DAY'S CONVOLUTION FOR PRO-∞-CATEGORIES

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Abstract. We generalize the known construction of Day convolution to the case where the source monoidal (∞-)category is fibered over some pro-∞-category.

Contents

1. Introduction

Let $\mathcal{A}$ be a (∞-)category and let $(\mathcal{E}, \otimes_{\mathcal{E}})$ be a symmetric monoidal (∞-)category. The category of functors $\mathrm{Fun}(\mathcal{A}, \mathcal{E})$ inherits from $\mathcal{E}$ a tensor product, simply defined by

$$(F \otimes G)(a) \coloneqq F(a) \otimes_{\mathcal{E}} G(a) \ .$$

On the other hand, when $\mathcal{A}$ is on its own equipped with a (symmetric) monoidal structure $(\mathcal{A}, \otimes_{\mathcal{A}})$ and $\mathcal{E}$ has enough colimits, one can equip $\mathrm{Fun}(\mathcal{A}, \mathcal{E})$ with a second monoidal structure $\otimes_{\mathrm{Day}}$, known as the *Day convolution*, given by the concrete formula

$$(F \otimes_{\mathrm{Day}} G)(a) \coloneqq \operatorname*{colim}_{a_1 \otimes_{\mathcal{A}} a_2 \to a} F(a_1) \otimes_{\mathcal{E}} G(a_2) \ . \tag{1.1}$$

This formula can be seen as a categorification of the usual convolution product for group rings. For instance, if G is a finite (commutative) group then $\mathbf{Mod}_{\mathbb{C}[G]}$ has two (symmetric) monoidal structures: the one given by tensor product relative to $\mathbb{C}[G]$ and the one induced by the Hopf algebra structure of $\mathbb{C}[G]$. Under the identification $\mathbf{Mod}_{\mathbb{C}[G]} \simeq \mathrm{Fun}(G, \mathbf{Mod}_{\mathbb{C}})$, the latter corresponds exactly to Day's convolution. In the ∞-categorical setting some extra care is required to take care of all the necessary homotopy coherences needed to define a (symmetric) monoidal structure. This has been done by Glasman in [3], and later on was further generalized by Lurie in [6, §2.2.6].

We came across Day convolution while working on the companion paper [**?**] on the derived moduli stacks of higher dimensional Stokes structures. The basic geometric setup in there can be described as follows: start with a pair (X, D) where X is a complex manifold and D is a simple normal crossing divisor, and a set of irregular values $\mathcal{I}$ (that is, meromorphic functions with poles along D, up to a certain equivalence relation). Out of this, one defines a *stratification by Stokes lines* $P_{\mathcal{I}}$ on the infinitesimal homotopy type $\Pi_\infty(\widehat{X} \smallsetminus D)$ of X along D, *and* a cocartesian fibration in posets $\mathcal{A}_{\mathcal{I}} \to \Pi_\infty^\Sigma(\widehat{X} \smallsetminus D, P_{\mathcal{I}})$ over the exit path ∞-category associated to the stratification $P_{\mathcal{I}}$.[1] Stokes data with coefficients in some presentable ∞-category $\mathbf{Mod}_A$ are then defined as a certain subcategory of $\mathrm{Fun}(\mathcal{A}_{\mathcal{I}}, \mathbf{Mod}_A)$. On the other side of the Riemann-Hilbert correspondence, these objects are identified with flat vector bundles with irregular connections and specified formal type around D. Forgetting the formal type, flat vector bundles can be tensored together, so it is natural to ask whether a tensor structure can be intrinsically defined on the Stokes side. This is not an entirely trivial matter, as already observed in [7, Remark 1.11], and yet it is an important step needed to obtain a satisfactory characterization of the cotangent complex of the moduli stack of higher dimensional Stokes data.

The tensor product of Stokes data should be seen as a particular instance of Day convolution: at the moral level, the fibers of the fibration $\mathcal{A}_{\mathcal{I}} \to \Pi_\infty^\Sigma(\widehat{X} \smallsetminus D, P_{\mathcal{I}})$ have a monoidal structure induced by the sum of irregular values. There are two technical obstructions to make this idea precise:

(1) the monoidal structure is only defined relatively to some base;

(2) in order to be able to sum irregular values, we cannot just fix a set $\mathcal{I}$, but we have to consider them all at once.

Unfortunately, working with all irregular values at once it not an option, because in this case the stratification by Stokes lines would cover the entire space. So we have to resort "filtering" irregular values or, in more categorical terms, consider them as an ind-object. At the level of stratified spaces, this amounts to consider a *pro-object* of exit paths ∞-categories.

This leads to the main result of this paper. Let $\mathfrak{X} = \text{“lim”}_{i\in I} \mathcal{X}_i \in \mathrm{Pro}(\mathbf{Cat}_\infty)$ be a pro-∞-category. Define

$$\mathbf{CoCart}_{\mathfrak{X}} \coloneqq \operatorname*{colim}_{i\in I} \mathbf{CoCart}_{\mathcal{X}_i} \ .$$

Its objects can informally be identified with equivalence classes of pairs $(i, \mathcal{A})$ where $i \in I$, $\mathcal{A} \to \mathcal{X}_i$ is a cocartesian fibration, and two such pairs $(i, \mathcal{A})$ and $(j, \mathcal{B})$ are equivalent if there exists an index k mapping to i and j together with an equivalence

$$\mathcal{X}_k \times_{\mathcal{X}_i} \mathcal{A} \simeq \mathcal{X}_k \simeq_{\mathcal{X}_j} \mathcal{B} \ .$$

Besides, $\mathbf{CoCart}_{\mathfrak{X}}$ has a natural symmetric monoidal (cartesian) structure, so it makes sense to consider algebra objects $\mathcal{A}^\otimes \in \mathrm{CAlg}(\mathbf{CoCart}_{\mathfrak{X}}^\times)$. Fix $\mathcal{A}^\otimes$ and assume that the underlying cocartesian fibration is defined over $\mathcal{X}_i$. We can therefore form the ∞-category

$$\operatorname*{colim}_{j\to i} \mathrm{Fun}(\mathcal{X}_j \times_{\mathcal{X}_i} \mathcal{A}, \mathcal{E}) \ , \tag{1.2}$$

The main result of this paper is then:

Theorem 1.3 (see Corollary 4.17)**.** *The ∞-category* (1.2) *has a symmetric monoidal ∞-category, generalizing the classical Day convolution product.*

One of the main difficulties comes from the fact that the canonical morphism

$$\operatorname*{colim}_{i\in I} \mathrm{CAlg}(\mathbf{CoCart}_{\mathcal{X}_i}) \longrightarrow \mathrm{CAlg}(\mathbf{CoCart}_{\mathfrak{X}})$$

is not an equivalence. In other words, we cannot simply use the classical Day convolution to assert that every term in (1.2) has a symmetric monoidal structure. Rather, one has to mimic the construction of Day's convolution in this more elaborated setting. It turns out that neither the approach of Glasman [3] nor the operadic norm method of Lurie [6, §2.2.6] lend to an immediate

[1] Here, we are ignoring the technical issue of non-conicality of $P_{\mathcal{I}}$, which is irrelevant to the current paper. We refer to the introduction of [?] for more on this situation.

generalization. We therefore take a different perspective, that can already be found in [6, Remark 4.8.1.13]: given $\mathcal{E} \in \mathrm{CAlg}(\mathbf{Pr}^{\mathrm{L}})$, the functor

$$\mathrm{Fun}_!(-, \mathcal{E}) \colon \mathbf{Cat}_\infty^\times \longrightarrow \mathbf{Pr}^{\mathrm{L},\otimes}$$

(whose functoriality is given by left Kan extensions) has a natural lax symmetric monoidal structure. In particular, it lifts to a functor

$$\mathrm{Fun}_!(-, \mathcal{E}) \colon \mathrm{CAlg}(\mathbf{Cat}_\infty^\times) \longrightarrow \mathrm{CAlg}(\mathbf{Pr}^{\mathrm{L},\otimes}) \ ,$$

providing a different realization of Day convolution, which is well suited to tackle the above theorem.

