widetilde{X}$ be the canonical inclusion. For $f, g \in j_*\mathcal{O}_U$, we write

$$f \leq g \text{ if and only if } e^{f-g} \in \mathcal{A}^{\mathrm{mod}}_{\widetilde{X}} .$$

By Lemma 16.1.10, the relation $\leq$ induces an order on $(j_*\mathcal{O}_U)/(j_*\mathcal{O}_U)^{\mathrm{lb}}$. From now on, we view $(j_*\mathcal{O}_U)/(j_*\mathcal{O}_U)^{\mathrm{lb}}$ as an object of $\mathrm{Sh}^{\mathrm{hyp}}(\widetilde{X}, \mathbf{Poset})$.

Remark 16.1.12. Viewing $\pi^*\mathcal{O}_X(*D)$ inside $j_*\mathcal{O}_U$, we have

$$\pi^*\mathcal{O}_X(*D) \cap (j_*\mathcal{O}_U)^{\mathrm{lb}} = \pi^*\mathcal{O}_X .$$

Hence, $\pi^*(\mathcal{O}_X(*D)/\mathcal{O}_X)$ can be seen as a subsheaf of $(j_*\mathcal{O}_U)/(j_*\mathcal{O}_U)^{\mathrm{lb}}$. From now on, we view it as an object of $\mathrm{Sh}^{\mathrm{hyp}}(\widetilde{X}, \mathbf{Poset})$.

16.2. **Sheaf of unramified irregular values.**

Definition 16.2.1. Let X be a topological space. Let $\mathcal{F} \in \mathrm{Sh}^{\mathrm{hyp}}(X, \mathbf{Cat}_\infty)$. We say that $\mathcal{F}$ is *locally generated* if there is a cover by open subsets $U \subset X$ such that for every $x \in U$, the functor $\mathcal{F}(U) \to \mathcal{F}_x$ is essentially surjective. We say that $\mathcal{F}$ is *globally generated* if for every $x \in X$, the functor $\mathcal{F}(X) \to \mathcal{F}_x$ is essentially surjective.

Local and global generation enjoy the following functoriality property:

Lemma 16.2.2. *Let $f\colon Y \to X$ be a morphism of topological spaces. Let $\mathcal{F} \in \mathrm{Sh}^{\mathrm{hyp}}(X, \mathbf{Cat}_\infty)$. If $\mathcal{F}$ is locally (resp. globally) generated, then so is $f^{*,\mathrm{hyp}}(\mathcal{F})$.*

Proof. We argue in the locally generated situation, the globally generated situation being similar. Let $y \in Y$ and put $x = f(y)$. Let $U \subset X$ be an open neighbourhood of x as in Definition 16.2.1. Let $V \subset Y$ be an open neighbourhood of y such that $f(V) \subset U$. For $z \in V$, there is a factorization

$$\mathcal{F}(U) \to (f^{*,\mathrm{hyp}}(\mathcal{F}))(V) \to (f^{*,\mathrm{hyp}}(\mathcal{F}))_z \simeq \mathcal{F}_{f(z)} \ .$$

Since the composition is essentially surjective, so is the second functor. $\square$

Recollection 16.2.3 ([37, Definition 2.4.2])**.** Let (X, D) be a strict normal crossing pair. A *sheaf of unramified irregular values* is a locally generated subsheaf of finite sets $\mathscr{I} \subset \mathcal{O}_X(*D)/\mathcal{O}_X$ in the sense of Definition 16.2.1.

The goal of what follows is to show that a sheaf of unramified irregular values is automatically constructible on (X, D). Before this, a couple of lemmas are needed.

Lemma 16.2.4. *Let X be a complex manifold. Let $i\colon Y \hookrightarrow X$ be a submanifold. Let $\mathcal{E}$ be a quasi-coherent sheaf on X and consider the presheaf-theoretic restriction $i^{-1}\mathcal{E}$. Then,*

(1) *the presheaf $i^{-1}\mathcal{E}$ is a sheaf.*

(2) *Assume furthermore that for every connected open subset $U \subset X$ and every $x \in U \cap Y$, the map $\mathcal{E}(U) \to \mathcal{E}_x$ is injective. Then for every connected open subset $U \subset Y$ and every $x \in U$, the map $(i^{-1}\mathcal{E})(U) \to \mathcal{E}_x$ is injective.*

Proof. We first prove (2). Let $s \in (i^{-1}\mathcal{E})(U)$ such that $s_x = 0$. Then, there is an open subset $V \subset X$ containing U and $e \in \mathcal{E}(V)$ such that $s = [(V, e)]$ in $(i^{-1}\mathcal{E})(U)$. At the cost of replacing V by a union of sufficiently small balls centred at points of U, we can suppose that V is connected. By assumption on $\mathcal{E}$ we have $e = 0$ so that $s = 0$. We now prove (1). Consider the sheaf-theoretic restriction $i^*\mathcal{E}$. We have to show that the sheafification morphism $i^{-1}\mathcal{E} \to i^*\mathcal{E}$ is an isomorphism of presheaves. The question is local. Since sheafification commutes with colimits, it is enough to treat the case where $\mathcal{E} = \mathcal{O}_X$. Hence, we can suppose that $X \subset \mathbb{C}^n$ is a polydisc with coordinates $(z_1, \dots, z_n)$ and Y defined by $z_1 = \cdots = z_l = 0$ for some $1 \leq l \leq n$. By (2), we know that $i^{-1}\mathcal{O}_X$ is separated, so that $i^{-1}\mathcal{O}_X \to i^*\mathcal{O}_X$ is injective. We show the surjectivity. Let $U \subset Y$ be an open subset and let $s \in (i^*\mathcal{O}_X)(U)$. Then, s is an equivalence class of open cover $\mathcal{U}$ of U in Y, and for every $V \in \mathcal{U}$ the datum of a section $s_V = [(W, f_W)] \in (i^{-1}\mathcal{O}_X)(V)$ such that for every $V, V' \in \mathcal{U}$, we have $s_V|_{V \cap V'} = s_{V'}|_{V \cap V'}$. At the cost of shrinking the W's, we can suppose that each W is a union of balls centred at points of V and $Y \cap W = V$. At the cost of passing to a finer cover, we can suppose that each W is a ball and $V = Y \cap W$. In that case, for every such W, W', the intersection $W \cap W'$ is either empty or connected containing a point of U. In the latter case, the analytic functions f_W and $f_{W'}$ coincide on an open subset of $W \cap W'$. Since $W \cap W'$ is connected, they coincide on $W \cap W'$ and thus glue as an analytic function f on the union Ω of the W's. Then, the class $[(\Omega, f)] \in (i^{-1}\mathcal{O}_X)(U)$ maps to s. $\square$

Lemma 16.2.5. *Let $V \subset \mathbb{C}^n$ be an open subset. Let $y, z \in V$ such that $[y, z] \subset V$. Then, there is a polydisc $\delta \subset V$ containing $[y, z]$.*

Proof. At the cost of applying a suitable affine transformation, we can suppose that $y = 0$ and $z = (1, 0, \dots, 0)$. For $n > 0$, consider the polydisc

$$\Delta_k \coloneqq \{x \in \mathbb{C}^n \text{ such that } |x_1 - 1/2| < 1/2 + 1/k, |x_i| < 1/k, i = 2, \dots, n\} \ .$$

If $\Delta_k \not\subseteq V$ for every $k > 0$, one can construct a sequence of $\mathbb{C}^n \setminus V$ converging to a point of $[y, z]$, thus producing a point of $[y, z]$ not in V. Contradiction. Hence, Δ_k for k big enough does the job. □

Proposition 16.2.6. *Let (X, D) be a strict normal crossing pair. Let $\mathscr{I} \subset \mathcal{O}_X(*D)/\mathcal{O}_X$ be a sheaf of unramified irregular values. Then, $\mathscr{I} \in \mathrm{Cons}_D^{\mathrm{hyp}}(X, \mathbf{Set})$.*

Proof. Let $D_1, \dots, D_l$ be the irreducible components of D. Let $I \subset \{1, \dots, l\}$ be a subset. We have to show that $i_I^{\circ,*}(\mathscr{I})$ is locally constant. The question is local. Hence, we can suppose that $X \subset \mathbb{C}^n$ is a polydisc with coordinates $(z_1, \dots, z_n)$ and D defined by $z_1 \cdots z_l = 0$ for some $1 \leq l \leq n$ with $I = \{1, \dots, l\}$. Let $x \in D_I^\circ$. Let $B \subset X$ be a polydisc centred at x. At the cost of shrinking B, we can suppose that $B \cap D_I^\circ$ is a polydisc as well. We have to show that at the cost of shrinking B further, the restriction $(i_I^{\circ,*}\mathscr{I})|_{B \cap D_I^\circ}$ is a constant sheaf, which amounts to show that

$$(i_I^{\circ,*}\mathscr{I})(B \cap D_I^\circ) \to \mathscr{I}_y \tag{16.2.7}$$

is bijective for every $y \in B \cap D_I^\circ$. Since $\mathscr{I}$ is locally generated, we can suppose that (16.2.7) is surjective for every $y \in B \cap D_I^\circ$. Fix $y \in B \cap D_I^\circ$. We have to show that (16.2.7) is injective. Consider the quasi-coherent sheaf $\mathcal{E} \coloneqq \mathcal{O}_B(*(B \cap D_I^\circ))/\mathcal{O}_B$ on B. To show that (16.2.7) is injective, it is enough to show that

$$(i_I^{\circ,*}\mathcal{E})(B \cap D_I^\circ) \to \mathcal{E}_y \tag{16.2.8}$$

is injective. By Lemma 16.2.4, this further amounts to show that

$$(i_I^{\circ,-1}\mathcal{E})(B \cap D_I^\circ) \to \mathcal{E}_y \tag{16.2.9}$$

is injective. Take $s \in (i_I^{\circ,-1}\mathcal{E})(B \cap D_I^\circ)$ and assume that $s_y = 0$. Let $z \in B \cap D_I^\circ$. We want to show that $s_z = 0$. Let us represent s by (V, t) where $t \in \mathcal{E}(V)$ and $B \cap D_I^\circ \subset V \subset B$. By Lemma 16.2.5, there is a polydisc $\Delta \subset V$ containing $[y, z]$. Hence, $t|_\Delta \in \mathcal{E}(\Delta)$ is represented by a meromorphic function $f \in (\mathcal{O}_X(*D))(\Delta)$ which is holomorphic in a neighbourhood of y. Thus, f is holomorphic, so that $s_z = 0$ and the proof of Proposition 16.2.6 is complete. □

Remark 16.2.10. In the setting of Proposition 16.2.6, let us denote by $(\widetilde{X}, \widetilde{D})$ the space $\widetilde{X}$ endowed with the stratification induced by that of D on X. Then, Proposition 16.2.6 yields $\pi^*\mathscr{I} \in \mathrm{Cons}_{\widetilde{D}}^{\mathrm{hyp}}(\widetilde{X}, \mathbf{Set})$.

Under constructibility assumption, local generation can sometimes be upgraded into global generation, due to the following

Lemma 16.2.11. *Let (M, X, P) be a subanalytic stratified space such that $\Pi_\infty(X, P)$ admits an initial object. Then, every locally generated constructible sheaf $\mathcal{F} \in \mathrm{Cons}_P^{\mathrm{hyp}}(X, \mathbf{Cat}_\infty)$ is globally generated.*

Proof. Let $y \in X$. We want to show that $\mathcal{F}(X) \to \mathcal{F}_y$ is essentially surjective. Let $x \in X$ initial in $\Pi_\infty(X, P)$ and let $U \subset X$ be an open neighbourhood of x on which $\mathcal{F}$ is globally generated. At the cost of shrinking U, we can further suppose by Proposition 2.5.6 that x is initial in $\Pi_\infty(U, P)$. Choose a morphism $\gamma : x \to y$ in $\Pi_\infty(X, P)$. At the cost of replacing y by a point of γ distinct from x and sufficiently close to x, we can suppose that $y \in U$. Let $F \colon \Pi_\infty(X, D) \to \mathbf{Cat}_\infty$ be the functor corresponding to $\mathcal{F}$ via the exodromy equivalence (2.3.6). By assumption, the second arrow of

$$\lim_{\Pi_\infty(X,P)} F \to \lim_{\Pi_\infty(U,P)} F \to F(y)$$

is essentially surjective, while the first one is an equivalence since x is initial in both $\Pi_\infty(X,P)$ and $\Pi_\infty(U,P)$. Lemma 16.2.11 thus follows. □

Example 16.2.12. Let $\Delta \subset \mathbb{C}^l$ be a polydisc with coordinates $(z_1,\dots,z_l)$, let Y be a weakly contractible complex manifold and put $X = \Delta \times Y$. Let D be the divisor defined by $z_1 \cdots z_l = 0$. Then 0 is initial in $\Pi_\infty(X,D)$.

Example 16.2.13. Let $\Delta \subset \mathbb{C}^l$ be a polydisc of radius $r > 0$ with coordinates $(z_1,\dots,z_l)$, let Y be a weakly contractible complex manifold and put $X = \Delta \times Y$. Let D be the divisor defined by $z_1 \cdots z_l = 0$. Let $\pi\colon \widetilde{X} \to X$ be the real blow-up of X along D. Let $I_1,\dots,I_l \subset S^1$ be strict open intervals. Then, any point of $[0,r)^l \times I_1 \times \cdots \times I_l \times Y \subset \widetilde{X}$ above the origin is initial in

$$\Pi_\infty([0,r)^l \times I_1 \times \cdots \times I_l \times Y, \widetilde{D}) .$$

Corollary 16.2.14. *Let Y be a weakly contractible complex manifold. Let $\Delta \subset \mathbb{C}^l$ be a polydisc with coordinates $(z_1,\dots,z_l)$ and put $X = \Delta \times Y$. Let D be the divisor defined by $z_1 \cdots z_l = 0$. Let $\mathscr{I} \subset \mathcal{O}_X(*D)/\mathcal{O}_X$ be a sheaf of unramified irregular values. Then, $\mathscr{I}$ is globally generated.*

Proof. Combine Proposition 16.2.6 with Lemma 16.2.11 applied to Example 16.2.12. □

Corollary 16.2.15. *Let $\Delta \subset \mathbb{C}^l$ be a polydisc with coordinates $(z_1,\dots,z_l)$, let Y be a weakly contractible complex manifold and put $X = \Delta \times Y$. Let D be the divisor defined by $z_1 \cdots z_l = 0$. Let $\pi\colon \widetilde{X} \to X$ be the real blow-up of X along D. Let $\mathscr{I} \subset \mathcal{O}_X(*D)/\mathcal{O}_X$ be a sheaf of unramified irregular values. Then, the canonical restriction map*

$$\mathscr{I}(X) \to (\pi^*\mathscr{I})(\widetilde{X})$$

is bijective.

Proof. By Proposition 16.2.6, the sheaf $\mathscr{I}$ is constructible on (X,D), so that $\pi^*\mathscr{I}$ is constructible on $(\widetilde{X},\widetilde{D})$ (see Remark 16.2.10). Let $F\colon \Pi_\infty(X,D) \to \mathbf{Set}$ be the functor corresponding to $\mathscr{I}$ via the exodromy equivalence (2.3.6). By Recollection 2.3.5, we have to show that

$$\lim_{\Pi_\infty(X,D)} F \to \lim_{\Pi_\infty(\widetilde{X},\widetilde{D})} F \circ \pi$$

is an equivalence. Since Y is weakly contractible, we can suppose that Y is a point. Since $\mathbf{Set}$ is a 1-category, the functor $F\colon \Pi_\infty(X,D) \to \mathbf{Set}$ factors uniquely through the homotopy category $\mathrm{ho}(\Pi_\infty(X,D))$ as a functor $G\colon \mathrm{ho}(\Pi_\infty(X,D)) \to \mathbf{Set}$. Hence we are left to show that

$$\lim_{\mathrm{ho}(\Pi_\infty(X,D))} G \to \lim_{\mathrm{ho}(\Pi_\infty(\widetilde{X},\widetilde{D}))} G \circ \pi$$

is an equivalence. To do this, it is enough to show that

$$\mathrm{ho}(\Pi_\infty(X,D)) \to \mathrm{ho}(\Pi_\infty(\widetilde{X},\widetilde{D})) \tag{16.2.16}$$

is final in the 1-categorical sense. If $r > 0$ denotes the radius of Δ, we have

$$\widetilde{X} = [0,r)^l \times (S^1)^l .$$

Since ho commutes with finite products, we obtain

$$\mathrm{ho}(\Pi_\infty(\widetilde{X},\widetilde{D})) \simeq \mathrm{ho}(\Pi_\infty([0,r)^l, D)) \times \mathrm{ho}(\Pi_\infty((S^1)^l)) .$$

Via this equivalence, the functor (16.2.16) identified with the projection on the first term. By [31, 4.1.1.13], we are thus left to show that $\mathrm{ho}(\Pi_\infty((S^1)^l))$ is connected, which is obvious. □

Corollary 16.2.15 implies immediately the following

Corollary 16.2.17. *Let (X, D) be a strict normal crossing pair. Let $\mathscr{I} \subset \mathcal{O}_X(*D)/\mathcal{O}_X$ be a sheaf of unramified irregular values. Let $\pi \colon \widetilde{X} \to X$ be the real blow-up along D. Then the unit transformation*

$$\mathscr{I} \to \pi_* \pi^* \mathscr{I}$$

is an equivalence.

16.3. **Good sheaf of unramified irregular values.**

Definition 16.3.1. Let $X \subset \mathbb{C}^n$ be a polydisc with coordinates $(z, y) \coloneqq (z_1, \dots, z_l, y_1, \dots y_{n-l})$. Let D be the divisor defined by $z_1 \cdots z_l = 0$. Let $a \in \mathcal{O}_{X,0}(*D)/\mathcal{O}_{X,0}$ and consider the Laurent expansion

$$\sum_{m \in \mathbb{Z}^l} a_m(y) z^m \ .$$

We say that a *admits an order* if the set

$$\{m \in \mathbb{Z}^l \text{ with } a_m \neq 0\} \cup \{0\}$$

admits a smallest element, denoted by $\operatorname{ord} a$.

Remark 16.3.2. The existence of an order does not depend on a choice of coordinates on X.

Recollection 16.3.3 ([37, Definition 2.1.2])**.** Let (X, D) be a strict normal crossing pair. Let $x \in X$. A subset $I \subset \mathcal{O}_{X,x}(*D)/\mathcal{O}_{X,x}$ is *good* if

(1) every non zero $a \in I$ admits an order with $a_{\operatorname{ord} a}$ invertible in $\mathcal{O}_{X,x}$.