Objects in $\mathrm{Pro}(\mathbf{Cat}_\infty)$ can be rather mysterious. For this reason, we offer a second generalization of Day convolution, replacing $\mathrm{Pro}(\mathbf{Cat}_\infty)$ by $\mathrm{Cond}(\mathbf{Cat}_\infty)$.

2. Reminders on tensor product of ∞-categories

Definition 2.1. Fix a collection of small simplicial sets $\mathcal{K}$.

(1) We say that an ∞-category $\mathcal{A}$ is *$\mathcal{K}$-cocomplete* if it admits K-indexed colimits for every $K \in \mathcal{K}$.

(2) We say that a functor $f \colon \mathcal{A} \to \mathcal{B}$ between $\mathcal{K}$-cocomplete ∞-categories is *$\mathcal{K}$-cocontinuous* if it commutes with K-indexed colimits for every $K \in \mathcal{K}$.

Following [6, Definition 4.8.1.1], we let $\mathbf{Cat}_\infty(\mathcal{K})$ denote the (non full) subcategory of $\mathbf{Cat}_\infty$ spanned by $\mathcal{K}$-cocomplete ∞-categories and $\mathcal{K}$-cocontinuous functors between them.

Notation 2.2.

(1) Given $\mathcal{A}, \mathcal{B} \in \mathbf{Cat}_\infty(\mathcal{K})$, we let $\mathrm{Fun}^{\mathcal{K}}(\mathcal{A}, \mathcal{B})$ be the full subcategory of $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$ spanned by $\mathcal{K}$-cocontinuous functors.

(2) Given an ∞-category $\mathcal{A} \in \mathbf{Cat}_\infty$, we let $\mathcal{P}^{\mathcal{K}}(\mathcal{A})$ be the $\mathcal{K}$-completion of $\mathcal{A}$, introduced in [5, Proposition 5.3.6.2]. The universal property proved in *loc. cit.* shows that $\mathcal{P}^{\mathcal{K}}$ provides a left adjoint to the natural functor $\mathbf{Cat}_\infty(\mathcal{K}) \to \mathbf{Cat}_\infty$.

(3) More generally, let $\mathcal{K}' \subseteq \mathcal{K}$ be two collections of simplicial sets. Then the relative completion functor $\mathcal{P}^{\mathcal{K}}_{\mathcal{K}'} \colon \mathbf{Cat}_\infty(\mathcal{K}') \to \mathbf{Cat}_\infty(\mathcal{K})$ of [5, Proposition 5.3.6.2] provides a left adjoint to the natural forgetful functor $\mathbf{Cat}_\infty(\mathcal{K}) \to \mathbf{Cat}_\infty(\mathcal{K}')$.

Remark 2.3. Since $\mathbf{Cat}_\infty(\mathcal{K})$ is not a full subcategory of $\mathbf{Cat}_\infty$, we cannot deduce just from the existence of the $\mathcal{K}$-completion functor that $\mathbf{Cat}_\infty(\mathcal{K})$ is presentable. This is nevertheless true, as it is shown in [6, Lemma 4.8.4.2].

Example 2.4.

(1) Taking $\mathcal{K} = \mathcal{K}_\top$ the collection of all small simplicial sets, $\mathbf{Cat}_\infty(\mathcal{K})$ becomes the ∞-category of cocomplete ∞-categories. The $\mathcal{K}$-completion functor in this case simply coincides with the presheaf functor $\mathrm{PSh}(\mathcal{C}) \coloneqq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$. Notice that the covariant functoriality in $\mathcal{C}$ is given by left Kan extensions.

(2) Taking $\mathcal{K}$ to be the single simplicial set Idem, $\mathbf{Cat}_\infty(\mathcal{K})$ becomes the ∞-category of idempotent-complete ∞-categories. The $\mathcal{K}$-completion functor becomes the idempotent completion, also known as Karoubi completion.

(3) Taking $\mathcal{K}$ be the collection of all small κ-filtered simplicial sets, the $\mathcal{K}$-completion functor becomes Ind_κ, the ind-object construction of cardinal κ. When $\kappa = \omega$, this gives back the usual ind-construction.

(4) Fix a regular cardinal κ and let $\mathcal{K}$ be the collection of all κ-small simplicial sets, plus Idem. Then $\mathbf{Cat}_\infty(\mathcal{K})$ is equivalent to $\mathbf{Pr}^{\mathrm{L},\kappa}$, and in this case the $\mathcal{K}$-completion functor becomes the functor sending $\mathcal{C}$ to $\mathrm{Ind}_\kappa(\mathcal{A})^\kappa$, the full subcategory of $\mathrm{Ind}_\kappa(\mathcal{A})$ spanned by κ-compact objects. See [5, Proposition 5.5.7.8].

Definition 2.5. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be three $\mathcal{K}$-cocomplete ∞-categories. We say that a functor

$$F \colon \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C}$$

is $\mathcal{K}$-*bicocontinuous* if it is $\mathcal{K}$-cocontinuous in both variables: for every $X \in \mathcal{A}$ and for every $Y \in \mathcal{B}$, the induced functors

$$F(X,-) \colon \mathcal{B} \longrightarrow \mathcal{C} \qquad \text{and} \qquad F(-,Y) \colon \mathcal{A} \longrightarrow \mathcal{C}$$

are $\mathcal{K}$-cocontinuous. We write $\mathrm{Fun}^{\mathcal{K}\times\mathcal{K}}(\mathcal{A}\times\mathcal{B},\mathcal{C})$ for the full subcategory of $\mathrm{Fun}(\mathcal{A}\times\mathcal{B},\mathcal{C})$ spanned by $\mathcal{K}$-bicocontinuous functors.

Fix two $\mathcal{K}$-cocomplete ∞-categories $\mathcal{A}$ and $\mathcal{B}$. Consider the functor

$$\mathrm{Fun}^{\mathcal{K}\times\mathcal{K}}(\mathcal{A}\times\mathcal{B},-) \colon \mathbf{Cat}_\infty(\mathcal{K}) \longrightarrow \mathbf{Cat}_\infty \ .$$

It is a simple exercise to check that this functor is accessible and commutes with arbitrary limits. Since $\mathbf{Cat}_\infty(\mathcal{K})$ is presentable, it follows that this functor is representable. In other words, we deduce the existence of an object $\mathcal{A} \otimes_{\mathcal{K}} \mathcal{B} \in \mathbf{Cat}_\infty(\mathcal{K})$ equipped with a $\mathcal{K}$-bicocontinuous functor

$$\mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{A} \otimes_{\mathcal{K}} \mathcal{B}$$

which is universal in the sense that for every $\mathcal{E} \in \mathbf{Cat}_\infty(\mathcal{K})$ the induced functor

$$\mathrm{Fun}^{\mathcal{K}}(\mathcal{A}\otimes_{\mathcal{K}}\mathcal{B},\mathcal{C}) \longrightarrow \mathrm{Fun}^{\mathcal{K}\times\mathcal{K}}(\mathcal{A}\times\mathcal{B},\mathcal{C})$$

is an equivalence. This construction is natural in both $\mathcal{A}$ and $\mathcal{B}$, and therefore we obtain a $\mathcal{K}$-tensor product

$$- \otimes_{\mathcal{K}} - \colon \mathbf{Cat}_\infty(\mathcal{K}) \times \mathbf{Cat}_\infty(\mathcal{K}) \longrightarrow \mathbf{Cat}_\infty(\mathcal{K}) \ .$$

It is furthermore shown in [6, Lemma 4.8.4.2] that this functor commutes with colimits in both variables. On the other hand, [6, Proposition 4.8.1.3] shows that the construction of the $\mathcal{K}$-tensor product can be promoted to a symmetric monoidal structure on $\mathbf{Cat}_\infty(\mathcal{K})$.

3. The relative setup

3.1. **$\mathcal{K}$-cocomplete cocartesian fibrations.** Consider the cartesian fibration

$$\mathrm{t} \colon \mathbf{Cat}_\infty^{[1]} \coloneqq \mathrm{Fun}(\Delta^1, \mathbf{Cat}_\infty) \longrightarrow \mathbf{Cat}_\infty \ .$$

sending a functor $\mathcal{A} \to \mathcal{X}$ to its target ∞-category. We need to suitably twist $\mathbf{Cat}_\infty^{[1]}$ by passing to the dual cocartesian fibration, in the following sense:

Definition 3.1. Let $p \colon \mathcal{A} \to \mathcal{X}$ be a cartesian fibration and let $\Phi_{\mathcal{A}} \colon \mathcal{X}^{\mathrm{op}} \to \mathbf{Cat}_\infty$ be its cartesian straightening. The *dual cocartesian fibration* $p^\star \colon \mathcal{A}^\star \to \mathcal{X}^{\mathrm{op}}$ is the cocartesian fibration classified by $\Phi_{\mathcal{A}}$.

Recollection 3.2. In the setting of the above definition, recall from [1] that objects of $\mathcal{A}^\star$ coincide with the objects of $\mathcal{A}$, while 1-morphisms $a \to b$ in $\mathcal{A}^\star$ are given by spans

$$a \xleftarrow{\ u\ } c \xrightarrow{\ v\ } b$$

where u is p-cocartesian and $p(v)$ is equivalent to the identity of $p(b)$.

We let

$$\mathbb{B}\colon \mathbf{Cat}_\infty^{[1]\star} \longrightarrow \mathbf{Cat}_\infty^{\mathrm{op}}$$

be the cocartesian fibration dual to t. Specializing Recollection 3.2 to this setting, we see that objects of $\mathbf{Cat}_\infty^{[1]\star}$ are functors $\mathcal{A} \to \mathcal{X}$, and morphisms $\mathbf{f} = (f, u, v)$ from $\mathcal{B} \to \mathcal{Y}$ to $\mathcal{A} \to \mathcal{X}$ are commutative diagrams in $\mathbf{Cat}_\infty$ of the form

$$\begin{array}{ccccc} \mathcal{B} & \xleftarrow{u} & \mathcal{B}_{\mathcal{X}} & \xrightarrow{v} & \mathcal{A} \\ \downarrow & & \downarrow & \swarrow & \\ \mathcal{Y} & \xleftarrow{f} & \mathcal{X} & & \end{array} \tag{3.3}$$

where the square is a pullback. With respect to this description, $\mathbb{B}$ sends $\mathcal{A} \to \mathcal{X}$ to its target (or base) $\mathcal{X}$, and a diagram as above defines a $\mathbb{B}$-cocartesian morphism if and only if v is an equivalence.

We define **CoCart** to be the (non-full) subcategory of $\mathbf{Cat}_\infty^{[1]\star}$ whose objects are cocartesian fibrations, and whose 1-morphisms are commutative diagrams as above where v is required to preserve cocartesian edges. In this way, **CoCart** becomes a cocartesian fibration over $\mathbf{Cat}_\infty^{\mathrm{op}}$ such that $\mathbf{CoCart} \to \mathbf{Cat}_\infty^{[1]\star}$ preserves cocartesian edges. Notice that the fiber at $\mathcal{X} \in \mathbf{Cat}_\infty^{\mathrm{op}}$ simply coincide with the ∞-category $\mathbf{CoCart}_{\mathcal{X}}$.

Definition 3.4. Let $\mathcal{K}$ be a collection of small simplicial sets.

(1) A cocartesian fibration $p\colon \mathcal{A} \to \mathcal{X}$ is said to be $\mathcal{K}$-*cocomplete* if its straightening

$$\Phi_{\mathcal{A}}\colon \mathcal{X} \longrightarrow \mathbf{Cat}_\infty$$

factors through $\mathbf{Cat}_\infty(\mathcal{K})$.

(2) Let $\mathbf{f} = (f, u, v)$ be a morphism in **CoCart** as in (3.3). We say that $\mathbf{f}$ is $\mathcal{K}$-*cocontinuous* if for every $x \in \mathcal{X}$ the induced functor $v_x\colon \mathcal{B}_{f(x)} \to \mathcal{A}_x$ commutes with $\mathcal{K}$-colimits.

We let $\mathbf{CoCart}(\mathcal{K})$ be the (non-full) subcategory of **CoCart** spanned by $\mathcal{K}$-cocomplete cocartesian fibrations and $\mathcal{K}$-cocontinuous functors between them. In this way, $\mathbf{CoCart}(\mathcal{K})$ becomes a cocartesian fibration over $\mathbf{Cat}_\infty^{\mathrm{op}}$ such that $\mathbf{CoCart}(\mathcal{K}) \to \mathbf{CoCart}$ preserves cocartesian edges.

Remark 3.5. A cocartesian fibration $p\colon \mathcal{A} \to \mathcal{X}$ is $\mathcal{K}$-cocomplete if and only if it admits $\mathcal{K}$-colimits relative to $\mathcal{X}$, in the sense of [5, Definition 4.3.1.1]. See Proposition 4.3.1.10 in *loc. cit.*

For a fixed ∞-category $\mathcal{X}$, we let $\mathbf{CoCart}(\mathcal{K})_{/\mathcal{X}}$ be the fiber of $\mathbf{CoCart}(\mathcal{K}) \to \mathbf{Cat}_\infty^{\mathrm{op}}$ at $\mathcal{X}$. Notice that by definition, we have a canonical equivalence

$$\mathbf{CoCart}(\mathcal{K})_{/\mathcal{X}} \simeq \mathrm{Fun}(\mathcal{X}, \mathbf{Cat}_\infty(\mathcal{K}))$$

induced by the straightening / unstraightening equivalence for cocartesian fibrations over $\mathcal{X}$.