(2) For every distinct $a, b \in I$, $a - b$ admits an order with $(a-b)_{\operatorname{ord}(a-b)}$ invertible in $\mathcal{O}_{X,x}$.

(3) The set $\{\operatorname{ord}(a-b), a, b \in I\} \subset \mathbb{Z}^l$ is totally ordered.

Recollection 16.3.4 ([37, Definition 2.4.2])**.** Let (X, D) be a strict normal crossing pair. A *good sheaf of unramified irregular values* is a sheaf of unramified irregular values such that for every $x \in X$, the set $\mathscr{I}_x \subset \mathcal{O}_{X,x}(*D)/\mathcal{O}_{X,x}$ is good in the sense of Recollection 16.3.3.

When restricted to good sheaves of irregular values, the order from Recollection 16.1.11 admits a handy characterisation that we now describe.

Recollection 16.3.5 ([37, §3.1.2])**.** Let $\Delta \subset \mathbb{C}^l$ be a polydisc with coordinates $(z_1, \dots, z_l)$, let Y be a complex manifold and put $X = \Delta \times Y$ and $U \coloneqq X \setminus D$. Let $\pi \colon \widetilde{X} \to X$ be the real blow-up along D and let $x \in \widetilde{X}$. Let $a, b \in (\pi^{-1}(\mathcal{O}_X(*D)/\mathcal{O}_X))_x$ and let $\mathfrak{a}$ and $\mathfrak{b}$ be lifts of a and b to $\mathcal{O}_X(*D)$ on some open subset $V \subset X$. By Remark 16.1.6, the function

$$\operatorname{Re}(\mathfrak{a} - \mathfrak{b})|z^{-\operatorname{ord}(a-b)}| \colon V \setminus D \to \mathbb{R}$$

extends as a real analytic function

$$F_{a,b} \colon \pi^{-1}(V) \to \mathbb{R} \ .$$

Then, the following are equivalent:

(1) $a \leq_x b$ in the sense of Recollection 16.1.11;

(2) $a = b$ or $a \neq b$ and $F_{a,b}(x) < 0$.

The goal of what follows is to show that for every *good* sheaf of unramified irregular values $\mathscr{I} \subset \pi^*(\mathcal{O}_X(*D)/\mathcal{O}_X)$, there exists a *finite* subanalytic stratification $\widetilde{X} \to P$ such that $\pi^* \mathscr{I} \in \operatorname{Cons}_P^{\mathrm{hyp}}(\widetilde{X}, \mathbf{Poset})$. Before that, a couple of intermediate steps are needed. To this end, we introduce the following

Definition 16.3.6. Let (M, X) be a subanaltyic stratified space. Let $\mathcal{F} \in \operatorname{Sh}\operatorname{hyp}(X, \mathbf{Poset})$. Note that for $x \in X$, the stalk

$$\mathcal{F}_x = \operatorname*{colim}_{x \in U} \mathcal{F}(U)$$

is naturally endowed with an order $\leq_x$ by performing the above colimit in Poset instead of Set. For an open subset $U \subset X$ and for $a, b \in \mathcal{F}(U)$, we put

$$U_{a<b} := \{x \in U \text{ such that } a_x <_x b_x \text{ in } \mathcal{F}_x\}$$

and

$$U_{a=b} := \{x \in U \text{ such that } a_x = b_x \text{ in } \mathcal{F}_x\}$$

and

$$U_{a*b} := \{x \in U \text{ such that } a_x \text{ and } b_x \text{ cannot be compared in } \mathcal{F}_x\} \ .$$

Remark 16.3.7. Let (M, X) be a subanaltyic stratified space. Let $\mathcal{F} \in \operatorname{Sh}\operatorname{hyp}(X, \mathbf{Poset})$. For every open subset $U \subset X$ and for every $a, b \in \mathcal{F}(U)$, the set $U_{a=b}$ is open and $U_{a<b}$ and $U_{a*b} = U \setminus (U_{a<b} \cup U_{a>b} \cup U_{a=b})$ are locally closed.

Example 16.3.8. Let $\Delta \subset \mathbb{C}^l$ be a polydisc with coordinates $(z_1, \ldots, z_l)$, let Y be a complex manifold and put $X = \Delta \times Y$. Let D be the divisor defined by $z_1 \cdots z_l = 0$. Let $\mathscr{I} \subset \mathcal{O}_X(*D)/\mathcal{O}_X$ be a sheaf of unramified irregular values. Let $\pi \colon \widetilde{X} \to X$ be the real blow-up along D. Let $\alpha, \beta \in \mathscr{I}(X)$ and put $a = \pi^*\alpha \in (\pi^*\mathscr{I})(\widetilde{X})$ and $b = \pi^*\beta \in (\pi^*\mathscr{I})(\widetilde{X})$. Let $A \subset \{1, \ldots, l\}$ be the set of indices i such that $\alpha - \beta$ has a pole along D_i. By Recollection 16.3.5, we have

$$\widetilde{X}_{a=b} = \bigsqcup_{I \subset \{1,\ldots,l\} \setminus A} \widetilde{D}_I^\circ$$

and

$$\widetilde{X}_{a<b} = \bigsqcup_{\substack{I \subset \{1,\ldots,l\} \\ I \cap A \neq \emptyset}} \widetilde{D}_I^\circ \cap \{F_{a,b} < 0\} \ .$$

Furthermore,

$$\widetilde{X}_{a*b} = \widetilde{X} \setminus (\widetilde{X}_{a<b} \cup \widetilde{X}_{a>b} \cup \widetilde{X}_{a=b}) \ .$$

In particular the three sets above are subanalytic in $S^1L(D)$.

Lemma 16.3.9. *Let (M, X, P) be a subanaltyic stratified space where X is closed. Let $\mathcal{F} \in \operatorname{Sh}^{\mathrm{hyp}}(X, \mathbf{Poset})$. Let $? \in \{<, =, *\}$. Assume that*

(1) $\mathcal{F}^{\mathrm{set}} \in \operatorname{Cons}_P^{\mathrm{hyp}}(X, \mathbf{Set})$;

(2) $\mathcal{F}$ is locally generated (Definition 16.2.1) ;

(3) there exists a fundamental system of open neighbourhoods $W \subset M$ such that for every $a, b \in \mathcal{F}(W \cap X)$, the set $(W \cap X)_{a?b}$, is subanalytic in W.

Then, for every open subset $U \subset X$ subanalytic in M, for every $a, b \in \mathcal{F}(U)$, the set $U_{a?b}$ is locally closed subanalytic in M.

Proof. Local closeness is automatic by Remark 16.3.7. Let $x \in M$. We need to show that $U_{a<b}$ is subanalytic in a neighbourhood of x in M. Since X is closed, we can suppose that $x \in X$. At the cost of replacing M by a sufficiently small open neighbourhood of x in M, we can suppose by (2) that $\mathcal{F}$ is globally generated. At the cost of shrinking M further, we can suppose that P is finite. Since U is a subanalytic subset of M, so are the $U_p = U \cap X_p$ for $p \in P$. On the other hand, the set of connected components of a subanalytic subset is locally finite. Hence, at the cost of replacing M by a smaller neighbourhood of x, we can suppose that the U_p have only a finite number of connected components $C_{1,p}, \ldots, C_{n(p),p}$. By global generation, for $p \in P$

and $1 \leq i \leq n(p)$, the sections $a|_{C_{i,p}}, b|_{C_{i,p}}$ extend to X as sections $\alpha_{i,p}, \beta_{i,p}$ of $\mathcal{F}$. At the cost of replacing M by a smaller neighbourhood of x, we can suppose by (3) that the $X_{\alpha_{i,p}?\beta_{i,p}}$ are subanalytic in M. On the other hand,

$$U_{a?b} = \bigsqcup_{p\in P} U_{a?b} \cap U_p = \bigsqcup_{p\in P} \bigsqcup_{i=1}^{n(p)} (C_{i,p})_{a|_{C_{i,p}}?b|_{C_{i,p}}} = \bigsqcup_{p\in P} \bigsqcup_{i=1}^{n(p)} X_{\alpha_{i,p}?\beta_{i,p}} \cap C_{i,p} \ .$$

Since a finite union and intersection of subanalytic subsets is again subanalytic, Lemma 16.3.9 is thus proved. □

Corollary 16.3.10. *Let (X, D) be a strict normal crossing pair. Let $\mathscr{I} \subset \mathcal{O}_X(*D)/\mathcal{O}_X$ be a good sheaf of unramified irregular values. Let $\pi\colon \widetilde{X} \to X$ be the real blow-up along D and consider $\pi^*\mathscr{I} \in \mathrm{Sh}^{\mathrm{hyp}}(\widetilde{X}, \mathbf{Poset})$. For every open subset $U \subset \widetilde{X}$ subanalytic in $S^1L(D)$, for every $a, b \in (\pi^*\mathscr{I})(U)$, the sets $U_{a<b}$,$U_{a=b}$,U_{a*b} are locally closed subanalytic in $S^1L(D)$.*

Proof. Let $? \in \{<, =, *\}$. We prove that $U_{a?b}$ is locally closed subanalytic in M. We check that the conditions of Lemma 16.3.9 are satisfied. First observe that $\widetilde{X}$ is closed in $S^1L(D)$. Condition (1) is satisfied by Remark 16.2.10. Condition (2) is satisfied by Lemma 16.2.2. To check (3), we can suppose that $X \subset \mathbb{C}^n$ is a polydisc with D defined by $z_1 \cdots z_l = 0$. Let $x \in \widetilde{X}$. We want to find a fundamental system of open neighbourhoods of x in $S^1L(D)$ satisfying (3). By Proposition 2.5.6, it is enough to show that any open subset $W \subset S^1L(D)$ such that x is initial in $\Pi_\infty(W \cap \widetilde{X}, \widetilde{D})$ does the job. Indeed let $W \subset S^1L(D)$ be such an open subset and put $U \coloneqq W \cap \widetilde{X}$. Let $a, b \in (\pi^*\mathscr{I})(U)$. By Corollary 16.2.15, the canonical restriction map

$$\mathscr{I}(X) \to (\pi^*\mathscr{I})(\widetilde{X})$$

is bijective with $\mathscr{I}$ and $\pi^*\mathscr{I}$ globally generated in virtue of Corollary 16.2.14 and Lemma 16.2.2. Hence, there is $\alpha, \beta \in \mathscr{I}(X)$ such that $a_x = (\pi^*\alpha)_x$ and $b_x = (\pi^*\beta)_x$. Since x is initial in $\Pi_\infty(W \cap \widetilde{X}, \widetilde{D})$, we obtain $a = (\pi^*\alpha)|_U$ and $b = (\pi^*\beta)|_U$. Thus, we have

$$U_{a<b} = \widetilde{X}_{\pi^*\alpha?\pi^*\beta} \cap W \ .$$

Hence, to show that $U_{a?b}$ is subanalytic in W, it is enough to show that $\widetilde{X}_{\pi^*\alpha?\pi^*\beta}$ is subanalytic in $S^1L(D)$. This case follows from Example 16.3.8. □

Lemma 16.3.11. *Let (M, X, P) be a subanalytic stratified space where P is finite. Let $\mathcal{F} \in \mathrm{Sh}^{\mathrm{hyp}}(X, \mathbf{Poset})$ such that $\mathcal{F}^{\mathrm{set}}$ is P-hyperconstructible and takes values in finite sets. Assume the existence of a finite cover of X by open subanalytic subsets $U \subset X$ such that*

(1) $\mathcal{F}|_U$ is globally generated ;

(2) for every $a, b \in \mathcal{F}(U)$, the sets $U_{a<b}$, $U_{a=b}$ and U_{a*b} are locally closed subanalytic in M.

Then, there is a finite subanalytic refinement $Q \to P$ such that $\mathcal{F} \in \mathrm{Cons}^{\mathrm{hyp}}_Q(X, \mathbf{Poset})$.

Proof. Let $U \subset X$ be an open subanalytic subset satisfying (1) and (2). For $f\colon \mathcal{F}(U) \times \mathcal{F}(U) \to \{<, =, *, >\}$ and $p \in P$, put

$$U_{f,p} \coloneqq U_p \bigcap \bigcap_{(a,b)\in\mathcal{F}(U)^2} U_{af(a,b)b} \ .$$

Note that U_p is a subanalytic subset of M since U and X_p are. Since $\mathcal{F}(U)$ is finite, item (2) implies that $U_{f,p}$ is a locally closed subanalytic subset of M. By assumption, we have $\mathcal{F}^{\mathrm{set}}|_{U_p} \in \mathrm{Loc}^{\mathrm{hyp}}(X_p, \mathbf{Set})$. By (1), we deduce $\mathcal{F}|_{U_{f,p}} \in \mathrm{Loc}^{\mathrm{hyp}}(U_{f,p}, \mathbf{Poset})$. Then, Lemma 16.3.11 follows from Lemma 2.5.8. □

Corollary 16.3.12. *Let (X, D) be a strict normal crossing pair where X admits a smooth compactification. Let $\mathscr{I} \subset \mathcal{O}_X(*D)/\mathcal{O}_X$ be a good sheaf of unramified irregular values. Let $\pi\colon \widetilde{X} \to X$ be the real blow-up along D. Then, there exists a finite subanalytic stratification $\widetilde{X} \to P$ refining $(\widetilde{X}, \widetilde{D})$ such that $\pi^*\mathscr{I} \in \mathrm{Cons}_P^{\mathrm{hyp}}(\widetilde{X}, \mathbf{Poset})$.*

Proof. By Remark 16.2.10, $(\pi^*\mathscr{I})^{\mathrm{set}}$ is hyperconstructible on $(\widetilde{X}, \widetilde{D})$. Let $X \hookrightarrow Y$ be a smooth compactification of X. At the cost of applying resolution of singularities, we can suppose that $Z \coloneqq Y \setminus X$ is a divisor such that $E \coloneqq Z + D$ has strict normal crossings. Hence, X admits a finite cover by open subanalytic subsets $U \simeq \Delta^{n-k} \times (\Delta^*)^k$ with coordinates (z, y) such that $D \cap U$ is defined by $z_1 \cdots z_l = 0$, where $\Delta \subset \mathbb{C}$ is the unit disc. Let $S_+, S_- \subset \Delta^*$ be a cover by open sectors. For $\varepsilon\colon \{1, \dots, k\} \to \{-, +\}$, put

$$U_\varepsilon \coloneqq \Delta^{n-k} \times S_{\varepsilon(1)} \times \cdots \times S_{\varepsilon(k)}$$

and $\widetilde{U}_\varepsilon \coloneqq \pi^{-1}(U_\varepsilon)$. Note that $\widetilde{U}_\varepsilon$ is a subanalytic subset of $S^1L(D)$ since $U_\varepsilon \subset X$ is subanalytic. To conclude, it is enough to show that $\widetilde{U}_\varepsilon$ satisfies the conditions (1) and (2) of Lemma 16.3.11. By Lemma 16.2.11, the sheaf $\mathscr{I}|_{U_\varepsilon}$ is globally generated. By Lemma 16.2.2, we deduce that $(\pi^*\mathscr{I})|_{\widetilde{U}_\varepsilon}$ is globally generated. Let $a, b \in (\pi^*\mathscr{I})(\widetilde{U}_\varepsilon)$ and $? \in \{<, =, *\}$. By Corollary 16.3.10, the set $\widetilde{U}_{\varepsilon, a?b}$ is subanalytic in $S^1L(D)$. By Remark 16.3.7, it is locally closed in $S^1L(D)$. Then, Corollary 16.3.12 follows from Lemma 16.3.11. □

16.4. Level structure.

Construction 16.4.1. The goal of what follows is to construct a local level structure for good sheaves of unramified irregular values. Assume that $X \subset \mathbb{C}^n$ is a polydisc with coordinates $(z, y) = (z_1, \dots, z_l, y_1, \dots, y_{n-l})$. Let D be the divisor defined by $z_1 \cdots z_l = 0$. Let $\mathscr{I} \subset \mathcal{O}_X(*D)/\mathcal{O}_X$ be a good sheaf of unramified irregular values. Let $\pi\colon \widetilde{X} \to X$ be the real-blow up along D and let $\widetilde{X} \to P$ be a finite subanalytic stratification adapted to $\mathscr{I}$. Let $\widetilde{X} \to P$ be a finite subanalytic stratification such that $\pi^{-1}\mathscr{I}$ is P-constructible. By condition (3) from Recollection 16.3.3, the set $\{a - b, a, b \in I\}$ is totally ordered with respect to the partial order on $\mathbb{Z}^l$. Hence, there exists a sequence

$$m(0) < m(1) < \cdots < m(d) = 0 \tag{16.4.2}$$

in $\mathbb{Z}^l$ such that for every $k = 0, \dots, d-1$, the vectors $m(k)$ and $m(k+1)$ differ only by 1 at exactly one coordinate and every $\mathrm{ord}(a - b)$ for $a, b \in \mathscr{I}(X)$ distinct appears in this sequence (such a sequence is referred to as an auxiliary sequence in [37, §2.1.2]). Fix $k = 0, \dots, d$ and put

$$\mathscr{I}^k \coloneqq \mathrm{Im}(\mathscr{I} \to \mathcal{O}_X(*D)/z^{m(k)}\mathcal{O}_X) \ .$$

Then, $\mathscr{I}^k$ is a constructible sheaf in finite sets on (X, D). The goal of what follows is to endow $\pi^*\mathscr{I}^k$ with a canonical structure of sheaves in finite posets. For a section $a \in \mathscr{I}$ we denote by $[a]_k$ its image under $\mathscr{I} \to \mathscr{I}^k$.

Lemma 16.4.3. *Let $x \in \widetilde{X}$. Let $a, b \in \mathscr{I}_{\pi(x)}$, such that $a <_x b$ and $[a]_k \neq [b]_k$. Then for every $a', b' \in \mathscr{I}_{\pi(x)}$ with $[a]_k = [a']_k$ and $[b]_k = [b']_k$, we have $a' \leq_x b'$.*

Proof. We can suppose that $x = 0$. By assumption $a \neq b$. Write

$$a - b \coloneqq f(y) z^{\mathrm{ord}(a-b)} + \sum_{m > \mathrm{ord}(a-b)} (a-b)_m(y) z^m \ .$$

where $f(0) \neq 0$. Put $x = (\theta_1, \dots, \theta_l) \in \pi^{-1}(0)$ and write $\mathrm{ord}(a - b) = (m_1, \dots, m_l)$. Then, the assumption $a <_x b$ means

$$\Re(f(0) e^{m_1\theta_1 + \cdots + m_l\theta_l}) < 0 \ .$$

Now let $a', b' \in \mathscr{I}_{\pi(x)}$ with $[a]_k = [a']_k$ and $[b]_k = [b']_k$. In particular $[a-b]_k = [a'-b']_k$, that is

$$\begin{aligned} a' - b' &= a - b + z^{m(d-k)}g \ , g \in \mathcal{O}_{X,0} \\ &= f(y)z^{\operatorname{ord}(a-b)} + z^{m(d-k)}g + \sum_{m > \operatorname{ord}(a-b)} (a-b)_m(y)z^m \end{aligned}$$

Since $[a]_k \neq [b]_k$, we have $m(d-k) > \operatorname{ord}(a-b)$. Hence $\operatorname{ord}(a-b) = \operatorname{ord}(a'-b')$ and

$$a' - b' = f(y)z^{\operatorname{ord}(a-b)} + \sum_{m > \operatorname{ord}(a'-b')} (a'-b')_m(y)z^m$$

Hence, we also have $a' \leq_x b'$. □

Corollary 16.4.4. *Let $x \in \widetilde{X}$. Then, there is a unique order $\leq_x^k$ on $\mathscr{I}^k_{\pi(x)}$ such that*

$$(\mathscr{I}_{\pi(x)}, \leq_x) \to (\mathscr{I}^k_{\pi(x)}, \leq_x^k)$$

is a level morphism of posets in the sense of Definition 10.1.1.

Proof. The uniqueness is obvious since $\mathscr{I}_{\pi(x)} \to \mathscr{I}^k_{\pi(x)}$ is surjective. For $\alpha, \beta \in \mathscr{I}^k_{\pi(x)}$, put $\alpha \leq_x^k \beta$ if $\alpha = \beta$ or if $\alpha \neq \beta$ and there exists $a, b \in \mathscr{I}_{\pi(x)}$ with $\alpha = [a]_k$ and $\beta = [b]_k$ such that $a <_x b$ and $[a]_k \neq [b]_k$. Then, Corollary 16.4.4 follows from Lemma 16.4.3. □

We stay in the setting of Construction 16.4.1. For every open subset $U \subset \widetilde{X}$, we define a partial order $\leq_U$ on $(\pi^*\mathscr{I}^k)(U)$ by

$$a \leq_U b \text{ if and only if } a \leq_x^k b \text{ in } \mathscr{I}^k_{\pi(x)} \text{ for every } x \in U.$$

Then, $\pi^*\mathscr{I}^k \in \operatorname{Cons}_P(\widetilde{X}, \mathbf{Poset})$ and the canonical morphism

$$\pi^*\mathscr{I} \to \pi^*\mathscr{I}^k$$

is a morphism of P-constructible sheaves in finite posets on $\widetilde{X}$. Hence, the chain

$$\mathcal{O}_X(*D)/\mathcal{O}_X \to \mathcal{O}_X(*D)/z^{m(1)}\mathcal{O}_X \to \cdots \to \mathcal{O}_X(*D)/z^{m(d)}\mathcal{O}_X$$

induces a chain of constructible sheaves on (X, D)

$$\mathscr{I} = \mathscr{I}^d \to \mathscr{I}^{d-1} \to \cdots \to \mathscr{I}^0 = *$$

which in turn induces a chain

$$\pi^*\mathscr{I} = \pi^*\mathscr{I}^d \to \pi^*\mathscr{I}^{d-1} \to \cdots \to \pi^*\mathscr{I}^0 = *$$

of P-constructible sheaves in finite posets over $\widetilde{X}$. By Corollary 16.4.4, the corresponding chain of cocartesian fibrations in finite posets on $\Pi_\infty(\widetilde{X}, P)$

$$\mathcal{I} = \mathcal{I}^d \to \mathcal{I}^{d-1} \to \cdots \to \mathcal{I}^0 = * \tag{16.4.5}$$

is a level structure on $(\widetilde{X}, P, \mathcal{I})$ relative to (X, D) in the sense of Definition 11.5.3.

Remark 16.4.6. The level structure (16.4.5) depends on a choice of auxiliary sequence (16.4.2).

16.5. **Piecewise elementarity.**

Lemma 16.5.1. *Fix $X \subset \mathbb{C}^n$ be a polydisc with coordinates $(z, y) = (z_1, \dots, z_l, y_1, \dots, y_{n-l})$. Let D be the divisor defined by $z_1 \cdots z_l = 0$ and put $I = \{1, \dots, l\}$. Let $\mathscr{I} \subset \mathcal{O}_X(*D)/\mathcal{O}_X$ be a good sheaf of unramified irregular values. Let $\pi\colon \widetilde{X} \to X$ be the real-blow up along D. Let $x \in \widetilde{X}$ such that $\pi(0) = 0$. Let $m \in \mathbb{Z}_{\leq 0}$ non zero. Then, there is a closed subanalytic neighbourhood $S \subset \widetilde{X}_I^{\circ}$ of x mapping to a closed subanalytic neighbourhood $\overline{B} \subset D_I^{\circ}$ of 0 such that for every $y \in \overline{B}$, the following holds ;*

1. *the fibre $S_y = S \cap \pi^{-1}(y)$ is homeomorphic to a closed cube in $\mathbb{R}^l$,*
2. *Via the homeomorphism from (1), for every $a, b \in \mathscr{I}$ defined on $\overline{B}$ with $\operatorname{ord}(a-b) = m$, the Stokes locus $(S_y)_{a,b}$ is a hyperplane whose complement has exactly two components C_1 and C_2 such that $a <_x b$ for every $x \in C_1$ and $b <_x a$ for every $x \in C_2$.*

Proof. We have $\widetilde{X}_I^{\circ} = (S^1)^l \times \mathbb{C}^{n-l}$ and we see $(S^1)^l$ as the quotient $\mathbb{R}^l$. Put $m = (m_1, \dots, m_l)$. Let $A \subset \mathbb{R}$ be a finite set. For $\alpha \in A$, the locus of points $\theta \in (S^1)^l$ satisfying

$$\cos(\alpha + m_1\theta_1 + \cdots + m_l\theta_l) = 0$$

is the image under the canonical projection $\mathbb{R}^l \to (S^1)^l$ of the set of affine hyperplanes $H(\alpha, k) \subset \mathbb{R}^l$,$k \in \mathbb{Z}$ defined by

$$\alpha + m_1\theta_1 + \cdots + m_l\theta_l = \pi/2 + k\pi \ .$$

Let $\tilde{x} \in \mathbb{R}^l$ mapping to x. Note that for every $\alpha \in A$ and $k \in \mathbb{Z}$, the hyperplanes $H(\alpha, k)$ and $H(\alpha, k+1)$ are parallel and distant by $\pi/\|m\|$. Hence, for every sufficiently generic choice of point z close enough to $\tilde{x}$, the closed cube $C(x, A) \subset \mathbb{R}^l$ centred at z with edges of length $\pi/\|m\|$ and with two faces parallel to the above hyperplanes satisfies

(a) for every $\alpha \in A$, there is a unique $k_\alpha \in \mathbb{Z}$ such that $C(x, A)$ meets $H(\alpha, k_\alpha)$.

(b) $C(x, A) \setminus H(\alpha, k_\alpha)$ has exactly two connected components.

Since $p\colon \mathbb{R}^l \to (S^1)^l$ is a diffeomorphism in a neighbourhood of $C(x, A)$, its image $p(C(x, A))$ is a closed subanalytic subset of $(S^1)^l$. For $a, b \in \mathscr{I}$ defined in a neighbourhood of 0, write

$$a - b := f_{a,b}(y) z^{\operatorname{ord}(a-b)} + \sum_{m' > m} (a-b)_{m'}(y) z^{m'} \ .$$

Choose some argument $\alpha_{a,b} \in \mathbb{R}$ for $f_{a,b}(0)$ and put

$$A := \{\alpha_{a,b}, a, b \in \mathscr{I} \text{ defined in a neighbourhood of 0 with } \operatorname{ord}(a-b) = m\} \ .$$

Fix $\varepsilon > 0$ small enough and put

$$S := p(C(x, A)) \times \overline{B(0, \varepsilon)} \subset \widetilde{X}_I^{\circ}$$

where $B(0, \varepsilon) \subset \mathbb{C}^{n-l}$ is the ball of radius ε centred at 0. Note that (1) is satisfied for every $y \in \overline{B(0, \varepsilon)}$. Since the conditions (a) and (b) are satisfied for $C(x, A)$, observe that S satisfies (2) for $y = 0$. Since the conditions (a) and (b) are open in the choice of A, we deduce the existence of $\varepsilon > 0$ such that (2) holds for every $y \in \overline{B(0, \varepsilon)}$. This concludes the proof of Lemma 16.5.1. □

Proposition 16.5.2. *Fix $X \subset \mathbb{C}^n$ be a polydisc with coordinates $(z, y) = (z_1, \dots, z_l, y_1, \dots, y_{n-l})$. Let D be the divisor defined by $z_1 \cdots z_l = 0$. Let $\mathscr{I} \subset \mathcal{O}_X(*D)/\mathcal{O}_X$ be a good sheaf of unramified irregular values. Let $\pi\colon \widetilde{X} \to X$ be the real-blow up along D and let $\widetilde{X} \to P$ be a finite subanalytic stratification adapted to $\mathscr{I}$. Let $m(0) < m(1) < \cdots < m(d) = 0$ be an auxiliary sequence as in* (16.4.2)*. Then, the level structure* (16.4.5) *is strongly piecewise elementary (Definition 11.5.3).*

Proof. Let $k = 1, \dots, d$, note $p \colon \mathcal{I}^k \to \mathcal{I}^{k-1}$. Consider the pullback square

$$\begin{tikzcd} \mathscr{I}^k_p \arrow[r] \arrow[d, "\pi"] & \mathscr{I}^k \arrow[d, "p"] \\ \mathscr{I}^{k-1,\mathrm{set}} \arrow[r] & \mathscr{I}^{k-1} \end{tikzcd} , \tag{16.5.3}$$

Denote by $D_1, \dots, D_l$ the components of D and fix $I \subset \{1, \dots, l\}$. Then, we have to show that $(\widetilde{X}^\circ_I, P, \mathcal{I}^k_p|_{\widetilde{X}^\circ_I})$ is piecewise elementary at every point $x \in \widetilde{X}$ in the sense of Definition 11.3.19. Since this is a local question on D°_I, we can suppose that $\pi(x) = 0$ and that $\mathscr{I}^{\mathrm{set}}$ is the constant sheaf on D°_I. That is, we can suppose that $I = \{1, \dots, l\}$. We want to apply Theorem 15.2.4. For this, we need to compute the Stokes loci of the sections of $\mathscr{I}^k_p$ on $\widetilde{X}^\circ_I = (S^1)^l \times D^\circ_I$. Since $\mathscr{I}^{\mathrm{set}}$ is the constant sheaf on D°_I, so is $\mathscr{I}^{k,\mathrm{set}} = \mathscr{I}^{k,\mathrm{set}}_p$. Hence, the map

$$\mathscr{I}^{\mathrm{set}}(D^\circ_I) \to \mathscr{I}^{k,\mathrm{set}}_p(D^\circ_I)$$

is surjective. Let $a, b \in \mathscr{I}^{\mathrm{set}}(D^\circ_I)$ such that their images $[a]_k, [b]_k \in \mathscr{I}^{k,\mathrm{set}}_p(D^\circ_I)$ are distinct. In particular, we have

$$\operatorname{ord}(a-b) < m(k) \ . \tag{16.5.4}$$

Thus,

$$\begin{aligned} [a]_{k-1} = [b]_{k-1} &\Leftrightarrow a - b \in z^{m(k-1)}\mathcal{O}_X \\ &\Leftrightarrow \operatorname{ord}(a-b) \geq m(k-1) \\ &\Leftrightarrow \operatorname{ord}(a-b) = m(k-1) && \text{By (16.5.4)} \end{aligned}$$

In particular,

$$\begin{aligned} \operatorname{ord}(a-b) \neq m(k-1) &\Leftrightarrow [a]_{k-1} \neq [b]_{k-1} \\ &\Leftrightarrow \text{the Stokes locus of } [a]_k, [b]_k \text{ viewed as sections of } \pi^*\mathscr{I}^k_p \text{ is } \widetilde{X}^\circ_I. \end{aligned}$$

Thus, by Lemma 16.5.1, there is a closed subanalytic neighbourhood $S \subset \widetilde{X}^\circ_I$ of x mapping to a closed subanalytic neighbourhood $\overline{B} \subset D^\circ_I$ of 0 such that for every $y \in \overline{B}$, the following holds ;

(1) the fibre $S_y = S \cap \pi^{-1}(y)$ is homeomorphic to a closed cube in $\mathbb{R}^l$,

(2) via the homeomorphism from (1), for every $a, b \in \mathscr{I}$ defined on $\overline{B}$, the Stokes locus $(S_y)_{a,b}$ is either S_y or a hyperplane whose complement has exactly two components C_1 and C_2 such that $a <_x b$ for every $x \in C_1$ and $b <_x a$ for every $x \in C_2$.

By Theorem 15.2.4, the Stokes stratified space $(S_y, P, \mathcal{I}^k_p|_{S_y})$ is elementary for every $y \in D^\circ_I$, which concludes the proof of Proposition 16.5.2. □

Corollary 16.5.5. *Let (X, D) be a normal crossing pair where X admits a smooth compactification. Let $\mathscr{I} \subset \mathcal{O}_X(*D)/\mathcal{O}_X$ be a good sheaf of unramified irregular values. Let $\pi \colon \widetilde{X} \to X$ be the real-blow up along D and let $\widetilde{X} \to P$ be a finite subanalytic stratification such that $\pi^*\mathscr{I} \in \mathrm{Cons}_P(\widetilde{X}, \mathbf{Poset})$. Let $(\widetilde{X}, P, \mathcal{I})$ be the associated Stokes analytic stratified space. Then, $\pi \colon (\widetilde{X}, P, \mathcal{I}) \to (Y, Q)$ is a strongly proper family of Stokes analytic stratified spaces in finite posets locally admitting a piecewise elementary level structure.*

Proof. Combine Lemma 16.1.8 with Proposition 16.5.2. □

16.6. **Sheaf of (ramified) irregular values.** The goal of this subsection is to enhance Section 16.2 to the ramified setting. Since this requires to work directly on $\widetilde{X}$, we start by transporting the notion of sheaf of unramified irregular values from X to $\widetilde{X}$.

Lemma 16.6.1. *Let (X, D) be a strict normal crossing pair. Let $\pi\colon \widetilde{X} \to X$ be the real blow-up along D. Let $\mathscr{I} \subset \pi^*(\mathcal{O}_X(*D)/\mathcal{O}_X)$ be a sheaf. Then, the following are equivalent:*

(1) There is a sheaf of unramified irregular values $\mathscr{J} \subset \mathcal{O}_X(*D)/\mathcal{O}_X$ such that $\mathscr{I} \simeq \pi^*\mathscr{J}$.

(2) the direct image $\pi_*\mathscr{I} \subset \mathcal{O}_X(*D)/\mathcal{O}_X$ is a sheaf of unramified irregular values and the counit transformation $\pi^*\pi_*\mathscr{I} \to \mathscr{I}$ is an equivalence.

Proof. Immediate from Corollary 16.2.17. □

Definition 16.6.2. If the equivalent conditions of Lemma 16.6.10 are satisfied, we say that $\mathscr{I} \subset \pi^*(\mathcal{O}_X(*D)/\mathcal{O}_X)$ is a sheaf of unramified irregular values. If furthermore $\pi_*\mathscr{I}$ is a good sheaf of unramified irregular values, we say that $\mathscr{I}$ is a good sheaf of unramified irregular values.

Remark 16.6.3. By design, Lemma 16.6.1 and Corollary 16.2.17 imply that (π^*, π_*) induce a bijection between (good) sheaves of irregular values on $\widetilde{X}$ and (good) sheaves of irregular values on X.

Construction 16.6.4 ([44, 9.c]). Let $X \subset \mathbb{C}^n$ be a polydisc with coordinates $(z_1, \ldots, z_n)$. Let D be the divisor defined by $z_1 \cdots z_l = 0$ and put $U \coloneqq X \setminus D$. Let $\pi\colon \widetilde{X} \to X$ be the real blow-up along D. Let $j\colon U \hookrightarrow \widetilde{X}$ be the canonical inclusion. Define $\rho\colon X_d \to X$ by $(z_1, \ldots, z_n) \to (z_1^d, \ldots, z_l^d, z_{l+1}, \ldots, z_n)$ for $d \geq 1$ and consider the (not cartesian for $d > 1$) commutative square

$$\begin{tikzcd} \widetilde{X}_d \arrow[r, "\widetilde{\rho}"] \arrow[d, "\pi_d"] & \widetilde{X} \arrow[d, "\pi"] \\ X_d \arrow[r, "\rho"] & X \end{tikzcd}$$

of real blow-up along D. The unit transformation $\mathcal{O}_U \hookrightarrow \rho_*\mathcal{O}_{U_d}$ yields an inclusion

$$j_*\mathcal{O}_U \hookrightarrow j_*\rho_*\mathcal{O}_{U_d} \ .$$

On the other hand, the unit transformation $\pi_d^*\mathcal{O}_{X_d}(*D) \hookrightarrow j_{d,*}\mathcal{O}_{U_d}$ yields

$$\widetilde{\rho}_*\pi_d^*\mathcal{O}_{X_d}(*D) \hookrightarrow \widetilde{\rho}_* j_{d,*}\mathcal{O}_{U_d} = j_*\rho_*\mathcal{O}_{U_d} \ .$$

Put

$$IV_d \coloneqq j_*\mathcal{O}_U \cap \widetilde{\rho}_*\pi_d^*\mathcal{O}_{X_d}(*D) \subset j_*\mathcal{O}_U \ .$$

As in Remark 16.1.12, we have

$$IV_d \cap (j_*\mathcal{O}_U)^{\mathrm{lb}} = j_*\mathcal{O}_U \cap \widetilde{\rho}_*\pi_d^*\mathcal{O}_{X_d} \ .$$

We put

$$\mathscr{I}\mathscr{V}_d \coloneqq IV_d/(IV_d \cap (j_*\mathcal{O}_U)^{\mathrm{lb}}) \subset (j_*\mathcal{O}_U)/(j_*\mathcal{O}_U)^{\mathrm{lb}} \ .$$

For an arbitrary strict normal crossing pair (X, D), the $\mathscr{I}\mathscr{V}_d$, $d \geq 1$ are defined locally and glue into subshseaves

$$\mathscr{I}\mathscr{V}_d(X, D) \subset (j_*\mathcal{O}_U)/(j_*\mathcal{O}_U)^{\mathrm{lb}}$$

for $d \geq 1$. By Recollection 16.1.11, we view $\mathscr{I}\mathscr{V}_d(X, D)$ as an object of $\mathrm{Sh}^{\mathrm{hyp}}(\widetilde{X}, \mathbf{Poset})$.