3.2. **Relative completions.** Let $\mathcal{K}' \subseteq \mathcal{K}$ be two collections of simplicial sets. There is an obvious forgetful functor

$$\mathbb{U}_{\mathcal{K}'}^{\mathcal{K}}\colon \mathbf{CoCart}(\mathcal{K}) \longrightarrow \mathbf{CoCart}(\mathcal{K}') \tag{3.6}$$

which forgets $\mathcal{K}$-cocompleteness to $\mathcal{K}'$-cocompleteness. Besides, $\mathbb{U}_{\mathcal{K}'}^{\mathcal{K}}$ preserves cocartesian edges over $\mathbf{Cat}_\infty^{\mathrm{op}}$. We are going to see that $\mathbb{U}_{\mathcal{K}'}^{\mathcal{K}}$ has a left adjoint $\mathbb{P}_{\mathcal{K}'}^{\mathcal{K}}$ relative to $\mathbf{Cat}_\infty^{\mathrm{op}}$ ([6, Definition 7.3.2.2]). We first construct $\mathbb{P}_{\mathcal{K}'}^{\mathcal{K}}$ over each object of $\mathbf{Cat}_\infty^{\mathrm{op}}$. To this end, for every $\mathcal{X} \in \mathbf{Cat}_\infty$, denote by

$$\mathcal{U}_{\mathcal{K}'}^{\mathcal{K}}(-/\mathcal{X})\colon \mathbf{CoCart}(\mathcal{K})_{/\mathcal{X}} \longrightarrow \mathbf{CoCart}(\mathcal{K}')_{/\mathcal{X}}$$

the restriction of $\mathbb{U}_{\mathcal{K}'}^{\mathcal{K}}$ over $\mathcal{X}$.

Definition 3.7. Let $\mathcal{K}' \subseteq \mathcal{K}$. We define a functor

$$\mathcal{P}^{\mathcal{K}}_{\mathcal{K}'}(-/\mathcal{X}) \colon \mathbf{CoCart}(\mathcal{K}')_{/\mathcal{X}} \longrightarrow \mathbf{CoCart}(\mathcal{K})_{/\mathcal{X}}$$

as the composition

$$\mathbf{CoCart}(\mathcal{K}')_{/\mathcal{X}} \simeq \mathrm{Fun}(\mathcal{X}, \mathbf{Cat}_\infty(\mathcal{K}')) \xrightarrow{\mathcal{P}^{\mathcal{K}}_{\mathcal{K}'} \circ -} \mathrm{Fun}(\mathcal{X}, \mathbf{Cat}_\infty(\mathcal{K})) \simeq \mathbf{CoCart}(\mathcal{K})_{/\mathcal{X}}$$

For an object $p \colon \mathcal{A} \to \mathcal{X}$ of $\mathbf{CoCart}(\mathcal{K}')_{/\mathcal{X}}$, we denote by $\mathcal{P}^{\mathcal{K}}_{\mathcal{K}'}(\mathcal{A}/\mathcal{X})$ its image.

One checks that $\mathcal{P}^{\mathcal{K}}_{\mathcal{K}'}(-/\mathcal{X})$ is left adjoint to $\mathcal{U}^{\mathcal{K}}_{\mathcal{K}'}(-/\mathcal{X})$. The following lemma is obvious:

Lemma 3.8. *Let $\mathcal{K}' \subseteq \mathcal{K}$. Let $f \colon \mathcal{Y} \to \mathcal{X}$ be a morphism in $\mathbf{Cat}_\infty$. Let $p \colon \mathcal{A} \to \mathcal{X}$ be a $\mathcal{K}'$-cocomplete cocartesian fibration. Then the commutative diagram*

$$\begin{array}{ccc} \mathbf{CoCart}(\mathcal{K})_{/\mathcal{X}} & \xrightarrow{\mathcal{U}^{\mathcal{K}}_{\mathcal{K}'}(-/\mathcal{X})} & \mathbf{CoCart}(\mathcal{K}')_{/\mathcal{X}} \\ \downarrow{\scriptstyle \mathcal{Y}\times_{\mathcal{X}} -} & & \downarrow{\scriptstyle \mathcal{Y}\times_{\mathcal{X}} -} \\ \mathbf{CoCart}(\mathcal{K})_{/\mathcal{Y}} & \xrightarrow{\mathcal{U}^{\mathcal{K}}_{\mathcal{K}'}(-/\mathcal{Y})} & \mathbf{CoCart}(\mathcal{K}')_{/\mathcal{Y}} \end{array}$$

is horizontally left adjointable. In other words, the canonical Beck-Chevalley transformation

$$\mathcal{P}^{\mathcal{K}}_{\mathcal{K}'}(\mathcal{Y} \times_{\mathcal{X}} \mathcal{A}/\mathcal{Y}) \longrightarrow \mathcal{Y} \times_{\mathcal{X}} \mathcal{P}^{\mathcal{K}}_{\mathcal{K}'}(\mathcal{A}/\mathcal{X})$$

is an equivalence.

Corollary 3.9. *The functor $\mathbb{U}^{\mathcal{K}}_{\mathcal{K}'}$ admits a left adjoint*

$$\mathbb{P}^{\mathcal{K}}_{\mathcal{K}'} \colon \mathbf{CoCart}(\mathcal{K}') \longrightarrow \mathbf{CoCart}(\mathcal{K}) \tag{3.10}$$

relative to $\mathbf{Cat}_\infty^{\mathrm{op}}$. In particular, the fiber of $\mathbb{P}^{\mathcal{K}}_{\mathcal{K}'}$ at $\mathcal{X} \in \mathbf{Cat}_\infty^{\mathrm{op}}$ is equivalent to $\mathcal{P}^{\mathcal{K}}_{\mathcal{K}'}(-/\mathcal{X})$.

Proof. It suffices to apply Proposition B.1 to $\mathbb{U}^{\mathcal{K}}_{\mathcal{K}'}$. Condition (1) is automatically satisfied, while condition (2) is ensured by Lemma 3.8, so the conclusion follows. □

Following [6, §4.8.1], we let P denote the collection of all sets of small simplicial sets, ordered by inclusion. We now assemble the ∞-categories $\mathbf{CoCart}(\mathcal{K}), \mathcal{K} \in \mathsf{P}$. Define $\underline{\mathbf{CoCart}}$ to be the (non full) subcategory of $\mathbf{CoCart} \times \mathsf{P}$ defined by the following two conditions:

1. objects of $\underline{\mathbf{CoCart}}$ are pairs $(p, \mathcal{K})$, where $p \in \mathbf{CoCart}(\mathcal{K})$;
2. if $\mathcal{K}' \subset \mathcal{K}$, a morphism from $(q, \mathcal{K}')$ to $(p, \mathcal{K})$ is a $\mathcal{K}'$-*cocontinuous* morphism in $\mathbf{CoCart}$. Put otherwise,

$$\mathrm{Map}_{\underline{\mathbf{CoCart}}}(q, p) \simeq \mathrm{Map}_{\mathbf{CoCart}(\mathcal{K}')}(q, \mathbb{U}^{\mathcal{K}}_{\mathcal{K}'}(p))$$

In particular, the fiber of $\underline{\mathbf{CoCart}} \to \mathsf{P}$ over $\mathcal{K}$ coincides with $\mathbf{CoCart}(\mathcal{K})$.

Proposition 3.11. *The projection*

$$\pi \colon \underline{\mathbf{CoCart}} \longrightarrow \mathbf{Cat}_\infty^{\mathrm{op}} \times \mathsf{P}$$

is a cocartesian fibration. Additionally, the further projection

$$\pi' \colon \underline{\mathbf{CoCart}} \longrightarrow \mathsf{P}$$

is a cartesian fibration as well.

Proof. We first prove that $\pi'\colon \underline{\mathbf{CoCart}} \to \mathsf{P}$ is both a cartesian and a cocartesian fibration. Let $\mathcal{K}' \subseteq \mathcal{K}$ be a morphism in P. Let $p\colon \mathcal{A} \to \mathcal{X}$ be an object in $\mathbf{CoCart}(\mathcal{K})$. Then, the morphism $\mathbb{U}^{\mathcal{K}}_{\mathcal{K}'}(p) \to p$ in $\underline{\mathbf{CoCart}}$ corresponding to the identity of $\mathbb{U}^{\mathcal{K}}_{\mathcal{K}'}(p)$ in $\mathbf{CoCart}(\mathcal{K}')$ is readily checked to be cartesian over $\mathcal{K}' \subseteq \mathcal{K}$. Hence, π' is a cartesian fibration. To check that it is a cocartesian fibration as well, [5, Proposition 5.2.2.5] ensures that it is enough to check that for every $\mathcal{K}' \subseteq \mathcal{K}$, the functor $\mathbb{U}^{\mathcal{K}}_{\mathcal{K}'}\colon \mathbf{CoCart}(\mathcal{K}) \longrightarrow \mathbf{CoCart}(\mathcal{K}')$ admits a left adjoint. Since this is guaranteed by Corollary 3.9, we conclude that π' is also a cocartesian fibration.

We now prove that $\pi\colon \underline{\mathbf{CoCart}} \to \mathbf{Cat}_\infty^{\mathrm{op}} \times \mathsf{P}$ is a cocartesian fibration. Write

$$\pi_2\colon \mathbf{Cat}_\infty^{\mathrm{op}} \times \mathsf{P} \longrightarrow \mathsf{P}$$

for the natural projection, so that $\pi' = \pi_2 \circ \pi$. Observe that for every $p\colon \mathcal{A} \to \mathcal{X}$ in $\mathbf{CoCart}(\mathcal{K})$, the π'-cocartesian morphism $\mathbf{f}\colon p \to \mathbb{P}^{\mathcal{K}}_{\mathcal{K}'}(p)$ is sent to the morphism $(\mathrm{id}_{\mathcal{X}}, \mathcal{K}' \subseteq \mathcal{K})$ in $\mathbf{Cat}_\infty^{\mathrm{op}} \times \mathsf{P}$. Since this is obviously π_2-cocartesian, [5, Proposition 2.4.1.3-(3)] guarantees that $\mathbf{f}$ is π-cocartesian as well. Notice that every morphism $(\mathcal{Y}, \mathcal{K}') \to (\mathcal{X}, \mathcal{K})$ in $\mathbf{Cat}_\infty^{\mathrm{op}} \times \mathsf{P}$ can be written as the composite of a morphism

$$(\mathcal{Y}, \mathcal{K}') \longrightarrow (\mathcal{X}, \mathcal{K}')$$

whose P-component is the identity, and a morphism

$$(\mathcal{X}, \mathcal{K}') \longrightarrow (\mathcal{X}, \mathcal{K}) \ ,$$

whose $\mathbf{Cat}_\infty^{\mathrm{op}}$-component is the identity. We just proved that morphisms of the second kind admit π-cocartesian lifts. Since π-cocartesian morphisms are stable under composition, we are reduced to check that morphisms of the first kind admit π-cocartesian lifts as well.

To prove this last statement, it is enough to argue that for every $\mathcal{K}'$-cocomplete cocartesian fibration $p\colon \mathcal{B} \to \mathcal{Y}$ and for every functor $\mathcal{X} \to \mathcal{Y}$ (seen as a morphism from $\mathcal{Y}$ to $\mathcal{X}$ in $\mathbf{Cat}_\infty^{\mathrm{op}}$), the natural functor $u\colon \mathcal{B}_{\mathcal{X}} \leftarrow \mathcal{B}$ (seen as a morphism from $(\mathcal{B} \to \mathcal{Y}, \mathcal{K}')$ to $(\mathcal{B}_{\mathcal{X}} \to \mathcal{X}, \mathcal{K}')$ in $\underline{\mathbf{CoCart}}$) is π-cocartesian. Unraveling the definitions, we see that u, seen as a morphism in $\mathbf{CoCart} \times \mathsf{P}$, is cocartesian with respect to the natural projection $\mathbf{CoCart} \times \mathsf{P} \to \mathbf{Cat}_\infty^{\mathrm{op}} \times \mathsf{P}$. Since $\underline{\mathbf{CoCart}} \to \mathbf{CoCart} \times \mathsf{P}$ is faithful, [5, Proposition 2.4.4.3] shows that it is enough to verify that a morphism

$$(\mathcal{B}_{\mathcal{X}} \to \mathcal{X}, \mathcal{K}') \longrightarrow (\mathcal{C} \to \mathcal{Z}, \mathcal{K})$$

in $\mathbf{CoCart} \times \mathsf{P}$ belongs to $\underline{\mathbf{CoCart}}$ if and only if the morphism induced by composition with u

$$(\mathcal{B} \to \mathcal{Y}, \mathcal{K}') \longrightarrow (\mathcal{C} \to \mathcal{Z}, \mathcal{K})$$

belongs to $\underline{\mathbf{CoCart}}$. Unraveling the definitions, this follows from the very definition of morphisms in $\underline{\mathbf{CoCart}}$ and the identification

$$\mathcal{B} \times_{\mathcal{Y}} \mathcal{Z} \simeq (\mathcal{B} \times_{\mathcal{Y}} \mathcal{X}) \times_{\mathcal{X}} \mathcal{Z} \ .$$

The conclusion follows. □

The goal of what follows is to endow $\underline{\mathbf{CoCart}}$ with a structure of $\mathbf{Cat}_\infty^{\mathrm{op}} \times \mathsf{P}$-family of symmetric monoidal ∞-categories in the sense of Definition A.1. See Proposition 3.13.

3.3. **Relative cartesian product.** Let $p\colon \mathcal{A} \to \mathcal{C}$ be a cocartesian fibration. Let $\Phi_{\mathcal{A}}\colon \mathcal{C} \to \mathbf{Cat}_\infty$ be its unstraightening. Assume that for every $c \in \mathcal{C}$, the ∞-category $\mathcal{A}_c$ has all finite products and that for every $\gamma\colon c \to d$, the induced functor $\mathcal{A}_c \to \mathcal{A}_d$ commutes with finite products. Let $\mathbf{Cat}_\infty^{\mathbf{Cart}} \subset \mathbf{Cat}_\infty$ denotes the subcategory spanned by ∞-categories having all finite products and those functors commuting with finite products. Then, the functor $\Phi_{\mathcal{A}}\colon \mathcal{C} \to \mathbf{Cat}_\infty$ factors uniquely through $\mathbf{Cat}_\infty^{\mathbf{Cart}}$ into a functor

$$\Phi_{\mathcal{A}}^{\mathbf{Cart}}\colon \mathcal{C} \to \mathbf{Cat}_\infty^{\mathbf{Cart}}$$

Let $\mathrm{SMCat}_\infty^\times \subset \mathrm{SMCat}_\infty$ be the full subcategory spanned by cartesian symmetric monoidal categories. Then, [6, Corollary 2.4.1.9] ensures that the forgetful functor $\theta\colon \mathrm{SMCat}_\infty^\times \to \mathbf{Cat}_\infty^{\mathbf{Cart}}$ is an equivalence of ∞-categories. Applying Proposition A.7 to the composition

$$\theta^{-1} \circ \Phi_{\mathcal{A}}^{\mathbf{Cart}}\colon \mathcal{C} \to \mathrm{SMCat}_\infty$$

thus yields a $\mathcal{C}$-family of symmetric monoidal ∞-categories

$$p^\otimes\colon \mathcal{A}^\otimes \to \mathrm{Fin}_* \times \mathcal{C}$$

in the sense of Definition A.1. Concretely, objects of $\mathcal{A}^\otimes$ are $(S, c, (a_i)_{i\in S})$ where S is a finite set, where $c \in \mathcal{C}$ and where $a_i \in \mathcal{C}_c$ for every $i \in S$. Morphisms from $(T, d, (b_j)_{j\in T})$ to $(S, c, (a_i)_{i\in S})$ in $\mathcal{A}^\otimes$ is the data $(\alpha, \gamma, (f_i)_{i\in S})$ where $\alpha\colon T \to S$ is a map, where $\gamma\colon d \to c$ is a morphism in $\mathcal{C}$ and where

$$f_i\colon \prod_{\alpha(j)=i} b_j \longrightarrow a_i$$

is a morphism of $\mathcal{A}$ above γ for every $i \in S$.