Example 16.6.5. In the setting of Construction 16.6.4, we have

$$\mathscr{I}\mathscr{V}_1(X, D) = \pi^*(\mathcal{O}_X(*D)/\mathcal{O}_X)$$

in virtue of Remark 16.1.12.

Construction 16.6.4 suggests to introduce the following

Definition 16.6.6. Let (X,D) be a strict normal crossing pair. Let $d \geq 1$ be an integer. A *d-Kummer cover of* (X,D) is an holomorphic map $\rho\colon X \to X$ such that there is a cover of X by open subsets U with $\rho(U) \subset U$ such that $\rho|_U$ reads as

$$(z_1,\dots,z_n) \to (z_1^d,\dots,z_l^d,z_{l+1},\dots,z_n) \tag{16.6.7}$$

for some choice of local coordinates $(z_1,\dots,z_n)$ with D defined by $z_1\cdots z_l = 0$.

Remark 16.6.8. Following [44], in the setting of Definition 16.6.6, we will denote the source of ρ by X_d instead of X.

Lemma 16.6.9 ([44, Lemma 9.6])**.** *Let (X,D) be a strict normal crossing pair. Let $\pi\colon \widetilde{X} \to X$ be the real blow-up along D. Let $j\colon U \hookrightarrow \widetilde{X}$ be the canonical inclusion. Let $d \geq 1$ be an integer and let $\rho\colon X_d \to X$ be a d-Kummer cover of (X,D). Then, via the inclusion*

$$\widetilde{\rho}^* j_*\rho_*\mathcal{O}_{U_d} = \widetilde{\rho}^*\widetilde{\rho}_* j_*\mathcal{O}_{U_d} \hookrightarrow j_*\mathcal{O}_{U_d} ,$$

we have

$$\widetilde{\rho}^*(\mathscr{I}\mathscr{V}_d(X,D)) = \pi_d^*(\mathcal{O}_{X_d}(*D)/\mathcal{O}_{X_d})$$

in $\mathrm{Sh}^{\mathrm{hyp}}(\widetilde{X}_d,\mathbf{Poset})$.

Lemma 16.6.10. *Let (X,D) be a strict normal crossing pair. Let $d \geq 1$ be an integer and let $\mathscr{I} \subset \mathscr{I}\mathscr{V}_d(X,D)$ be a sheaf. Then, the following are equivalent:*

(1) For every $x \in X$, there exist local coordinates $(z_1,\dots,z_n)$ centred at x with D defined by $z_1 \dots z_l = 0$ such that for the map ρ given by (16.6.7)*, the pullback $\widetilde{\rho}^*\mathscr{I}$ is a sheaf of unramified irregular values in the sense of Definition 16.6.2.*

(2) For every open subset $U \subset X$ and every d-Kummer cover $\rho\colon U_d \to U$, the pullback $\widetilde{\rho}^*\mathscr{I}$ is a sheaf of unramified irregular values in the sense of Definition 16.6.2.

Proof. Left to the reader. □

Definition 16.6.11. If the equivalent conditions of Lemma 16.6.10 are satisfied, we say that $\mathscr{I} \subset \mathscr{I}\mathscr{V}_d(X,D)$ is a *sheaf of irregular values*. If furthermore the $\widetilde{\rho}^*\mathscr{I}$ are good sheaves of unramified irregular values, we say that $\mathscr{I}$ is a good sheaf of irregular values.

Lemma 16.6.12. *Let $f\colon (N,Y,Q) \to (M,X,P)$ be a morphism of analytic stratified spaces such that the induced morphism $f\colon Y \to X$ is open surjective. Let $\mathcal{F} \in \mathrm{Cons}_P^{\mathrm{hyp}}(X,\mathbf{Cat}_\infty)$. Then, $\mathcal{F}$ is locally generated if and only if $f^*(\mathcal{F})$ is locally generated.*

Proof. The direct implication follows from Lemma 16.2.2. Assume that $f^*(\mathcal{F})$ is locally generated. To show that $\mathcal{F}$ is locally generated, it is enough to show in virtue of Proposition 2.5.6 that every open subset $U \subset X$ such that $\Pi_\infty(U,P)$ admits an initial object x contains an open neighbourhood of x on which $\mathcal{F}$ is globally generated. By surjectivity, choose $x' \in Y$ above x. Since $f^*(\mathcal{F})$ is locally generated, we can choose an open subset $V' \subset Y$ containing x' on which $f^*(\mathcal{F})$ is globally generated. By Lemma 16.2.2, we can suppose that $V' \subset f^{-1}(U)$. At the cost of shrinking V' further, we can suppose by Proposition 2.5.6 that x' is initial in $\Pi_\infty(V',Q)$. Put $V \coloneqq f(V') \subset U$. Note that V is an open neighbourhood of x by openness of $f\colon Y \to X$. To conclude, we are left to show that $\mathcal{F}|_V$ is globally generated. Let $y \in V$ and let us show that $\mathcal{F}(V) \to \mathcal{F}_y$ is essentially surjective. Choose $y' \in V'$ above y. Then, by design of U and V' there is a commutative diagram

$$\begin{tikzcd}
& \mathcal{F}(U) \arrow[r, "\sim"] \arrow[d] \arrow[ddl, bend right] & \mathcal{F}_x \simeq (f^*(\mathcal{F}))_{x'} \\
& \mathcal{F}(V) \arrow[r] \arrow[dl] & (f^*(\mathcal{F}))(V') \arrow[u, "\wr"'] \arrow[dll, bend left, two heads] \\
\mathcal{F}_y \simeq (f^*(\mathcal{F}))_{y'} & &
\end{tikzcd}$$

The conclusion thus follows. □

Proposition 16.6.13. *Let (X, D) be a strict normal crossing pair. Let $\mathscr{I} \subset \mathscr{I}\mathscr{V}_d(X, D)$ be a sheaf of irregular values for some $d \geq 1$. Then, the following hold:*

(1) $\mathscr{I}^{\mathrm{set}}$ is hyperconstructible on $(\widetilde{X}, \widetilde{D})$;

(2) $\mathscr{I}$ is locally generated;

If furthermore $\mathscr{I}$ is good, then

(3) for every open subset $U \subset \widetilde{X}$ subanalytic in $S^1L(D)$, for every $a, b \in \mathscr{I}(U)$, the sets $U_{a<b}$,$U_{a=b}$,U_{a*b} are locally closed subanalytic in $S^1L(D)$;

If furthermore X admits a smooth compactification, then

(4) there exists a finite subanalytic stratification $\widetilde{X} \to P$ refining $(\widetilde{X}, \widetilde{D})$ such that $\mathscr{I} \in \mathrm{Cons}_P^{\mathrm{hyp}}(\widetilde{X}, \mathbf{Poset})$.

Proof. Item (1) follows from the fact that local hyperconstancy can be check locally for the étale topology. Item (2) is a local question. Hence, we can assume the existence of a surjective d-Kummer cover $\rho\colon X_d \to X$ of (X, D) of the form (16.6.7) such that $\widetilde{\rho}^*\mathscr{I}$ is a sheaf of unramified irregular values. In particular $\widetilde{\rho}^{-1}\mathscr{I}$ is locally generated. Observe that $\widetilde{\rho}$ is open and surjective. Then, (2) follows from Lemma 16.6.12. Let us prove (3). We are going to apply Lemma 16.3.9. Conditions (1) and (2) from Lemma 16.3.9 are satisfied. To show that Lemma 16.3.9-(3) is satisfied, we can suppose the existence of a surjective Kummer cover $\rho\colon X_d \to X$ such that $\widetilde{\rho}^*\mathscr{I}$ is a sheaf of unramified irregular values. Let $W \subset S^1L(D)$ be an open *subanalytic*subset. Let $? \in \{<, =, *\}$ and let $a, b \in \mathscr{I}(W \cap X)$. We want to show that $(W \cap \widetilde{X})_{a?b}$ is a subanalytic subset of W. Since W and $\widetilde{X}$ are subanalytic in $S^1L(D)$, so is $W \cap \widetilde{X}$. Hence $\widetilde{\rho}^*(W \cap \widetilde{X}) \subset \widetilde{X}_d$ is subanalytic as well. By Corollary 16.3.10 applied to $\widetilde{\rho}^{-1}\mathscr{I}$, we know that $(\widetilde{\rho}^*(W \cap \widetilde{X}))_{\pi^*a?\pi^*b}$ is subanalytic. On the other hand, we have

$$(W \cap \widetilde{X})_{a?b} = \widetilde{\rho}((\widetilde{\rho}^*(W \cap \widetilde{X}))_{\pi^*a?\pi^*b})$$

Since the image of a subanalytic subset by a proper map is again subanalytic, we conclude that $(W \cap \widetilde{X})_{a?b}$ is subanalytic and (3) is proved. We know prove (4). Let $X \hookrightarrow Y$ be a smooth compactification of X. At the cost of applying resolution of singularities, we can suppose that $Z \coloneqq Y \setminus X$ is a divisor such that $E \coloneqq Z + D$ has strict normal crossings. Hence, X admits a finite cover by open subanalytic subsets $U \simeq \Delta^{n-k} \times (\Delta^*)^k$ with coordinates (z, y) such that $\Delta \cap U$ is defined by $z_1 \cdots z_l = 0$, where $\Delta \subset \mathbb{C}$ is the unit disc. Let $S_+, S_- \subset \Delta^*$ be a cover by open sectors. For $\varepsilon\colon \{1, \dots, k\} \to \{-, +\}$, put

$$U_\varepsilon \coloneqq \Delta^{n-k} \times S_{\varepsilon(1)} \times \cdots \times S_{\varepsilon(k)} \subset U$$

Let $(I_+, I_-) \subset S^1$ be a cover by strict open intervals. For $\varepsilon\colon \{1,\dots,k\} \to \{-,+\}$ and $\eta\colon \{1,\dots,l\} \to \{-,+\}$, put

$$V_{\varepsilon,\eta} \coloneqq [0,1)^l \times I_{\eta_1} \times \cdots \times I_{\eta_l} \times \Delta^{n-l-k} \times S_{\varepsilon(1)} \times \cdots \times S_{\varepsilon(k)} \subset \pi^{-1}(U_\varepsilon)$$

Note that $V_{\varepsilon,\eta}$ is a subanalytic subset of $S^1L(D)$. To prove (4), it is enough to show that the $V_{\varepsilon,\eta}$ satisfy the conditions of Lemma 16.3.11. This follows from the above points (1) (2) (3) and Lemma 16.2.11 applied to Example 16.2.13. □

Proposition 16.6.14. *Let (X, D) be a normal crossing pair where X admits a smooth compactification. Let $\pi\colon \widetilde{X} \to X$ be the real-blow up along D. Let $\mathscr{I} \subset \mathscr{I}\mathscr{V}_d(X, D)$ be a good sheaf of irregular values for some $d \geq 1$. Let $\widetilde{X} \to P$ be a finite subanalytic stratification such that $\mathscr{I} \in \mathrm{Cons}_P(\widetilde{X}, \mathbf{Poset})$. Let $(\widetilde{X}, P, \mathcal{I})$ be the associated Stokes analytic stratified space. Then, $\pi\colon (\widetilde{X}, P, \mathcal{I}) \to (Y, Q)$ is a strongly proper family of Stokes analytic stratified spaces in finite posets locally admitting a ramified piecewise elementary level structure in the sense of Definition 11.5.12.*

Proof. Immediate from Corollary 16.5.5. □

Proposition 16.6.14 unlock all the results proved in Section 12 and Section 13. In particular, we have the following

Theorem 16.6.15. *In the setting of Proposition 16.6.14, let k be an animated commutative ring. Then, $\mathbf{St}_{\mathcal{I}}$ is locally geometric of finite presentation. Moreover, for every animated commutative k-algebra A and every morphism*

$$x\colon \mathrm{Spec}(A) \to \mathbf{St}_{\mathcal{I}}$$

classifying a Stokes functor $F\colon \mathcal{I} \to \mathrm{Perf}_A$, there is a canonical equivalence

$$x^*\mathbb{T}_{\mathbf{St}_{\mathcal{I}}} \simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{I},\mathrm{Mod}_A)}(F, F)[1] \ ,$$

where $\mathbb{T}_{\mathbf{St}_{\mathcal{I}}}$ denotes the tangent complex of $\mathbf{St}_{\mathcal{I}}$ and the right hand side denotes the Mod_A-enriched Hom of $\mathrm{Fun}(\mathcal{I}, \mathrm{Mod}_A)$.

Proof. Combine Corollary 16.5.5 with Theorem 13.1.4. □

Part 4. Categorical complements

This fourth part is to be understood as an appendix, collecting auxiliary results needed in the main body, mostly of categorical flavor. At the same time, we use in a couple of points the language of the specialization equivalence that has been developed in Section 6 to obtain important structural results for cocartesian fibrations, that are interesting in their own right. See in particular Theorem 17.1.2, Theorem 18.2.1 and Corollary 20.2.8.

17. Compactness results for ∞-categories

17.1. **Compactness in the unstable setting.** Inspired by the usual terminology in non-commutative geometry (see e.g. [33, Chapter 11]), we introduce:

Definition 17.1.1. We say that an ∞-category $\mathcal{C}$ is

(1) *compact* if it is a compact object in $\mathbf{Cat}_\infty$;

(2) *proper* if for every $c, c' \in \mathcal{C}$, the mapping space $\mathrm{Map}_{\mathcal{C}}(c, c')$ is a compact object in $\mathbf{Spc}$.

The first goal of this section is to prove the following:

Theorem 17.1.2. *Let $\mathcal{X}$ be an ∞-category and let $\mathcal{A} \to \mathcal{X}$ be a cocartesian fibration. Assume that $\mathcal{X}$ is compact and that for every $x \in \mathcal{X}$, the fiber $\mathcal{A}_x$ is compact in $\mathbf{Cat}_\infty$. Then $\mathcal{A}$ is compact in $\mathbf{Cat}_\infty$ as well.*

Remark 17.1.3. See [10, Remark 6.5.4] for an analogous statement for finite ∞-categories instead of compact ones.

The proof will use the specialization equivalence. Before giving it, we need a couple of preliminaries.

Lemma 17.1.4.

(1) Compact objects in $\mathbf{Cat}_\infty$ are closed under finite products.

(2) An ∞-category $\mathcal{X} \in \mathbf{Cat}_\infty$ is compact if and only if for every filtered diagram $\mathcal{C}_\bullet \colon I \to \mathbf{Cat}_\infty$ with colimit $\mathcal{C}$, the canonical map

$$\operatorname*{colim}_i \mathrm{Fun}(\mathcal{X}, \mathcal{C}_i) \to \mathrm{Fun}(\mathcal{X}, \mathcal{C}) \tag{17.1.5}$$

is an equivalence in $\mathbf{Cat}_\infty$.

Proof. First we prove (1). Fix therefore two compact ∞-categories $\mathcal{X}$ and $\mathcal{Y}$. We can suppose that $\mathcal{X}$ and $\mathcal{Y}$ are retract of finite ∞-categories $\mathcal{X}'$ and that $\mathcal{Y}'$, respectively. Then $\mathcal{X} \times \mathcal{Y}$ is a retract of $\mathcal{X}' \times \mathcal{Y}$, which in turn is a retract of $\mathcal{X}' \times \mathcal{Y}'$. It is therefore sufficient to prove that the latter is again a finite ∞-category. This latter statement follows immediately from the fact that the products $\Delta^n \times \Delta^m$ are again finite.

We now prove point (2). Since $*$ is compact, we see that the stated condition implies the compactness of $\mathcal{X}$ by applying $\mathrm{Map}_{\mathbf{Cat}_\infty}(*, -)$ to (17.1.5). As for the converse, since $\mathbf{Cat}_\infty$ is compactly generated by the standard simplexes and since $- \times \mathcal{X} \dashv \mathrm{Fun}(\mathcal{X}, -)$, it is in fact enough to prove that for every $[n] \in \mathbf{\Delta}$, the canonical map

$$\operatorname*{colim}_i \mathrm{Map}_{\mathbf{Cat}_\infty}(\Delta^n \times \mathcal{X}, \mathcal{C}_i) \to \mathrm{Map}_{\mathbf{Cat}_\infty}(\Delta^n \times \mathcal{X}, \mathcal{C})$$

is an equivalence. Since point (1) guarantees that $\Delta^n \times \mathcal{X}$ is again compact, the conclusion follows. $\square$

Lemma 17.1.6. *Let $\mathcal{X}$ be an ∞-category. Then:*

(1) the forgetful functor

$$\mathrm{U}_{\mathcal{X}} \colon \mathbf{Cart}_{\mathcal{X}} \to \mathbf{Cat}_{\infty/\mathcal{X}}$$

commutes with filtered colimits;

(2) if $\mathcal{X}$ is compact in $\mathbf{Cat}_\infty$, then the functor

$$\Sigma_{\mathcal{X}} \colon \mathbf{Cat}_{\infty/\mathcal{X}} \to \mathbf{Cat}_\infty$$

commutes with filtered colimits.

Proof. Notice that $\mathrm{U}_{\mathcal{X}}$ is by definition faithful. Thus, to prove (1) it is enough to prove that for any filtered diagram $\mathcal{C}_\bullet \colon I \to \mathbf{Cart}_{\mathcal{X}}$, the following two statements hold:

(i) the colimit $p \colon \mathcal{C} \to \mathcal{X}$ of $\mathrm{U}_{\mathcal{X}}(\mathcal{C}_\bullet) \colon I \to \mathbf{Cat}_{\infty/\mathcal{X}}$ is a cartesian fibration;

(ii) for every other cartesian fibration $q \colon \mathcal{D} \to \mathcal{X}$ equipped with a cone $f_\bullet \colon \mathcal{C}_\bullet \to \mathcal{D}$ in $\mathbf{Cart}_{\mathcal{X}}$, the induced functor $f \colon \mathcal{C} \to \mathcal{D}$ preserves cartesian edges.