3.4. **Relative tensor product.** We now endow $\underline{\mathbf{CoCart}}$ with a structure of $\mathbf{Cat}_\infty^{\mathrm{op}} \times \mathsf{P}$-family of symmetric monoidal ∞-categories in the sense of Definition A.1. This can be viewed as a relative construction of Lurie's tensor product of ∞-categories [6, §4.8.1], Lurie's construction being the case where the base category $\mathcal{X}$ is the point. We thus follow Lurie's exposition. The fiber of

$$\mathbb{B}\colon \mathbf{CoCart} \longrightarrow \mathbf{Cat}_\infty^{\mathrm{op}}$$

at an ∞-category $\mathcal{X}$ coincides with the ∞-category $\mathbf{CoCart}_{\mathcal{X}}$ of cocartesian fibrations over $\mathcal{X}$. Observe that [5, Proposition 2.4.4.3] guarantees that $\mathbf{CoCart}_{\mathcal{X}}$ has finite products, and for every morphism $\mathcal{Y} \to \mathcal{X}$ the base change

$$\mathcal{Y} \times_{\mathcal{X}} -\colon \mathbf{CoCart}_{\mathcal{X}} \longrightarrow \mathbf{CoCart}_{\mathcal{Y}}$$

preserves finite products. From Section 3.3, we deduce that $\mathbf{CoCart}$ yields a $\mathbf{Cat}_\infty^{\mathrm{op}}$-family of symmetric monoidal ∞-categories

$$\mathbf{CoCart}^\otimes \longrightarrow \mathrm{Fin}_* \times \mathbf{Cat}_\infty^{\mathrm{op}}$$

in the sense of Definition A.1. Informally, Section 3.3 describes $\mathbf{CoCart}^\otimes$ as follows:

(1) objects are $(S, \mathcal{X}, \{p_i\colon \mathcal{A}_i \to \mathcal{X}\}_{i\in S})$, where S is a finite set, where $\mathcal{X} \in \mathbf{Cat}_\infty$ and where $p_i\colon \mathcal{A}_i \to \mathcal{X}$ are cocartesian fibrations for every $i \in S$;

(2) a morphism from $(T, \mathcal{Y}, \{q_j : \mathcal{B}_j \to \mathcal{Y}\}_{j\in T})$ to $(S, \mathcal{X}, \{p_i\colon \mathcal{A}_i \to \mathcal{X}\}_{i\in S})$ in $\mathbf{CoCart}^\otimes$ is the data $(\alpha, f, (\mathbf{f}_i)_{i\in S})$ where $\alpha\colon T \to S$ is a map, where $f\colon \mathcal{X} \to \mathcal{Y}$ is a functor and where

$$\mathbf{f}_i\colon \prod_{\alpha(j)=i} q_j \longrightarrow p_i$$

is a morphism of $\mathbf{CoCart}$ above f for every $i \in S$, that is $\mathbf{f}_i = (f, u_i, v_i)$ as in (3.3) with

$$v_i\colon \prod_{\alpha(j)=i} f^*(\mathcal{B}_j) \to \mathcal{A}_i \tag{3.12}$$

morphism in $\mathbf{CoCart}_{\mathcal{X}}$.

Following [6, §4.8.1], let P be the collection of all sets of small simplicial sets, ordered by inclusion. Define $\underline{\mathbf{CoCart}}^\otimes$ to be the subcategory of $\mathbf{CoCart}^\otimes \times \mathsf{P}$ defined by the following conditions:

(1) an object $(S, \mathcal{X}, (p_i)_{i\in S}, \mathcal{K})$ of $\mathbf{CoCart}^\otimes \times \mathsf{P}$ belongs to $\underline{\mathbf{CoCart}}^\otimes$ if and only if $p_i \in \mathbf{CoCart}(\mathcal{K})_{/\mathcal{X}}$ for every $i \in S$.

(2) if $\mathcal{K}' \subseteq \mathcal{K}$, a morphism $(\alpha, f, (\mathbf{f}_i)_{i\in S})\colon (T, \mathcal{Y}, (q_j)_{j\in T}, \mathcal{K}') \to (S, \mathcal{X}, (p_i)_{i\in S}, \mathcal{K})$ belongs to $\underline{\mathbf{CoCart}}^{\otimes}$ if and only if (3.12) is $\mathcal{K}'$-cocontinuous in each variable for every $j \in S$.

Using Proposition 3.11 instead of [5, 5.3.6.2], the same proof of [6, 4.8.1.3] yields:

Proposition 3.13. *The functor*

$$\underline{\mathbf{CoCart}}^{\otimes} \longrightarrow \mathrm{Fin}_* \times \mathbf{Cat}_\infty^{\mathrm{op}} \times \mathsf{P}$$

is a cocartesian fibration.

In particular, the above cocartesian fibration endows $\underline{\mathbf{CoCart}}$ with a structure of $\mathbf{Cat}_\infty^{\mathrm{op}} \times \mathsf{P}$-family of symmetric monoidal ∞-categories. For $\mathcal{K} \in \mathsf{P}$, we denote by $\mathbf{CoCart}(\mathcal{K})^{\otimes}$ its fibre over $\mathcal{K}$. Observe that the fibre of the cocartesian fibration

$$\mathbf{CoCart}(\mathcal{K})^{\otimes} \to \mathrm{Fin}_* \times \mathbf{Cat}_\infty^{\mathrm{op}} \tag{3.14}$$

above $\langle 1 \rangle$ is $\mathbf{CoCart}(\mathcal{K})$. We deduce :

Corollary 3.15. *The cocartesian fibration* $\mathbf{CoCart}(\mathcal{K})^{\otimes} \to \mathrm{Fin}_* \times \mathbf{Cat}_\infty^{\mathrm{op}}$ *endows* $\mathbf{CoCart}(\mathcal{K})$ *with a structure of* $\mathbf{Cat}_\infty^{\mathrm{op}}$*-family of symmetric monoidal ∞-categories. Furthermore, if* $\mathcal{K}' \subseteq \mathcal{K}$, *the functor*

$$\mathbb{P}_{\mathcal{K}'}^{\mathcal{K}}\colon \mathbf{CoCart}(\mathcal{K}') \longrightarrow \mathbf{CoCart}(\mathcal{K})$$

is symmetric monoidal relative to $\mathbf{Cat}_\infty^{\mathrm{op}}$.

For $\mathcal{X} \in \mathbf{Cat}_\infty$ and $\mathcal{K} \in \mathsf{P}$, we denote by $\mathbf{CoCart}(\mathcal{K})^{\otimes}_{/\mathcal{X}}$ the fibre of (3.14) over $(\mathcal{X}, \mathcal{K})$. Observe that the fibre of the cocartesian fibration $\mathbf{CoCart}(\mathcal{K})^{\otimes}_{/\mathcal{X}} \to \mathrm{Fin}_*$ above $\langle 1 \rangle$ is $\mathbf{CoCart}(\mathcal{K})_{/\mathcal{X}}$. We deduce :

Corollary 3.16. *The cocartesian fibration* $\mathbf{CoCart}(\mathcal{K})^{\otimes}_{/\mathcal{X}} \to \mathrm{Fin}_*$ *endows* $\mathbf{CoCart}(\mathcal{K})_{/\mathcal{X}}$ *with a structure of symmetric monoidal ∞-categories. Furthermore, if* $\mathcal{K}' \subseteq \mathcal{K}$, *for* $f\colon \mathcal{X} \to \mathcal{Y}$ *a functor between ∞-categories, the functor*

$$\mathcal{X} \times_{\mathcal{Y}} \mathcal{P}_{\mathcal{K}'}^{\mathcal{K}}(-/\mathcal{Y})\colon \mathbf{CoCart}(\mathcal{K}')_{/\mathcal{Y}} \longrightarrow \mathbf{CoCart}(\mathcal{K})_{/\mathcal{X}}$$

is symmetric monoidal.

To complete this discussion, we let $\mathcal{K}_\top$ be the set of all small simplicial sets. We define $\mathbf{PrFib}^{\mathrm{L}}$ to be the full subcategory of $\mathbf{CoCart}(\mathcal{K}_\top)$ spanned by *presentable fibrations.* It is immediate that the induced functor

$$\mathbb{B}\colon \mathbf{PrFib}^{\mathrm{L}} \longrightarrow \mathbf{Cat}_\infty^{\mathrm{op}}$$

exhibits $\mathbf{PrFib}^{\mathrm{L}}$ as a cocartesian fibration over $\mathbf{Cat}_\infty^{\mathrm{op}}$. Moreover, [6, Proposition 4.8.1.15] immediately implies that $\mathbf{PrFib}^{\mathrm{L}}$ is closed under the symmetric monoidal structure of $\mathbf{CoCart}(\mathcal{K}_\top)$ relative to $\mathbf{Cat}_\infty^{\mathrm{op}}$. Thus, it follows that $\mathbf{PrFib}^{\mathrm{L}}$ can be promoted to a $\mathbf{Cat}_\infty^{\mathrm{op}}$-family of symmetric monoidal ∞-categories, that we will denote $\mathbf{PrFib}^{\mathrm{L},\otimes}$.

Remark 3.17. Fix an ∞-category $\mathcal{X} \in \mathbf{Cat}_\infty^{\mathrm{op}}$. Then $\mathbf{PrFib}^{\mathrm{L}}_{\mathcal{X}}$ coincides with the ∞-category of presentable fibrations over $\mathcal{X}$. Via straightening / unstraightening, this is equivalent to $\mathrm{Fun}(\mathcal{X}, \mathbf{Pr}^{\mathrm{L}})$. Under this identification, the symmetric monoidal structure $\mathbf{PrFib}^{\mathrm{L},\otimes}_{\mathcal{X}}$ supplied by the above constructions canonically coincide with the objectwise monoidal structure of $\mathrm{Fun}(\mathcal{X}, \mathbf{Pr}^{\mathrm{L},\otimes})$.

Notice that the functor

$$\mathbb{P}\mathrm{Sh} \coloneqq \mathbb{P}_{\emptyset}^{\mathcal{K}_\top}\colon \underline{\mathbf{CoCart}} \longrightarrow \mathbf{CoCart}(\mathcal{K}_\top)$$

factors through $\mathbf{PrFib}^{\mathrm{L}}$. Thus, as a consequence of the above corollary, we obtain:

Corollary 3.18. *The functor* $\mathbb{P}\mathrm{Sh}\colon \mathbf{CoCart} \to \mathbf{PrFib}^{\mathrm{L}}$ *can be promoted to a symmetric monoidal functor*

$$\mathbb{P}\mathrm{Sh}^{\otimes}\colon \mathbf{CoCart}^{\otimes} \longrightarrow \mathbf{PrFib}^{\mathrm{L},\otimes}$$

relative to $\mathbf{Cat}_{\infty}^{\mathrm{op}}$.

Variant 3.19. Given a cocartesian fibration $\mathcal{A} \to \mathcal{X}$ classifying a functor $\Phi_{\mathcal{A}}\colon \mathcal{X} \to \mathbf{Cat}_{\infty}$, write $\mathrm{op}(\mathcal{A}/\mathcal{X})$ for the cocartesian fibration over $\mathcal{X}$ classifying the functor $\Phi_{\mathcal{A}}^{\mathrm{op}}\colon \mathcal{X} \to \mathbf{Cat}_{\infty}$. The assignment sending $\mathcal{A} \to \mathcal{X}$ to the cocartesian fibration $\mathrm{op}(\mathcal{A}/\mathcal{X}) \to \mathcal{X}$ defines a global functor

$$\mathbb{O}\mathrm{p}\colon \mathbf{CoCart} \longrightarrow \mathbf{CoCart} \ ,$$

which acquires a natural symmetric monoidal structure $\mathbb{O}\mathrm{p}^{\otimes}$ relative to $\mathbf{Cat}_{\infty}^{\mathrm{op}}$. Write

$$\mathbb{C}\mathrm{o}\mathbb{P}\mathrm{Sh}^{\otimes} \coloneqq \mathbb{P}\mathrm{Sh}^{\otimes} \circ \mathbb{O}\mathrm{p}^{\otimes}\colon \mathbf{CoCart}^{\otimes} \longrightarrow \mathbf{PrFib}^{\mathrm{L},\otimes}$$

for the induced symmetric monoidal functor relative to $\mathbf{Cat}_{\infty}^{\mathrm{op}}$. We also write $\mathbb{C}\mathrm{o}\mathbb{P}\mathrm{Sh} \coloneqq \mathbb{P}\mathrm{Sh} \circ \mathbb{O}\mathrm{p}$ for the underlying functor $\mathbf{CoCart} \to \mathbf{PrFib}^{\mathrm{L}}$. Unraveling the definitions, we see that it takes a cocartesian fibration $\mathcal{A} \to \mathcal{X}$ classifying the functor $\Phi_{\mathcal{A}}$ to the cocartesian fibration classifying the functor

$$\mathrm{Fun}_!(\Phi_{\mathcal{A}}(-), \mathcal{S})\colon \mathcal{X} \longrightarrow \mathbf{Pr}^{\mathrm{L}} \ .$$

Variant 3.20. Let $\mathbb{E}$ be a (not necessarily cocartesian) section of the cocartesian fibration $\mathbb{B}\colon \mathbf{PrFib}^{\mathrm{L}} \to \mathbf{Cat}_{\infty}^{\mathrm{op}}$. Using the relative symmetric monoidal structure of $\mathbb{B}$, given by $\mathbf{PrFib}^{\mathrm{L},\otimes}$, we obtain a functor

$$\mathbb{E} \otimes -\colon \mathbf{PrFib}^{\mathrm{L}} \longrightarrow \mathbf{PrFib}^{\mathrm{L}}$$

over $\mathbf{Cat}_{\infty}^{\mathrm{op}}$. Informally speaking, this sends a presentable fibration $\mathcal{A} \to \mathcal{X}$ to the presentable fibration $\mathcal{A} \otimes_{\mathcal{X}} \mathbb{E}(\mathcal{X})$, where $\otimes_{\mathcal{X}}$ is the underlying tensor product of the symmetric monoidal structure on $\mathbf{PrFib}_{\mathcal{X}}^{\mathrm{L},\otimes} \simeq \mathrm{Fun}(\mathcal{X}, \mathbf{Pr}^{\mathrm{L}})^{\otimes}$, where the last symmetric monoidal structure is the one induced by $\mathbf{Pr}^{\mathrm{L},\otimes}$. We define

$$\exp_{\mathbb{E}} \coloneqq \mathbb{E} \otimes \mathbb{C}\mathrm{o}\mathbb{P}\mathrm{Sh}\colon \mathbf{CoCart} \longrightarrow \mathbf{PrFib}^{\mathrm{L}} \ ,$$

and we refer to $\exp_{\mathbb{E}}$ as the *exponential functor with coefficients in* $\mathbb{E}$.