For (i), it is enough to apply the definition. First, since the horns Λ^n_i and the simplexes Δ^n are compact in $\mathbf{Cat}_\infty$, we see that inner fibrations are stable under filtered colimits. Second, write $\lambda_i \colon \mathcal{C}_i \to \mathcal{C}$ for the canonical maps. Since the diagram was filtered, we see that every object $c \in \mathcal{C}$ is of the form $\lambda_i(c_i)$ for some $i \in I$ and some $c_i \in \mathcal{C}_i$. Let $\alpha \colon x \to p(c)$ be a morphism in $\mathcal{X}$. Since $p(c) \simeq p(\lambda_i(c_i)) \simeq p_i(c_i)$ and since p_i is a cartesian fibration, we can find a p_i-cartesian lift $\beta_i \colon d_i \to c_i$ of α inside $\mathcal{C}_i$. Set $d \coloneqq \lambda_i(d_i)$ and $\beta \coloneqq \lambda_i(\beta_i)$. We claim that β is a p-cartesian lift of α. To see this, for every $(j, u \colon i \to j) \in I_{i/}$, write $\lambda_u \colon \mathcal{C}_i \to \mathcal{C}_j$ for the induced functor. Consider then the following commutative square:

$$\begin{tikzcd}
\operatorname*{colim}_{(j,u)\in I_{i/}} \mathcal{C}_{j/\lambda_u(\beta_i)} \arrow[r] \arrow[d] & \operatorname*{colim}_{(j,u)\in I_{i/}} \big(\mathcal{C}_{j/\lambda_u(d_i)} \times_{\mathcal{X}_{/p(c)}} \mathcal{X}_{/\alpha}\big) \arrow[d] \\
\mathcal{C}_{/\beta} \arrow[r] & \mathcal{C}_{/d} \times_{\mathcal{X}_{/p(c)}} \mathcal{X}_{/\alpha} ,
\end{tikzcd}$$

where the colimits are computed in $\mathbf{Cat}_\infty$. Since λ_u preserves cartesian edges, we see that the top horizontal map is an equivalence. It is therefore enough to prove that the vertical arrows are equivalence. Since the colimit is filtered, it commutes with fiber products, and therefore we are reduced to check that the canonical functors

$$\operatorname*{colim}_{(j,u)\in I_{i/}} \mathcal{C}_{j/\lambda_u(\beta_i)} \to \mathcal{C}_{/\beta} \qquad \text{and} \qquad \operatorname*{colim}_{(j,u)\in I_{i/}} \mathcal{C}_{j/\lambda_u(d_i)} \to \mathcal{C}_{/d}$$

are equivalences. We deal with the one on the left, as the other follows by a similar argument. Since $\mathbf{Cat}_\infty$ is compactly generated by the standard simplexes, it is enough to prove that for every Δ^n, the canonical map

$$\operatorname*{colim}_{(j,u)\in I_{i/}} \mathrm{Map}_{\mathbf{Cat}_\infty}(\Delta^n, \mathcal{C}_{j/\lambda_u(\beta_i)}) \to \mathrm{Map}_{\mathbf{Cat}_\infty}(\Delta^n, \mathcal{C}_{/\beta})$$

is an equivalence. Unraveling the definition of the comma category and using the identification $\Delta^n \star \Delta^1 \simeq \Delta^{n+2}$, we see that this map is canonically identified with the upper left diagonal map in the following commutative cube:

$$\begin{tikzcd}
& \mathrm{Map}_\beta(\Delta^{n+2}, \mathcal{C}) \arrow[rr] \arrow[dd] & & \mathrm{Map}(\Delta^{n+2}, \mathcal{C}) \arrow[dd, "\mathrm{ev}_{n+1,n+2}"] \\
\operatorname*{colim}_{(j,u) \in I_{i/}} \mathrm{Map}_{\lambda_u(\beta_i)}(\Delta^{n+2}, \mathcal{C}_j) \arrow[ur] \arrow[rr] \arrow[dd] & & \operatorname*{colim}_{(j,u) \in I_{i/}} \mathrm{Map}(\Delta^{n+2}, \mathcal{C}_j) \arrow[ur] \arrow[dd, "\mathrm{ev}_{n+1,n+2}" near start] & \\
& * \arrow[rr, "\beta" near start] & & \mathrm{Map}(\Delta^1, \mathcal{C}) \\
* \arrow[ur, equal] \arrow[rr, "\lambda_u(\beta_j)"] & & \operatorname*{colim}_{(j,u) \in I_{i/}} \mathrm{Map}(\Delta^1, \mathcal{C}_j) \arrow[ur] &
\end{tikzcd}$$

Notice that the front and the back squares are pullback by definition. It is therefore sufficient to check that the other diagonal maps are equivalences, and this follows directly from the fact that both Δ^{n+2} and Δ^1 are compact in $\mathbf{Cat}_\infty$. This proves at the same time that $p \colon \mathcal{C} \to \mathcal{X}$ is a cartesian fibration, and that p-cartesian edges are exactly the morphisms of the form $\lambda_i(\beta_i)$ for some p_i-cartesian edge β_i inside $\mathcal{C}_i$. In particular, (ii) follows immediately.

We now prove (2). Notice that $\Sigma_{\mathcal{X}}$ is right adjoint to the functor $- \times \mathcal{X} \colon \mathbf{Cat}_\infty \to \mathbf{Cat}_{\infty/\mathcal{X}}$. It is therefore enough to verify that $- \times \mathcal{X}$ commutes with compact objects. Recall from [25, Lemma A.3.10] that an object in $\mathbf{Cat}_{\infty/\mathcal{X}}$ is compact if and only if it is compact in $\mathbf{Cat}_\infty$ after forgetting the structural map to $\mathcal{X}$. Since $\mathcal{X}$ itself is compact, the conclusion follows from Lemma 17.1.4-(1). $\square$

We are now ready for:

Proof of Theorem 17.1.2. Fix a filtered diagram $\mathcal{E}_\bullet \colon I \to \mathbf{Cat}_\infty$ with colimit $\mathcal{E}$. In virtue of Lemma 17.1.4-(2), we have to prove that the canonical map

$$\operatorname*{colim}_I \mathrm{Fun}(\mathcal{A}, \mathcal{E}_i) \to \mathrm{Fun}(\mathcal{A}, \mathcal{E})$$

is an equivalence. Write $\Upsilon_{\mathcal{A}}$ for the straightening of $\mathcal{A}$ and recall from Notation 6.2.4 that we write $\mathcal{E}^{\mathcal{A}}_{\mathrm{c}}$ for the *cartesian* fibration classifying the functor

$$\mathrm{Fun}(\Upsilon_{\mathcal{A}}(-), \mathcal{E}) \colon \mathcal{X}^{\mathrm{op}} \to \mathbf{Cat}_\infty \ .$$

We similarly define the cartesian fibrations $\mathcal{E}^{\mathcal{A}}_{i,\mathrm{c}}$. Consider the canonical map

$$\operatorname*{colim}_I \mathcal{E}^{\mathcal{A}}_{i,\mathrm{c}} \to \mathcal{E}^{\mathcal{A}}_{\mathrm{c}}$$

in $\mathbf{Cart}_{\mathcal{X}}$. To see that this map is an equivalence, it is enough to test that for each $x \in \mathcal{X}$, the induced map between the fibers at x is an equivalence. However, at the level of fibers at x, this map is canonically identified with

$$\operatorname*{colim}_I \mathrm{Fun}(\mathcal{A}_x, \mathcal{E}_i) \to \mathrm{Fun}(\mathcal{A}_x, \mathcal{E}) \ .$$

Since $\mathcal{A}_x$ is compact by assumption, we see Lemma 17.1.4-(2) guarantees that this map is indeed an equivalence.

We can now apply Lemma 17.1.6-(1) to deduce that the canonical map

$$\operatorname*{colim}_I \mathcal{E}^{\mathcal{A}}_{i,\mathrm{c}} \to \mathcal{E}^{\mathcal{A}}_{\mathrm{c}}$$

is an equivalence also when the colimit is computed in $\mathbf{Cat}_{\infty/\mathcal{X}}$. At this point, the conclusion follows from the identifications

$$\mathrm{Fun}(\mathcal{A},\mathcal{E}_i)\simeq \Sigma_{\mathcal{X}}(\mathcal{E}_{i,\mathrm{c}}^{\mathcal{A}}) \qquad \text{and} \qquad \mathrm{Fun}(\mathcal{A},\mathcal{E})\simeq \Sigma_{\mathcal{X}}(\mathcal{E}_{\mathrm{c}}^{\mathcal{A}})\ ,$$

and Lemma 17.1.6-(2). □

17.2. **Compact and proper (co)limits.** One of the most fundamental results in category theory is the commutation of filtered colimits with finite limits in **Set** and in **Spc**. In fact, the finiteness condition can be relaxed, using various combinations of compactness and properness.

Lemma 17.2.1. *Let $\mathcal{E}$ be a stable complete and cocomplete ∞-category. Let $\mathcal{C}$ be a compact ∞-category. Then:*

(1) *the functor* $\mathrm{colim}_{\mathcal{C}}\colon \mathrm{Fun}(\mathcal{C},\mathcal{E})\to\mathcal{E}$ *commutes with limits.*

(2) *the functor* $\lim_{\mathcal{C}}\colon \mathrm{Fun}(\mathcal{C},\mathcal{E})\to\mathcal{E}$ *commutes with colimits.*

Proof. The two statements are dual to each other. It is therefore enough to prove the second. Because $\mathcal{E}$ is stable, it is enough to prove that $\lim_{\mathcal{C}}$ commutes with filtered colimits, for which we refer to [41, Lemma 6.7.4]. □

Lemma 17.2.2. *Let $f\colon \mathcal{A}\to\mathcal{B}$ be a functor between ∞-categories. Let $b\in\mathcal{B}$. Assume that $\mathcal{A}$ is compact and that for every $b'\in\mathcal{B}$, the mapping space $\mathrm{Map}_{\mathcal{B}}(b,b')$ is compact. Then both $\mathcal{A}\times_{\mathcal{B}}\mathcal{B}_{b/}$ and $\mathcal{A}\times_{\mathcal{B}}\mathcal{B}_{/b}$ are compact.*

Proof. Replacing $\mathcal{A}$ and $\mathcal{B}$ by $\mathcal{A}^{\mathrm{op}}$ and $\mathcal{B}^{\mathrm{op}}$ respectively we see that it is enough to argue that $\mathcal{A}\times_{\mathcal{B}}\mathcal{B}_{b/}$ is compact. For this, observe first that since $\mathcal{B}_{b/}\to\mathcal{B}$ is a cocartesian fibration, the pullback $\mathcal{A}\times_{\mathcal{B}}\mathcal{B}_{b/}\to\mathcal{A}$ is a cocartesian fibration as well. Since $\mathcal{A}$ is compact, we are left from Theorem 17.1.2 to show that the fibers of $\mathcal{A}\times_{\mathcal{B}}\mathcal{B}_{b/}\to\mathcal{A}$ are compact, which holds by assumption on the mapping spaces of $\mathcal{B}$. □

Proposition 17.2.3. *Let $\mathcal{X}$ be an ∞-category and let $p\colon\mathcal{A}\to\mathcal{B}$ be a morphism of cocartesian fibrations over $\mathcal{X}$. Assume that for every $x\in\mathcal{X}$, the ∞-category $\mathcal{A}_x$ is compact and $\mathcal{B}_x$ is proper. Let $\mathcal{E}$ be a complete, cocomplete and stable ∞-category. Then the functor*

$$p_!\colon \mathrm{Fun}(\mathcal{A},\mathcal{E})\to\mathrm{Fun}(\mathcal{B},\mathcal{E})$$

commutes with limits.

Proof. From Corollary 6.1.6, it is enough to treat the case where $\mathcal{X}$ is a point. In that case for every $F\colon\mathcal{A}\to\mathcal{E}$ and every $b\in\mathcal{B}$, we have by definition of left Kan extension

$$(p_!(F))(b)\simeq \underset{\mathcal{A}\times_{\mathcal{B}}\mathcal{B}_{/b}}{\mathrm{colim}} F|_{\mathcal{A}\times_{\mathcal{B}}\mathcal{B}_{/b}}$$

From Lemma 17.2.2, the ∞-category $\mathcal{A}\times_{\mathcal{B}}\mathcal{B}_{/b}$ is compact. Thus, Proposition 17.2.3 follows from Lemma 17.2.1 applied to $\mathcal{C}=\mathcal{A}\times_{\mathcal{B}}\mathcal{B}_{/b}$. □

Remark 17.2.4. The assumption on $\mathcal{B}$ is always satisfied when the fibers of $\mathcal{B}$ are posets.

17.3. **Stability properties for smooth and proper stable ∞-categories.** Fix an animated ring k. Recall that $\mathrm{Mod}_k\in\mathrm{CAlg}(\mathbf{Pr}^{\mathrm{L},\omega})$ (see e.g. [7, Proposition 2.4]). We set

$$\mathbf{Pr}^{\mathrm{L},\omega}_k \coloneqq \mathrm{Mod}_{\mathrm{Mod}_k}(\mathbf{Pr}^{\mathrm{L},\omega}) \qquad \text{and} \qquad \mathbf{Pr}^{\mathrm{L}}_k\coloneqq \mathrm{Mod}_{\mathrm{Mod}_k}(\mathbf{Pr}^{\mathrm{L}})\ .$$

Given $\mathcal{C}\in\mathbf{Pr}^{\mathrm{L},\omega}_k$, we write

$$\mathrm{Hom}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}}\times\mathcal{C}\to\mathrm{Mod}_k$$

for the canonical enrichment over Mod_k. Recall also that $\mathcal{C}$ is dualizable in $\mathbf{Pr}_k^{\mathrm{L}}$, with dual $\mathcal{C}^\vee$ given by $\mathrm{Ind}((\mathcal{C}^\omega)^{\mathrm{op}})$ and write

$$\mathrm{coev}_{\mathcal{C}} \colon \mathrm{Mod}_k \to \mathcal{C}^\vee \otimes_k \mathcal{C}$$

for the coevaluation map in $\mathbf{Pr}_k^{\mathrm{L}}$. Recall the following definitions:

Definition 17.3.1. A compactly generated k-linear stable ∞-category $\mathcal{C} \in \mathbf{Pr}_k^{\mathrm{L},\omega}$ is said to be:

(1) *of finite type* if it is a compact object in $\mathbf{Pr}_k^{\mathrm{L},\omega}$;

(2) *proper* if for every compact objects $x, y \in \mathcal{C}^\omega$, $\mathrm{Hom}_{\mathcal{C}}(x,y)$ belongs to $\mathrm{Perf}(k)$;

(3) *smooth* if $\mathrm{coev}_{\mathcal{C}}$ preserves compact objects.

Remark 17.3.2. Let $\mathcal{C} \in \mathbf{Pr}_k^{\mathrm{L},\omega}$. If $\mathcal{C}$ is of finite type, then it is smooth. On the other hand, if $\mathcal{C}$ is smooth and proper, then it is of finite type.

Lemma 17.3.3. *Let $\mathcal{C}_\bullet \colon A \to \mathbf{Pr}_k^{\mathrm{L,R}}$ be a diagram such that $\mathcal{C}_a$ is compactly generated for every $a \in A$. Set*

$$\mathcal{C} \coloneqq \lim_{a \in A} \mathcal{C}_a \ ,$$

the limit being computed in $\mathbf{Pr}^{\mathrm{L}}$. Then $\mathcal{C}$ is compactly generated. Furthermore, if $\mathcal{C}_a$ is of finite type for every $a \in \mathcal{C}$ and A is a compact ∞-category, then $\mathcal{C}$ is of finite type as well.

Proof. Since the limit is computed in $\mathbf{Pr}_R^{\mathrm{L}}$, [32, Corollary 3.4.3.6] and [31, Proposition 5.5.3.13] show that it can alternatively be computed in $\mathbf{Cat}_\infty$. Since all the transition morphisms are in $\mathbf{Pr}^{\mathrm{R}}$ as well, [31, Theorem 5.5.3.18] guarantees that the limit can be also computed in $\mathbf{Pr}^{\mathrm{R}}$. Using the equivalence $\mathbf{Pr}^{\mathrm{R}} \simeq (\mathbf{Pr}^{\mathrm{L}})^{\mathrm{op}}$, we conclude that passing to left adjoints we can write

$$\mathcal{C} \simeq \operatorname*{colim}_{a \in A^{\mathrm{op}}} \mathcal{C}_a \ ,$$

the colimit being computed in $\mathbf{Pr}^{\mathrm{L}}$. Notice that the transition maps in this colimit diagram, being left adjoints to colimit-preserving functors, automatically preserve compact objects. Thus, [32, Lemma 5.3.2.9] shows that this colimit can be computed in $\mathbf{Pr}^{\mathrm{L},\omega}$. It follows that $\mathcal{C}$ is compactly generated. Besides, [32, Corollary 3.4.4.6] implies that this colimit can also be computed in $\mathbf{Pr}_k^{\mathrm{L},\omega}$, so the second half of the statement follows from the fact that compact objects are closed under finite colimits and retracts. □

Corollary 17.3.4. *Let $p \colon \mathcal{A} \to \mathcal{X}$ be a presentable cocartesian fibration with compact and proper fibers in the sense of Definition 17.1.1. Assume that $\mathcal{X}$ is compact. Let k be an animated ring and let $\mathcal{E}$ be a compactly generated k-linear stable ∞-category of finite type. Then, so is $\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E})$.*

Proof. Let $\Upsilon_{\mathcal{A}} : \mathcal{X} \to \mathbf{Pr}^{\mathrm{L}}$ be the straightening of $p \colon \mathcal{A} \to \mathcal{X}$ and consider the diagram

$$\mathrm{Fun}_!(\Upsilon_{\mathcal{A}}(-), \mathcal{E}) \colon \mathcal{X} \to \mathbf{Pr}^{\mathrm{L}}$$

where $\mathrm{Fun}_!$ denotes the functoriality given by left Kan extensions. From [31, 3.3.3.2], there is a canonical equivalence

$$\mathrm{Fun}^{\mathrm{cocart}}(\mathcal{A}, \mathcal{E}) \simeq \lim_{\mathcal{X}} \mathrm{Fun}_!(\Upsilon_{\mathcal{A}}(-), \mathcal{E})$$

By Proposition 17.2.3, the transition functors of the above diagram are left and right adjoints. Furthermore, $\mathrm{Fun}_!(\mathcal{A}_x, \mathcal{E})$ is of finite type for every $x \in \mathcal{X}$. Then, Corollary 17.3.4 follows from Lemma 17.3.3. □

18. Stability of localizations under cocartesian pullback

In [27, Proposition 2.1.4], Hinich proved that the pullback of a localization functor via a cocartesian fibration is again a localization functor. The theory surrounding the specialization equivalence and cocartesian functors developed so far allows for a model-independent proof, which we now give.

18.1. **Preliminaries.**

Lemma 18.1.1. *Let $p\colon \mathcal{B} \to \mathcal{Y}$ be a cocartesian fibration and let $\mathcal{E}$ be a presentable ∞-category. Let $\gamma\colon x \to y$ be a morphism in $\mathcal{Y}$. Let $F \in \mathrm{Fun}(\mathcal{B}_x, \mathcal{E})$ and $G \in \mathrm{Fun}(\mathcal{B}_y, \mathcal{E})$, and let $\alpha\colon F \to G$ be a morphism in $\exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y})$. The following statements are equivalent:*

1. *for every p-cocartesian lift $\phi\colon a \to b$ of γ in $\mathcal{B}$, the induced morphism (see Notation 7.1.9)*
$$\alpha(\phi)\colon F(a) \to G(b)$$
is an equivalence in $\mathcal{E}$;
2. *α is a $p_{\mathcal{E}}$-cartesian morphism in $\exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y})$.*

In addition, α is an equivalence in $\exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y})$ if and only if γ is an equivalence and condition (1) holds.

Proof. Since $p_{\mathcal{E}}\colon \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y}) \to \mathcal{Y}$ is a cartesian fibration, a morphism $\alpha\colon F \to G$ in $\exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y})$ is an equivalence if and only if it is $p_{\mathcal{E}}$-cartesian and its image in $\mathcal{Y}$ is an equivalence. So the second half of the statement follows automatically from the equivalence between statements (1) and (2). Choose a factorization of α as

$$\begin{tikzcd} F \arrow[rr, "\alpha"] \arrow[dr, "\alpha_0"'] & & G \\ & G' \arrow[ur, "\alpha_1"'] & \end{tikzcd}$$

where α_1 is $p_{\mathcal{E}}$-cartesian. Then as observed in Notation 7.1.9, any p-cocartesian lift $\phi\colon a \to b$ of γ induces via α_1 an equivalence
$$\alpha_1(\phi)\colon G'(a) \simeq G(b)\ .$$
It follows that condition (1) is equivalent to ask that for every $a \in \mathcal{B}_x$ the morphism
$$\alpha_0(a)\colon F(a) \to G'(a)$$
is an equivalence in $\mathcal{E}$. In turn, this condition is equivalent to ask that α_0 is an equivalence in $\exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y})$, and hence to condition (2). □

For later use, let us store the following consequence of Lemma 18.1.1

Corollary 18.1.2. *Let $p\colon \mathcal{A} \to \mathcal{X}$ be a locally constant cocartesian fibration (see Definition 19.1.4). Let $\mathcal{E}$ be a presentable ∞-category and let $F\colon \mathcal{A} \to \mathcal{E}$ be a cocartesian functor. Let $\sigma\colon \mathcal{X} \to \mathcal{A}$ be a cocartesian section. Then, $\sigma^*(F)\colon \mathcal{X} \to \mathcal{E}$ inverts every arrow of $\mathcal{X}$.*

Proof. Since $p\colon \mathcal{A} \to \mathcal{X}$ is locally constant, the same goes for the associated exponential fibration $p_{\mathcal{E}}\colon \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X}) \to \mathcal{X}$. Fix a morphism $\gamma\colon x \to y$ in $\mathcal{X}$, so that $\sigma(\gamma)\colon \sigma(x) \to \sigma(y)$ is a p-cocartesian lift of γ in $\mathcal{A}$. Choose a specialization morphism

$$(\operatorname{sp} F)_x \xrightarrow{\beta} G \xrightarrow{\alpha} (\operatorname{sp} F)_y$$

for F relative to γ. Then Proposition 19.1.8 guarantees that β is $p_{\mathcal{E}}$-cartesian in $\exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X})$. Thus, the result follows combining Lemma 18.1.1 and Corollary 7.1.10. □

18.2. **Hinich's theorem.** We are now ready for:

Theorem 18.2.1 (Hinich). *Let*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "u"] \arrow[d, "q"] & \mathcal{B} \arrow[d, "p"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

be a pullback square in $\mathbf{Cat}_\infty$*, where* p *is a cocartesian fibration. Assume that* f *exhibits* $\mathcal{Y}$ *as a localization of* $\mathcal{X}$ *at a collection of morphisms* W*. Then* u *is a localization functor as well, and exhibits* $\mathcal{B}$ *as localization of* $\mathcal{A}$ *at the collection* $W_{\mathcal{A}}$ *of cocartesian lifts of the arrows of* W*.*

Proof. We apply the criterion given in [11, Proposition 7.1.11]. To begin with, observe that if $\varphi \in W_{\mathcal{A}}$ then φ is q-cocartesian and therefore $u(\varphi)$ is p-cocartesian and lies over $f(q(\varphi))$ which is an equivalence in $\mathcal{X}$ since $q(\varphi) \in W$. Thus $u(\varphi)$ must be an equivalence as well, i.e. u inverts the arrows in $W_{\mathcal{A}}$.

Next, u is essentially surjective: indeed, if $b \in \mathcal{B}$ is an element, we can find $x \in \mathcal{X}$ and an equivalence $f(x) \simeq p(b)$, because f is essentially surjective. But then b defines an element in $\mathcal{B}_{f(x)}$ and since the given square is a pullback, we have $\mathcal{B}_{f(x)} \simeq \mathcal{A}_x$. Thus, we can write $b \simeq u(a)$ for some $a \in \mathcal{A}$.

Since a functor $g \colon \mathcal{C} \to \mathcal{D}$ is a localization if and only if $f^{\mathrm{op}} \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is a localization (see [11, Proposition 7.1.7]), to complete the proof it is enough to prove that

$$u^* \colon \operatorname{Fun}(\mathcal{B}, \mathbf{Spc}) \to \operatorname{Fun}(\mathcal{A}, \mathbf{Spc})$$

is fully faithful and the essential image consists of those functors $F \colon \mathcal{A} \to \mathbf{Spc}$ that invert the arrows in $W_{\mathcal{A}}$. We will more generally prove that this is the case for any presentable ∞-category $\mathcal{E}$ in place of $\mathbf{Spc}$. Proposition 6.1.2-(1) allows to rewrite u^* as

$$\Sigma(\mathcal{E}^u) \colon \operatorname{Fun}_{/\mathcal{Y}}(\mathcal{Y}, \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y})) \to \operatorname{Fun}_{/\mathcal{X}}(\mathcal{X}, \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{Y})) .$$

In virtue of Proposition 3.2.6-(1), we can rewrite

$$\operatorname{Fun}_{/\mathcal{X}}(\mathcal{X}, \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{Y})) \simeq \operatorname{Fun}_{/\mathcal{Y}}(\mathcal{X}, \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y})) ,$$

and under this identification $\Sigma(\mathcal{E}^u)$ simply becomes

$$f^* \colon \operatorname{Fun}_{/\mathcal{Y}}(\mathcal{Y}, \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y})) \to \operatorname{Fun}_{/\mathcal{Y}}(\mathcal{X}, \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y})) . \tag{18.2.2}$$

Consider now the following commutative cube:

$$\begin{tikzcd}[column sep=small]
 & \ast \arrow[rr, "\mathrm{id}_{\mathcal{Y}}"] \arrow[dd, equal] & & \operatorname{Fun}(\mathcal{Y}, \mathcal{Y}) \arrow[dd, "f^*"] \\
\operatorname{Fun}_{/\mathcal{Y}}(\mathcal{Y}, \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y})) \arrow[ur] \arrow[rr] \arrow[dd, "f^*"] & & \operatorname{Fun}(\mathcal{Y}, \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y})) \arrow[ur] \arrow[dd, "f^*"] & \\
 & \ast \arrow[rr, "f"] & & \operatorname{Fun}(\mathcal{X}, \mathcal{Y}) \\
\operatorname{Fun}_{/\mathcal{Y}}(\mathcal{X}, \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y})) \arrow[ur] \arrow[rr] & & \operatorname{Fun}(\mathcal{X}, \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y})) \arrow[ur] &
\end{tikzcd} \tag{18.2.3}$$

The bottom and the top squares are pullbacks by definition. Since f is a localization, the functor

$$f^* \colon \operatorname{Fun}(\mathcal{Y}, \mathcal{Y}) \to \operatorname{Fun}(\mathcal{X}, \mathcal{Y})$$

is fully faithful, which implies that the back square is a pullback as well. Thus, the front square is a pullback as well, and therefore the full faithfulness of (18.2.2) follows from the full faithfulness of

$$f^* \colon \operatorname{Fun}(\mathcal{Y}, \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y})) \to \operatorname{Fun}(\mathcal{X}, \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y})) \ ,$$

which holds because f is a localization.

Since the front square is a pullback, we also deduce that a section $s \in \operatorname{Fun}_{/\mathcal{Y}}(\mathcal{X}, \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{Y}))$ lies in the essential image of f^* if and only if it inverts all arrows in W. Via the specialization equivalence of Proposition 3.3.3, we deduce that a functor $F \in \operatorname{Fun}(\mathcal{B}, \mathcal{E})$ lies in the essential image of u^* if and only if $\mathcal{E}^u \circ (\operatorname{sp} F) \colon \mathcal{X} \to \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y})$ inverts all arrows in W. Fix $\gamma \colon x \to y$ in W. By assumption $f(\gamma)$ is an equivalence in $\mathcal{Y}$, so Lemma 18.1.1 shows that $\mathcal{E}^u \circ (\operatorname{sp} F)$ inverts γ if and only if

$$\big(\mathcal{E}^u(\operatorname{sp} F)\big)_\gamma \colon \big(\mathcal{E}^u(\operatorname{sp} F)\big)_x \to \big(\mathcal{E}^u(\operatorname{sp} F)\big)_y$$

is $p_{\mathcal{E}}$-cartesian in $\exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y})$. Since $p_{\mathcal{E}} \colon \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{Y}) \to \mathcal{Y}$ is a cartesian fibration, it is actually enough to check that the above morphism is locally cartesian. Therefore, we can replace $\mathcal{B} \to \mathcal{Y}$ by $\mathcal{B}_{f(\gamma)} \to \Delta^1$, and since $\mathcal{B}_{f(\gamma)} \simeq \mathcal{A}_\gamma$, Lemma 18.1.1 further shows that it is enough to check that for every q-cocartesian lift $\phi \colon a \to a'$ of γ in $\mathcal{A}$, the morphism

$$(\operatorname{sp} F)_\gamma(\phi) \colon (\operatorname{sp} F)_x(a) \to (\operatorname{sp} F)_y(a')$$

is an equivalence in $\mathcal{E}$. However, Corollary 7.1.10 provides a canonical identification of this morphism with $F(\phi)$. In other words, $\mathcal{E}^u \circ (\operatorname{sp} F)$ inverts γ if and only if F inverts all q-cocartesian lifts of γ. The conclusion follows. □

19. Locally constant and finite étale fibrations

We collect in this section some material on cocartesian fibrations that generalize the idea of local constancy and finite covering in topology.

19.1. **Local constancy.** We start with the following definition:

Definition 19.1.1. Let $\mathcal{X}$ be an ∞-category and let $\mathcal{E}$ be a presentable ∞-category. We write

$$\operatorname{Loc}(\mathcal{X}; \mathcal{E}) \coloneqq \operatorname{Fun}(\operatorname{Env}(\mathcal{X}), \mathcal{E}) \ .$$

Example 19.1.2. Let (X, P) be an exodromic stratified space. Then Proposition 2.3.8 implies that $\operatorname{Env}(\Pi_\infty(X, P)) \simeq \Pi_\infty(X)$. Therefore, $\operatorname{Loc}(\Pi_\infty(X, P); \mathcal{E})$ correspond via the exodromy equivalence exactly to $\mathcal{E}$-valued hyperconstructible hypersheaves on X.

Notation 19.1.3. Let $\mathcal{X}$ be an ∞-category and let $\lambda_{\mathcal{X}} \colon \mathcal{X} \to \operatorname{Env}(\mathcal{X})$ be the canonical localization morphism. Then for every presentable ∞-category $\mathcal{E}$, the functor

$$\lambda_{\mathcal{X}}^* \colon \operatorname{Loc}(\mathcal{X}; \mathcal{E}) \to \operatorname{Fun}(\mathcal{X}, \mathcal{E})$$

is fully faithful. Given $L \in \operatorname{Loc}(\mathcal{X}; \mathcal{E})$ we will often consider it implicitly as a functor $L \colon \mathcal{X} \to \mathcal{E}$ with the property of inverting every arrow in $\mathcal{X}$.

Definition 19.1.4. We say that a functor $p \colon \mathcal{A} \to \mathcal{X}$ of ∞-categories is *locally constant fibration* if it is a cocartesian fibration and its straightening $\Upsilon \colon \mathcal{X} \to \mathbf{Cat}_\infty$ belongs to $\operatorname{Loc}(\mathcal{X}; \mathbf{Cat}_\infty)$.

The following simply follows unraveling the definitions:

Lemma 19.1.5. *Locally constant fibrations are stable under pullback.*

It is possible to give a more intrinsic formulation of locally constant cocartesian fibrations as follows.

Recollection 19.1.6. Let $p\colon \mathcal{A} \to \Delta^1$ be a cartesian and cocartesian fibration and let

$$f\colon \mathcal{A}_0 \leftrightarrows \mathcal{A}_1 : g$$

be the induced adjunction. Write $\eta\colon \mathrm{id}_{\mathcal{A}_0} \to g \circ f$ and $\varepsilon\colon f \circ g \to \mathrm{id}_{\mathcal{A}_1}$ for the unit and the counit of this adjunction. It follows from [31, Proposition 5.2.2.8] that for every morphism $\phi\colon a \to b$ in $\mathcal{A}$ lying over $0 \to 1$ in Δ^1, there is a commutative diagram in $\mathcal{A}$

$$\begin{tikzcd} & g(f(a)) \arrow[r, "\alpha"] & f(a) \arrow[dr] & \\ a \arrow[ur, "\eta_a"] \arrow[rrr, "\phi"] \arrow[dr] & & & b \\ & g(b) \arrow[r, "\beta"] & f(g(b)) \arrow[ur, "\varepsilon_b"'] & \end{tikzcd} \tag{19.1.7}$$

where:

(1) α and $\varepsilon_b \circ \beta$ are p-cartesian;

(2) β and $\alpha \circ \eta_b$ are p-cocartesian.

Proposition 19.1.8. *Let $p\colon \mathcal{A} \to \mathcal{X}$ be a cocartesian fibration and let $\Upsilon\colon \mathcal{X} \to \mathbf{Cat}_\infty$ be its straightening. For every morphism $\gamma\colon x \to y$ in $\mathcal{X}$, the following statements are equivalent:*

(1) $p_\gamma\colon \mathcal{A}_\gamma \to \Delta^1$ (see Notation 7.1.4) is a cartesian fibration and an arrow in $\mathcal{A}_\gamma$ is cocartesian if and only it is cartesian;

(2) $\Upsilon(\gamma)\colon \Upsilon(x) \to \Upsilon(y)$ is an equivalence in $\mathbf{Cat}_\infty$;

In particular, p is locally constant if and only if condition (1) holds for every morphism γ in $\mathcal{X}$.

Proof. Assume first that (1) holds. Since p_γ is both Cartesian and coCartesian the functor $\Phi(\gamma)$ admits a right adjoint $R(\gamma)\colon \Upsilon(y) \to \Upsilon(x)$. Then Recollection 19.1.6 implies that in diagram (19.1.7) both α and $\alpha \circ \eta_a$ are p-cartesian lifts of γ, so the universal property of p-cartesian edges implies that η_a must be an equivalence. The dual argument shows that ε_b is an equivalence as well. It follows that $\Upsilon(\gamma)$ is an equivalence.

Suppose conversely that $\Phi(\gamma)$ is an equivalence. Then it admits a right adjoint, which in turn implies that p_γ is a cartesian fibration. Then in Recollection 19.1.6 both η and ε are equivalences. It immediately follows that the cocartesian lift $a \to f(a)$ is also cocartesian, and that the cocartesian lift $g(b) \to b$ is also cartesian, whence the conclusion. $\square$

19.2. **Finite étale fibrations.** We now introduce the following abstract formulation of the notion of finite covering in topology:

Definition 19.2.1. We say that a cocartesian fibration between ∞-categories $f\colon \mathcal{Y} \to \mathcal{X}$ is a *finite étale fibration* if:

(1) it is locally constant;

(2) it is a cartesian fibration;

(3) the fibers of f are finite sets.

Lemma 19.2.2. *Finite étale fibrations are closed under pullback.*

Finite étale fibrations satisfy another important stability property, that we are going to explain now.

Construction 19.2.3. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a functor of small ∞-categories. Recall from Recollection 6.2.1 the adjunction

$$f_!^{\mathrm{cc}}\colon \mathbf{CoCart}_{\mathcal{X}} \leftrightarrows \mathbf{CoCart}_{\mathcal{Y}} : f^* \ .$$

Evaluating the unit of this adjunction on a cocartesian fibration $p\colon \mathcal{A} \to \mathcal{X}$, we obtain the following commutative square:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "f_{\mathcal{A}}"] \arrow[d, "p"] & f_!^{\mathrm{cc}}(\mathcal{A}) \arrow[d, "q"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \ . \end{tikzcd} \tag{19.2.4}$$

When f is a localization, $f^*\colon \mathbf{CoCart}_{\mathcal{Y}} \to \mathbf{CoCart}_{\mathcal{X}}$ is fully faithful. In this case, the counit $f_!^{\mathrm{cc}}(\mathcal{X}) \simeq f_!^{\mathrm{cc}}(f^*(\mathcal{Y})) \to \mathcal{Y}$ is an equivalence. Therefore, in this case, the structural map $q\colon f_!^{\mathrm{cc}}(\mathcal{A}) \to \mathcal{Y}$ is canonically identified with $f_!^{\mathrm{cc}}(p)$.

Lemma 19.2.5. *Assume that f exhibits $\mathcal{Y}$ as the localization of $\mathcal{X}$ at a class of morphisms W. Let $\Upsilon_{\mathcal{A}}\colon \mathcal{X} \to \mathbf{Cat}_\infty$ be the straightening of $p\colon \mathcal{A} \to \mathcal{X}$. Then, the following are equivalent:*

(1) the square (19.2.4) *is a pullback;*

(2) the functor $\Upsilon_{\mathcal{A}}\colon \mathcal{X} \to \mathbf{Cat}_\infty$ maps W to equivalences;

(3) For every $\gamma \in W$, the pullback $p_\gamma\colon \mathcal{A}_\gamma \to \Delta^1$ (see Notation 7.1.4) is a cartesian fibration and an arrow in $\mathcal{A}_\gamma$ is cocartesian if and only it is cartesian.

Proof. The equivalence between (1) and (2) follows from the universal property of the localization. The equivalence between (2) and (3) follows from Proposition 19.1.8. ☐

Corollary 19.2.6. *Let $p\colon \mathcal{A} \to \mathcal{X}$ be a cocartesian fibration between ∞-categories. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a functor exhibiting $\mathcal{Y}$ as the localization of $\mathcal{X}$ at a class of morphisms W. Then, the following are equivalent:*

(1) $p\colon \mathcal{A} \to \mathcal{X}$ is a finite étale fibration;

(2) the square (19.2.4) *is a pullback and $f_!^{\mathrm{cc}}(p)\colon f_!^{\mathrm{cc}}(\mathcal{A}) \to \mathcal{Y}$ is a finite étale fibration.*

If these conditions are satisfied, the functor $f_{\mathcal{A}}\colon \mathcal{A} \to f_!^{\mathrm{cc}}(\mathcal{A})$ exhibits $f_!^{\mathrm{cc}}(\mathcal{A})$ as the localization of $\mathcal{A}$ at every morphism above W.

Proof. That (2) implies (1) follows from the preservation of finite étale fibrations under pullback from Lemma 19.2.2. Assume that (1) holds. Let $\lambda_{\mathcal{Y}}\colon \mathcal{Y} \to \mathrm{Env}(\mathcal{Y}) \simeq \mathrm{Env}(\mathcal{X})$ be the localization at every morphism. Since $p\colon \mathcal{A} \to \mathcal{X}$ is locally constant, Lemma 19.2.5-(2) is satisfied both for (p, W) and $(f_!^{\mathrm{cc}}(p), \mathrm{Mor}(\mathcal{Y}))$. Hence, there is a commutative diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d, "p"] & f_!^{\mathrm{cc}}(\mathcal{A}) \arrow[r] \arrow[d, "f_!^{\mathrm{cc}}(p)"] & \lambda_{\mathcal{X},!}^{\mathrm{cc}}(\mathcal{A}) \arrow[d, "\lambda_{\mathcal{X},!}^{\mathrm{cc}}(p)"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \arrow[r, "\lambda_{\mathcal{Y}}"] & \mathrm{Env}(\mathcal{X}) \end{tikzcd}$$

whose squares are pullback squares. By Lemma 19.2.2, we are thus left to show that

$$\lambda_{\mathcal{X},!}^{\mathrm{cc}}(p)\colon \lambda_{\mathcal{X},!}^{\mathrm{cc}}(\mathcal{A}) \to \mathrm{Env}(\mathcal{X}) \tag{19.2.7}$$

is a finite étale fibration. Since the outer square is a pullback, the fibres of (19.2.7) are finite sets. Local constancy is obvious since $\mathrm{Env}(\mathcal{X})$ is an ∞-groupoid. Note that (19.2.7) is an inner fibration as it is cocartesian. To show that it is cartesian, it is enough to show [31, Proposition 2.4.1.5] that $\lambda_{\mathcal{X},!}^{\mathrm{cc}}(\mathcal{A})$ is an ∞-groupoid. To do this, it is enough to show that $\mathcal{A} \to \lambda_{\mathcal{X},!}^{\mathrm{cc}}(\mathcal{A})$ exhibits

$\lambda^{\mathrm{cc}}_{\mathcal{X},!}(\mathcal{A})$ as the localization of $\mathcal{A}$ at every morphism. Hence, we are left to show more generally that $\mathcal{A} \to f^{\mathrm{cc}}_!(\mathcal{A})$ exhibits $f^{\mathrm{cc}}_!(\mathcal{A})$ as the localization of $\mathcal{A}$ at every morphism above a morphism of W. By Theorem 18.2.1, it is enough to show that every morphism in $\mathcal{A}$ is p-cocartesian. This follows immediately from the fact that the fibers of $p\colon \mathcal{A} \to \mathcal{X}$ are discrete. □

Corollary 19.2.8. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a localization functor. Then the adjunction*

$$f^{\mathrm{cc}}_!\colon \mathbf{CoCart}_{\mathcal{X}} \leftrightarrows \mathbf{CoCart}_{\mathcal{Y}} \colon f^*$$

restricts to an equivalence between the ∞-subcategories spanned by finite étale fibrations.

Proof. If $p\colon \mathcal{A} \to \mathcal{X}$ is a finite étale fibration, then so is $f^{\mathrm{cc}}_!(p)\colon f^{\mathrm{cc}}_!(\mathcal{A}) \to \mathcal{Y}$ in virtue Corollary 19.2.6 and the unit of $f^{\mathrm{cc}}_! \dashv f^*$ applied to $p\colon \mathcal{A} \to \mathcal{X}$ is an equivalence. If $p\colon \mathcal{B} \to \mathcal{Y}$ is a finite étale fibration, then so is $f^*(p)\colon f^*(\mathcal{A}) \to \mathcal{X}$ by Lemma 19.2.2. Since $f\colon \mathcal{X} \to \mathcal{Y}$ is a localization, the counit of $f_! \dashv f^*$ applied to $p\colon \mathcal{B} \to \mathcal{Y}$ is automatically an equivalence. □

The link with topological covering maps is expressed by the following:

Lemma 19.2.9. *Let (X, P) be a stratified space and let $f\colon Y \to X$ be a continuous morphism. Assume that:*

(1) $f\colon Y \to X$ is a finite covering map;

(2) (X, P) is conically refineable (Definition 2.3.10);

Then (Y, P) is conically refineable and the induced map

$$\Pi_\infty(Y, P) \to \Pi_\infty(X, P)$$

is a finite étale fibration.

Proof. Let $R \to P$ be a refinement such that (X, R) is conical with locally weakly contractible strata. Since f is a local homeomorphism, (Y, R) is also conical with locally weakly contractible strata. Therefore, there is a commutative diagram

$$\begin{tikzcd} \Pi_\infty(Y,R) \arrow[r] \arrow[d] & \Pi_\infty(Y,P) \arrow[d] \\ \Pi_\infty(X,R) \arrow[r, "r^{\mathrm{cc}}_!"] & \Pi_\infty(X,P) \end{tikzcd}$$

in $\mathbf{Cat}_\infty$. Assume that the left arrow is a finite étale fibration. By Corollary 19.2.6 we deduce the existence of a pullback square of finite étale fibrations

$$\begin{tikzcd} \Pi_\infty(Y,R) \arrow[r] \arrow[d] & r^{\mathrm{cc}}_!(\Pi_\infty(Y,R)) \arrow[d] \\ \Pi_\infty(X,R) \arrow[r, "r^{\mathrm{cc}}_!"] & \Pi_\infty(X,P) \end{tikzcd}$$

such that the top arrow exhibits $r_!(\Pi_\infty(Y, R))$ as the localization of $\Pi_\infty(Y, R)$ at every arrow above an equivalence of P. By Proposition 2.3.8, we deduce the existence of a canonical equivalence

$$r^{\mathrm{cc}}_!(\Pi_\infty(Y, P)) \simeq \Pi_\infty(Y, P) \ .$$

Hence, $\Pi_\infty(Y, P) \to \Pi_\infty(X, P)$ is a finite étale fibration. Thus, we are left to prove Lemma 19.2.9 in the case where (X, P) is conically stratified. In that case, so is (Y, P). Therefore, we have the following pullback square of simplicial sets

$$\begin{tikzcd} \mathrm{Sing}_Q(Y) \arrow[r] \arrow[d] & \mathrm{Sing}(Y) \arrow[d] \\ \mathrm{Sing}_P(X) \arrow[r] & \mathrm{Sing}(X) \end{tikzcd} .$$

Since f is a covering map, it is in particular a Serre fibration. Therefore, $\mathrm{Sing}(Y) \to \mathrm{Sing}(X)$ is a Kan fibration. It follows that the above square is a homotopy pullback, and therefore that

$$\begin{tikzcd} \Pi_\infty(Y,Q) \arrow[r] \arrow[d] & \Pi_\infty(Y) \arrow[d] \\ \Pi_\infty(X,P) \arrow[r] & \Pi_\infty(X) \end{tikzcd}$$

is a pullback in $\mathbf{Cat}_\infty$. By Lemma 19.2.2, we are left to prove Lemma 19.2.9 when $P = *$ is the trivial stratification. We know that $\mathrm{Sing}(Y) \to \mathrm{Sing}(X)$ is a Kan fibration, so that $\Pi_\infty(Y) \to \Pi_\infty(X)$ is both a left and a right fibration. Since the base is an ∞-groupoid, it follows that it is locally constant in the sense of Definition 19.1.4. Besides, for $x \in X$ we have a pullback

$$\begin{tikzcd} \mathrm{Sing}(Y_x) \arrow[r] \arrow[d] & \mathrm{Sing}(Y) \arrow[d] \\ \{x\} \arrow[r] & \mathrm{Sing}(X) \end{tikzcd}$$

of simplicial sets. Since the right vertical map is a Kan fibration, we deduce that it is a homotopy pullback, i.e. that

$$\{x\} \times_{\Pi_\infty(X)} \Pi_\infty(Y) \simeq \Pi_\infty(Y_x) \ .$$

Since f is a finite covering map, Y_x is discrete, whence the conclusion. □

20. Categorical actions

We collect some material on ∞-categorical actions that is needed throughout the text.

20.1. **Generalities.** We refer to [32, §4.8.1] for the theory of tensor products of presentable ∞-categories, that endows $\mathbf{Pr}^{\mathrm{L}}$ with a symmetric monoidal structure $\mathbf{Pr}^{\mathrm{L},\otimes}$. Fix an object $\mathcal{E}^\otimes \in \mathrm{CAlg}(\mathbf{Pr}^{\mathrm{L},\otimes})$. We refer to $\mathcal{E}^\otimes$ as a *presentably symmetric monoidal ∞-category.* In particular, we have an underlying tensor product

$$\otimes_{\mathcal{E}} \colon \mathcal{E} \times \mathcal{E} \to \mathcal{E}$$

commuting with colimits in both variables and a tensor unit $\mathbb{I}_{\mathcal{E}} \in \mathcal{E}$. We refer to an object in $\mathbf{Pr}^{\mathrm{L}}_{\mathcal{E}} \coloneqq \mathrm{Mod}_{\mathcal{E}^\otimes}(\mathbf{Pr}^{\mathrm{L},\otimes})$ as an *∞-categorical module over $\mathcal{E}^\otimes$.* Ignoring homotopy coherences, such an object can informally be described as an ∞-category $\mathcal{D}$ equipped with an external tensor product

$$\otimes \colon \mathcal{E} \times \mathcal{D} \to \mathcal{D}$$

that commutes with colimits in both variables and that satisfies the usual module relations. In particular, $\mathbb{I}_{\mathcal{E}} \otimes (-) \colon \mathcal{D} \to \mathcal{D}$ comes with an identification with $\mathrm{id}_{\mathcal{D}}$. Similarly, a morphism $f \colon \mathcal{D} \to \mathcal{D}'$ of ∞-categorical $\mathcal{E}^\otimes$-modules can informally be described as a functor f equipped with homotopy coherent identifications

$$f(E \otimes D) \simeq E \otimes f(D) \ ,$$

for $E \in \mathcal{E}$ and $D \in \mathcal{D}$. Finally, [32, Theorem 4.5.2.1] supplies $\mathbf{Pr}^{\mathrm{L}}_{\mathcal{E}}$ with an induced symmetric monoidal structure $\mathbf{Pr}^{\mathrm{L},\otimes}_{\mathcal{E}}$. In particular, given two ∞-categorical $\mathcal{E}^{\otimes}$-modules $\mathcal{D}$ and $\mathcal{D}'$, we can form the relative tensor product

$$\mathcal{D} \otimes_{\mathcal{E}} \mathcal{D}' \in \mathbf{Pr}^{\mathrm{L},\otimes}_{\mathcal{E}} .$$

Recollection 20.1.1. It follows from [32, Corollary 3.4.1.7] that a symmetric monoidal functor $f^{\otimes}\colon \mathcal{E}^{\otimes} \to \mathcal{D}^{\otimes}$ allows to see $\mathcal{D}^{\otimes}$ as a ∞-categorical module over $\mathcal{E}^{\otimes}$. The underlying tensor product is then informally defined as

$$E \otimes D \coloneqq f(E) \otimes_{\mathcal{D}} D .$$

Similarly, if

$$\begin{tikzcd} & \mathcal{E}^{\otimes} \arrow[dl, "f^{\otimes}"'] \arrow[dr, "g^{\otimes}"] & \\ \mathcal{D}^{\otimes} \arrow[rr, "h^{\otimes}"] & & \mathcal{D}'^{\otimes} \end{tikzcd}$$

is a commutative triangle in $\mathrm{CAlg}(\mathbf{Pr}^{\mathrm{L},\otimes})$, then $h\colon \mathcal{D} \to \mathcal{D}'$ inherits the structure of a $\mathcal{E}$-linear functor.

Recollection 20.1.2. Let $\mathcal{E}^{\otimes}$ be a presentably symmetric monoidal ∞-category. It follows from [32, Remark 2.1.3.4] that for every (small) ∞-category $\mathcal{A}$, $\mathrm{Fun}(\mathcal{A},\mathcal{E})$ inherits a symmetric monoidal structure, that we denote $\mathrm{Fun}(\mathcal{A},\mathcal{E})^{\otimes}$. Informally speaking, given two functors $F, G\colon \mathcal{A} \to \mathcal{E}$, their tensor product is defined by the rule

$$(F \otimes G)(a) \coloneqq F(a) \otimes_{\mathcal{E}} G(a) .$$

Similarly, if $f\colon \mathcal{B} \to \mathcal{A}$ is a functor of ∞-categories, then

$$f^*\colon \mathrm{Fun}(\mathcal{A},\mathcal{E}) \to \mathrm{Fun}(\mathcal{B},\mathcal{E})$$

inherits a canonical symmetric monoidal structure.

Lemma 20.1.3. *Let $\mathcal{E}^{\otimes}$ be a presentably symmetric monoidal ∞-category and let $f\colon \mathcal{A} \to \mathcal{B}$ be a cocartesian fibration. Reviewing* $\mathrm{Fun}(\mathcal{B},\mathcal{E})$ *as a* $\mathrm{Fun}(\mathcal{A},\mathcal{E})^{\otimes}$*-module via Recollections 20.1.1 and 20.1.2, the left Kan extension functor*

$$f_!\colon \mathrm{Fun}(\mathcal{B},\mathcal{E}) \to \mathrm{Fun}(\mathcal{A},\mathcal{E})$$

is $\mathrm{Fun}(\mathcal{A},\mathcal{E})^{\otimes}$*-linear.*

Proof. It follows from [33, Proposition 2.5.5.1] that $f_!$ is an oplax symmetric monoidal functor when we see both $\mathrm{Fun}(\mathcal{B},\mathcal{E})$ and $\mathrm{Fun}(\mathcal{A},\mathcal{E})$ as symmetric monoidal ∞-categories. Using [32, Corollary 3.4.1.5], we reduce ourselves to check that for every $F \in \mathrm{Fun}(\mathcal{A},\mathcal{E})$ and every $G \in \mathrm{Fun}(\mathcal{B},\mathcal{E})$, the canonical map

$$f_!(f^*(F) \otimes G) \to F \otimes f_!(G)$$

is an equivalence. Since the tensor product of $\mathcal{E}$ commutes with colimits in both variables, this follows from the formula for left Kan extensions provided by the dual of [41, Lemma 3.1.1]. $\square$

20.2. **Universal monadicity for finite étale fibrations.** To motivate the results of this section, consider the following:

Construction 20.2.1. Fix a presentably symmetric monoidal ∞-category $\mathcal{E}^{\otimes}$ and let

$$\begin{tikzcd} \mathcal{B} \arrow[r, "u"] \arrow[d, "q"] & \mathcal{A} \arrow[d, "p"] \\ \mathcal{Y} \arrow[r, "f"] & \mathcal{X} \end{tikzcd}$$

be a pullback square in $\mathbf{Cat}_\infty$. Via Recollection 20.1.2, we obtain a commutative square

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{X},\mathcal{E})^{\otimes} \arrow[r, "f^*"] \arrow[d, "p^*"] & \mathrm{Fun}(\mathcal{Y},\mathcal{E})^{\otimes} \arrow[d, "q^*"] \\ \mathrm{Fun}(\mathcal{A},\mathcal{E})^{\otimes} \arrow[r, "u^*"] & \mathrm{Fun}(\mathcal{B},\mathcal{E}) \ . \end{tikzcd}$$

Combining [32, Theorem 4.5.2.1 and Proposition 3.2.4.7], we obtain a canonical comparison map

$$\mu\colon \mathrm{Fun}(\mathcal{Y},\mathcal{E}) \otimes_{\mathrm{Fun}(\mathcal{X},\mathcal{E})} \mathrm{Fun}(\mathcal{A},\mathcal{E}) \to \mathrm{Fun}(\mathcal{B},\mathcal{E}) \ . \tag{20.2.2}$$

Warning 20.2.3. When $\mathcal{X} = *$, the comparison map (20.2.2) is an equivalence. If both f and p are cocartesian fibrations, one can easily prove that inside $\mathrm{Mod}_{\mathrm{Triv}_{\mathcal{X}}(\mathcal{E}^{\otimes})}(\mathbf{PrFib}_{\mathcal{X}}^{\mathrm{L},\otimes})$ there is a canonical equivalence

$$\exp_{\mathcal{E}}(\mathcal{Y}/\mathcal{X}) \otimes_{\mathrm{Triv}_{\mathcal{X}}(\mathcal{E})} \exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X}) \simeq \exp_{\mathcal{E}}(\mathcal{B}/\mathcal{X}) \ .$$

However, the global section functor

$$\Sigma_{\mathcal{X}}\colon \mathrm{Mod}_{\mathrm{Triv}_{\mathcal{X}}(\mathcal{E}^{\otimes})}(\mathbf{PrFib}_{\mathcal{X}}^{\mathrm{L},\otimes}) \to \mathrm{Mod}_{\mathrm{Fun}(\mathcal{X},\mathcal{E})}(\mathbf{Pr}^{\mathrm{L},\otimes})$$

is only lax monoidal. Because of this, the functor (20.2.2) is typically not an equivalence.

The goal of this section is to show that the situation gets considerably better if f is assumed to be a finite étale fibration and $\mathcal{E}$ to be stable. We start introducing some terminology:

Definition 20.2.4. Let $f\colon \mathcal{C} \to \mathcal{D}$ and $g\colon \mathcal{D} \to \mathcal{C}$ be functors between ∞-categories. We say that f and g are *biadjoints* if the adjunctions $f \dashv g$ and $g \dashv f$ hold.

Lemma 20.2.5. *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a finite étale fibration and let $\mathcal{E}$ be stable presentable ∞-category. Then the functors*

$$f_!\colon \mathrm{Fun}(\mathcal{Y},\mathcal{E}) \to \mathrm{Fun}(\mathcal{X},\mathcal{E}) \qquad \textit{and} \qquad f^*\colon \mathrm{Fun}(\mathcal{X},\mathcal{E}) \to \mathrm{Fun}(\mathcal{Y},\mathcal{E})$$

are biadjoints.

Proof. Fix a functor $F\colon \mathcal{Y} \to \mathcal{E}$. Since $f\colon \mathcal{Y} \to \mathcal{X}$ is a cocartesian fibration, the dual of [41, Lemma 3.1.1] provides for every $x \in \mathcal{X}$ a natural equivalence

$$f_!(F)(x) \simeq \operatorname*{colim}_{y \in \mathcal{Y}_x} F_y \ .$$

Since f is a finite étale fibration, $\mathcal{Y}_x \coloneqq \mathcal{Y} \times_{\mathcal{X}} \{x\}$ is a finite set. Thus, since $\mathcal{E}$ is stable, we deduce

$$f_!(F)(x) \simeq \bigoplus_{y \in \mathcal{Y}_x} F_y \ .$$

Since f is a cartesian fibration as well, [41, Lemma 3.1.1] yields

$$f_*(F)(x) \simeq \lim_{y \in \mathcal{Y}_x} F_y \simeq \bigoplus_{y \in \mathcal{Y}_x} F_y \ .$$

Thus, $f_!$ and f_* canonically agree, whence the conclusion. □

Lemma 20.2.6. *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a finite étale fibration and let $\mathcal{E}$ be stable presentable ∞-category. Then the composition*

$$\mathrm{id}_{\mathrm{Fun}(\mathcal{Y},\mathcal{E})} \to f^* \circ f_! \simeq f^* \circ f_* \to \mathrm{id}_{\mathrm{Fun}(\mathcal{Y},\mathcal{E})}$$

is an equivalence. In particular, $f_!$ is conservative.

Proof. Write α for the given composition. It is enough to prove that for every $x \in \mathcal{X}$, $j_x^*(\alpha)$ is an equivalence in $\mathrm{Fun}(\mathcal{Y}_x, \mathcal{E})$. Using Corollary 6.1.6 (applied with $\mathcal{A} = \mathcal{Y}$, $\mathcal{B} = \mathcal{X}$ and $\mathcal{Y} = \{x\}$), we can therefore reduce ourselves to the case where $\mathcal{X}$ consists of a single point.

In this case, $\mathcal{Y}$ is just a set. Unraveling the definitions, we see that the unit of $f_! \dashv f^*$ evaluated on $F \colon \mathcal{Y} \to \mathcal{E}$ sends $y \in \mathcal{Y}$ to the canonical inclusion

$$i_y \colon F(y) \to \bigoplus_{y' \in \mathcal{Y}_{f(y)}} F(y') \ ,$$

while the counit of $f^* \dashv f_*$ evaluated on F sends $y \in \mathcal{Y}$ to the canonical projection

$$\pi_y \colon \bigoplus_{y' \in \mathcal{Y}_{f(y)}} F_{y'} \to F_y \ ,$$

whence the conclusion. □

The following is the main result concerning finite étale fibrations:

Proposition 20.2.7 (Universal monadicity for finite étale fibrations)**.** *Let $f \colon \mathcal{Y} \to \mathcal{X}$ be a finite étale fibration and let $\mathcal{E}$ be a stable presentable ∞-category. For every categorical* $\mathrm{Fun}(\mathcal{X}, \mathcal{E})$*-module $\mathcal{D}$, the induced functor*

$$f_! \otimes \mathcal{D} \colon \ \mathrm{Fun}(\mathcal{Y}, \mathcal{E}) \otimes_{\mathrm{Fun}(\mathcal{X}, \mathcal{E})} \mathcal{D} \to \mathcal{D}$$

is monadic.

Proof. Using Lemma 20.1.3, we see that both $f^* \colon \mathrm{Fun}(\mathcal{X}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{Y}, \mathcal{E})$ and $f_! \colon \mathrm{Fun}(\mathcal{Y}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{X}, \mathcal{E})$ are $\mathrm{Fun}(\mathcal{X}, \mathcal{E})$-linear. Besides, they are biadjoints to each other thanks to Lemma 20.2.5. Therefore, we obtain well defined functors

$$f_! \otimes \mathcal{D} \colon \ \mathrm{Fun}(\mathcal{Y}, \mathcal{E}) \otimes_{\mathrm{Fun}(\mathcal{X}, \mathcal{E})} \mathcal{D} \to \mathcal{D} \qquad \text{and} \qquad f^* \otimes \mathcal{D} \colon \mathcal{D} \to \mathrm{Fun}(\mathcal{Y}, \mathcal{E}) \otimes_{\mathrm{Fun}(\mathcal{X}, \mathcal{E})} \mathcal{D} \ ,$$

that are still biadjoints to each other. Besides, Lemma 20.2.6 implies that the composition

$$\mathrm{id} \to (f_! \otimes \mathcal{D}) \circ (f^* \otimes \mathcal{D}) \to \mathrm{id}$$

is an equivalence, so it follows that $f_! \otimes \mathrm{id}_{\mathcal{D}}$ is conservative. Therefore, it is monadic thanks to Lurie-Barr-Beck's theorem [32, Theorem 4.7.3.5]. □

Corollary 20.2.8. *In the situation of Construction 20.2.1, assume that $f \colon \mathcal{Y} \to \mathcal{X}$ is a finite étale fibration. Then the comparison functor*

$$\mu \colon \ \mathrm{Fun}(\mathcal{Y}, \mathcal{E}) \otimes_{\mathrm{Fun}(\mathcal{X}, \mathcal{E})} \mathrm{Fun}(\mathcal{A}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{B}, \mathcal{E})$$

is an equivalence.

Proof. Notice that $u \colon \mathcal{B} \to \mathcal{A}$ is a finite étale fibration thanks to Lemma 19.2.2. Consider the following commutative triangle:

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{Y}, \mathcal{E}) \otimes_{\mathrm{Fun}(\mathcal{X}, \mathcal{E})} \mathrm{Fun}(\mathcal{A}, \mathcal{E}) \arrow[rr, "\mu"] \arrow[dr, "f_! \otimes \mathrm{Fun}(\mathcal{A}, \mathcal{E})"'] & & \mathrm{Fun}(\mathcal{B}, \mathcal{E}) \arrow[dl, "u_!"] \\ & \mathrm{Fun}(\mathcal{A}, \mathcal{E}) \ . & \end{tikzcd}$$

Using Proposition 20.2.7, we see that both diagonal morphisms are monadic. To conclude that the horizontal arrow is an equivalence, it is enough by [32, Corollary 4.7.3.16] to check that the Beck-Chevalley transformation

$$\mu \circ (f^* \otimes \mathrm{Fun}(\mathcal{A}, \mathcal{E})) \to u^*$$

is an equivalence. Since $u_!$ is conservative, it is enough to prove that the induced transformation

$$(f_! \circ f^*) \otimes \operatorname{Fun}(\mathcal{A}, \mathcal{E}) \simeq (f_! \otimes \operatorname{Fun}(\mathcal{A}, \mathcal{E})) \circ (f^* \otimes \operatorname{Fun}(\mathcal{A}, \mathcal{E})) \to u_! \circ u^*$$

is an equivalence. Fix a functor $F\colon \mathcal{A} \to \mathcal{E}$ and an object $a \in \mathcal{A}$. Set $x \coloneqq p(a)$ and write $\mathbb{I}$ for the tensor unit of $\operatorname{Fun}(\mathcal{A}, \mathcal{E})$ (that is, the constant functor associated to the tensor unit $\mathbb{I}_{\mathcal{E}}$ of $\mathcal{E}$). Evaluating the source of the above transformation at F and at a yields

$$\big(f_! f^*(\mathbb{I}) \otimes F\big)(a) \simeq \Big(\bigoplus_{y \in \mathcal{Y}_x} I \Big) \otimes F(a) \ ,$$

while

$$\big(u_! u^*(F)\big)(a) \simeq \bigoplus_{b \in \mathcal{B}_a} F(a) \ .$$

Since the square in Construction 20.2.1 is a pullback, $\mathcal{B}_a \simeq \mathcal{Y}_{p(a)} \simeq \mathcal{Y}_x$, whence the conclusion. $\square$

21. Additional properties of cocartesian fibrations

Finally, we collect some auxiliary results on cocartesian fibrations that are occasionally needed throughout the text.

21.1. **Global vs. local full faithfulness.** The following results provides a categorical local-to-global principle to test fully faithfulness:

Proposition 21.1.1. *Let $\mathcal{X}$ be an ∞-category and let $f\colon \mathcal{A} \to \mathcal{B}$ be a morphism in $\mathbf{PrFib}^{\mathrm{L}}_{\mathcal{X}}$. Then:*

(1) f is fully faithful if and only if for every $x \in \mathcal{X}$ the induced functor $f_x\colon \mathcal{A}_x \to \mathcal{B}_x$ is fully faithful;

(2) if f is fully faithful, then the same goes for

$$\Sigma_{\mathcal{X}}(f)\colon \operatorname{Fun}_{/\mathcal{X}}(\mathcal{X}, \mathcal{A}) \to \operatorname{Fun}_{/\mathcal{X}}(\mathcal{X}, \mathcal{B}) \ .$$

Proof. First we prove (1). Write $p\colon \mathcal{A} \to \mathcal{X}$ and $q\colon \mathcal{B} \to \mathcal{X}$ for the structural maps. Fix $a, a' \in \mathcal{A}$ and set $x \coloneqq p(a)$ and $x' \coloneqq p(a')$. The morphism f induces a canonical commutative triangle

$$\begin{tikzcd} \operatorname{Map}_{\mathcal{A}}(a, a') \arrow[rr, "\omega"] \arrow[dr] & & \operatorname{Map}_{\mathcal{B}}(f(a), f(a')) \arrow[dl] \\ & \operatorname{Map}_{\mathcal{X}}(x, x') & \end{tikzcd}$$

in $\mathbf{Spc}$. Thus, we see that ω is an equivalence if and only if for every $\gamma\colon x \to x'$ the fiber ω_γ is an equivalence. Let $a \to a_\gamma$ be a cocartesian lift of γ inside $\mathcal{A}$. Since f preserves cocartesian edges, we see that $f(a) \to f(a_\gamma)$ is cocartesian in $\mathcal{B}$. Thus, [31, Proposition 2.4.4.2] and the above commutative triangle supply a canonical identification of ω_γ with the map

$$\operatorname{Map}_{\mathcal{A}_{x'}}(a_\gamma, a') \to \operatorname{Map}_{\mathcal{B}_{x'}}(f(a_\gamma), f(a'))$$

induced by $f_{x'}\colon \mathcal{A}_{x'} \to \mathcal{B}_{x'}$. Thus, if $f_{x'}$ is fully faithful, we deduce that ω is an equivalence. As for the converse, it suffices to observe that with the above notations, the square

$$\begin{tikzcd} \operatorname{Map}_{\mathcal{A}_{x'}}(a_\gamma, a') \arrow[r] \arrow[d] & \operatorname{Map}_{\mathcal{B}_{x'}}(f(a_\gamma), f(a')) \arrow[d] \\ \operatorname{Map}_{\mathcal{A}}(a, a') \arrow[r, "\omega"] & \operatorname{Map}_{\mathcal{B}}(f(a), f(a')) \end{tikzcd}$$

is a pullback. Thus, when $\gamma = \mathrm{id}_x$, we see that the full faithfulness of f implies the full faithfulness of f_x.

We now prove (2). Consider the following commutative diagram

$$\begin{tikzcd} \mathrm{Fun}_{/\mathcal{X}}(\mathcal{X},\mathcal{A}) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{X},\mathcal{A}) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{X},\mathcal{X}) \arrow[d, equal] \\ \mathrm{Fun}_{/\mathcal{X}}(\mathcal{X},\mathcal{B}) \arrow[r] & \mathrm{Fun}(\mathcal{X},\mathcal{B}) \arrow[r] & \mathrm{Fun}(\mathcal{X},\mathcal{X}) \end{tikzcd} ,$$

whose rows are fibers sequences at $\mathrm{id}_{\mathcal{X}} \in \mathrm{Fun}(\mathcal{X},\mathcal{X})$. Since fully faithful functors are stable under pullbacks, it suffices to prove that the middle vertical functor is fully faithful. This follows immediately from the assumption and from [20, Proposition 5.1] (see also Lemma 5.2 in *loc. cit.*). ☐

Corollary 21.1.2. *Let $\mathcal{C}_\bullet \colon I \to \mathbf{Pr}^{\mathrm{L}}$ be a filtered diagram. Let*

$$\mathcal{C} \coloneqq \operatorname*{colim}_{i\in I} \mathcal{C}_i$$

be its colimit computed in $\mathbf{Pr}^{\mathrm{L}}$ and denote $\iota_i \colon \mathcal{C}_i \to \mathcal{C}$ for the canonical maps. If all the transition maps in $\mathcal{C}_\bullet$ are fully faithful, the same goes for each ι_i.

Proof. Fix an index $i \in I$. Up to replacing I by $I_{i/}$, we can suppose without loss of generality that i is the initial object of I. Thus, we obtain a transformation $\mathcal{C}_i \to \mathcal{C}_\bullet$, where $\mathcal{C}_i$ is seen as a constant diagram. Passing to the cocartesian unstraightenings, we obtain a morphism

$$f \colon \mathcal{C}_i \times I \to \mathrm{Un}_I(\mathcal{C}_\bullet)$$

of cocartesian fibrations over I. Our assumption implies that this functor is fully faithful fiberwise, and therefore Proposition 21.1.1 guarantees that f is itself fully faithful. Notice now that $\mathcal{C}_i \times I$ and $\mathrm{Un}_I(\mathcal{C}_\bullet)$ are also cartesian fibrations over I and that

$$\mathcal{C}_i \simeq \lim_{j\in I^{\mathrm{op}}} \mathcal{C}_i \simeq \mathrm{Fun}^{\mathrm{cart}}_{/I}(I, \mathcal{C}_i \times I) \qquad \text{and} \qquad \mathcal{C} \simeq \lim_{j\in I^{\mathrm{op}}} \mathcal{C}_j \simeq \mathrm{Fun}^{\mathrm{cart}}_{/I}(I, \mathrm{Un}_I(\mathcal{C}_\bullet)) \ .$$

Moreover, under these equivalences, ϕ induces the functor $\iota_i \colon \mathcal{C}_i \to \mathcal{C}$. We claim that ϕ preserves cartesian edges. Assuming this statement, we see that f induces the following commutative diagram:

$$\begin{tikzcd} \mathcal{C}_i \arrow[r] \arrow[d, "\iota_i"] & \mathrm{Fun}(I, \mathcal{C}_i) \arrow[d, "\Sigma_I(f)"] \\ \mathcal{C} \arrow[r] & \mathrm{Fun}_{/I}(I, \mathrm{Un}_I(\mathcal{C}_\bullet)) \end{tikzcd} ,$$

whose horizontal arrows are fully faithful. Since $\Sigma_I(f)$ is fully faithful by Proposition 21.1.1, we conclude that ι_i is fully faithful as well.

We are left to prove the claim. Let $j \to \ell$ be a morphism in I and let $f_{j,\ell} \colon \mathcal{C}_j \to \mathcal{C}_\ell$ be the induced functor. It fits in the following commutative triangle

$$\begin{tikzcd} & \mathcal{C}_i \arrow[dl, "f_j"'] \arrow[dr, "f_\ell"] & \\ \mathcal{C}_j \arrow[rr, "f_{j,\ell}"] & & \mathcal{C}_\ell \end{tikzcd} ,$$

where f_j and f_ℓ are the functors induced by $i \to j$ and $i \to \ell$, respectively. Write g_j, g_ℓ and $g_{j,\ell}$ for their right adjoints. Unraveling the definitions, we have to check that the Beck-Chevalley transformation

$$f_j \to g_{j,\ell} \circ f_\ell$$

is an equivalence. However, $f_\ell \simeq f_{j,\ell} \circ f_j$, and the unit $\mathrm{id}_{\mathcal{C}_j} \to g_{j,\ell} \circ f_{j,\ell}$ is an equivalence because $f_{j,\ell}$ is fully faithful by assumption. Thus, the conclusion follows. ☐

21.2. **Inducing left adjointability from the base.** The following lemma provides a general mechanism to deduce left adjointability involving cocartesian fibrations from the case of trivial fibrations. It plays an important role in the proof of the spreading out Theorem 11.4.1 for Stokes analytic stratified spaces.

Lemma 21.2.1. *Consider the commutative cube*

$$\begin{tikzcd}
\mathcal{D} \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{C} \arrow[dr, "j"] \arrow[dd, "p"] & \\
& \mathcal{B} \arrow[rr] \arrow[dd] & & \mathcal{A} \arrow[dd] \\
\mathcal{T} \arrow[rr] \arrow[dr] & & \mathcal{Z} \arrow[dr, "i"] & \\
& \mathcal{Y} \arrow[rr] & & \mathcal{X}
\end{tikzcd}$$

whose vertical faces are pull-back diagrams. Assume that the vertical arrows are cocartesian fibrations. Let $a \in \mathcal{C}$ and set $x := p(a) \in \mathcal{Z}$. Assume that the functor

$$\mathcal{T} \times_{\mathcal{Z}} \mathcal{Z}_{/x} \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{X}_{/i(x)}$$

is cofinal. Then, the functor

$$\mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{/a} \to \mathcal{B} \times_{\mathcal{A}} \mathcal{A}_{/j(a)}$$

is cofinal.

Proof. Since the vertical faces of the above cube are pull-back, the following square

$$\begin{tikzcd}
\mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{/a} \arrow[r] \arrow[d] & \mathcal{B} \times_{\mathcal{A}} \mathcal{A}_{/j(a)} \arrow[d] \\
\mathcal{T} \times_{\mathcal{Z}} \mathcal{Z}_{/x} \arrow[r] & \mathcal{Y} \times_{\mathcal{X}} \mathcal{X}_{/i(x)}
\end{tikzcd}$$

is a pull-back. From [31, 2.4.3.2], its vertical arrows are cocartesian fibrations. Since cocartesian fibrations are smooth [31, 4.1.2.15] and since the pull-back along a smooth map preserves cofinality [31, 4.1.2.10], Lemma 21.2.1 thus follows. □

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