Remark 3.21.

(1) If $\mathbb{E}$ is a cocartesian section of $\mathbb{B}$, then $\mathbb{E} \otimes -$ and $\exp_{\mathbb{E}}$ preserve $\mathbb{B}$-cocartesian edges. On the other hand, since $*$ is an initial object of $\mathbf{Cat}_{\infty}^{\mathrm{op}}$, we have

$$\mathrm{Fun}^{\mathrm{ccrt}}_{/\mathbf{Cat}_{\infty}^{\mathrm{op}}}(\mathbf{Cat}_{\infty}^{\mathrm{op}}, \mathbf{PrFib}^{\mathrm{L}}) \simeq \mathbf{Pr}^{\mathrm{L}} \ ,$$

so cocartesian sections of $\mathbb{B}$ are identified with presentable ∞-categories. For $\mathcal{E} \in \mathbf{Pr}^{\mathrm{L}}$, we will simply write $\exp_{\mathcal{E}}$, leaving it implicit that $\mathcal{E}$ is identified with a section of $\mathbb{B}$.

(2) Fix $\mathcal{E} \in \mathbf{Pr}^{\mathrm{L}}$. Unraveling the definitions, we see that $\exp_{\mathcal{E}}$ takes a cocartesian fibration $\mathcal{A} \to \mathcal{X}$ with straightening $\Phi_{\mathcal{A}}\colon \mathcal{X} \to \mathbf{Cat}_{\infty}$ to the presentable fibration $\exp_{\mathcal{E}}(\mathcal{A}/\mathcal{X}) \to \mathcal{X}$ classifying the functor

$$\mathrm{Fun}_!(\Phi_{\mathcal{A}}(-), \mathcal{E}) \simeq \mathrm{Fun}_!(\Phi_{\mathcal{A}}(-), \mathcal{S}) \otimes \mathcal{E}\colon \mathcal{X} \longrightarrow \mathbf{Pr}^{\mathrm{L}} \ .$$

Variant 3.22. Since the cocartesian fibration $\mathbb{B}\colon \mathbf{PrFib}^{\mathrm{L}} \to \mathbf{Cat}_{\infty}^{\mathrm{op}}$ has a relative symmetric monoidal structure, we can consider the induced cocartesian fibration

$$\mathrm{CAlg}(\mathbf{PrFib}^{\mathrm{L},\otimes}/\mathbf{Cat}_{\infty}^{\mathrm{op}}) \longrightarrow \mathbf{Cat}_{\infty}^{\mathrm{op}} \ ,$$

for whose construction we refer to §A.2. Notice that since $*$ is an initial object in $\mathbf{Cat}_{\infty}^{\mathrm{op}}$, we have

$$\mathrm{Fun}^{\mathrm{ccrt}}_{/\mathbf{Cat}_{\infty}^{\mathrm{op}}}(\mathbf{Cat}_{\infty}^{\mathrm{op}}, \mathrm{CAlg}(\mathbf{PrFib}^{\mathrm{L},\otimes}/\mathbf{Cat}_{\infty}^{\mathrm{op}})) \simeq \mathrm{CAlg}(\mathbf{Pr}^{\mathrm{L}}) \ .$$

Thus, given a presentably symmetric monoidal ∞-category $\mathcal{E}^{\otimes} \in \mathrm{CAlg}(\mathbf{Pr}^{\mathrm{L}})$, we interpret it as a cocartesian section of $\mathrm{CAlg}(\mathbf{PrFib}^{\mathrm{L},\otimes}/\mathbf{Cat}_{\infty}^{\mathrm{op}})$ and consider induced the $\mathbf{Cat}_{\infty}^{\mathrm{op}}$-lax symmetric monoidal functor

$$\mathbb{E} \otimes - \colon \mathbf{PrFib}^{\mathrm{L},\otimes} \longrightarrow \mathbf{PrFib}^{\mathrm{L},\otimes}$$

obtained in (A.10). Thus, the functor

$$\exp_{\mathbb{E}}^{\otimes} \coloneqq \mathbb{E} \otimes \mathbb{C}\mathrm{o}\mathbb{P}\mathrm{Sh}(-)^{\otimes} \colon \mathbf{CoCart}^{\otimes} \longrightarrow \mathbf{PrFib}^{\mathrm{L},\otimes}$$

has a naturally induced lax symmetric monoidal structure relative to $\mathbf{Cat}_{\infty}^{\mathrm{op}}$. In other words, when $\mathcal{E}$ has a symmetric monoidal structure $\mathcal{E}^{\otimes}$, the exponential functor $\exp_{\mathcal{E}}$ inherits a lax symmetric monoidal structure $\exp_{\mathcal{E}}^{\otimes}$.

3.5. **Sections.** Notice that since $\mathbf{Cat}_{\infty}^{\mathrm{op}}$ has $*$ as initial object, there is a canonical functor

$$\mathbb{T}\mathrm{riv}^{\otimes} \colon \mathbf{Pr}^{\mathrm{L},\otimes} \times \mathbf{Cat}_{\infty}^{\mathrm{op}} \longrightarrow \mathbf{PrFib}^{\mathrm{L},\otimes}$$

which is easily seen to be symmetric monoidal relatively to $\mathbf{Cat}_{\infty}^{\mathrm{op}}$ in the sense of Definition A.3. Concretely, if $\mathcal{X} \in \mathbf{Cat}_{\infty}^{\mathrm{op}}$ is a fixed ∞-category, the fiber

$$\mathbb{T}\mathrm{riv}_{\mathcal{X}}^{\otimes} \colon \mathbf{Pr}^{\mathrm{L},\otimes} \longrightarrow \mathbf{PrFib}_{\mathcal{X}}^{\mathrm{L},\otimes}$$

is the symmetric monoidal functor sending a presentable ∞-category $\mathcal{E}$ to the trivial presentable fibration $\mathcal{E} \times \mathcal{X} \to \mathcal{X}$. Under the straightening / unstraightening equivalence

$$\mathbf{PrFib}_{\mathcal{X}}^{\mathrm{L}} \simeq \mathrm{Fun}(\mathcal{X}, \mathbf{Pr}^{\mathrm{L}})$$

we see that the functor $\mathbb{T}\mathrm{riv}_{\mathcal{X}}$ underlying $\mathbb{T}\mathrm{riv}_{\mathcal{X}}^{\otimes}$ sends $\mathcal{E}$ to the associated constant functor. In particular, $\mathbb{T}\mathrm{riv}_{\mathcal{X}}$ has a right adjoint

$$\Sigma_{\mathcal{X}}^{\mathrm{ccrt}} \colon \mathbf{PrFib}_{\mathcal{X}}^{\mathrm{L}} \longrightarrow \mathbf{Pr}^{\mathrm{L}} ,$$

that sends a presentable fibration $\mathcal{A} \to \mathcal{X}$ to the presentable ∞-category of cocartesian sections

$$\Sigma_{\mathcal{X}}^{\mathrm{ccrt}}(\mathcal{A}) \simeq \mathrm{Fun}_{/\mathcal{X}}^{\mathrm{ccrt}}(\mathcal{X}, \mathcal{A}) .$$

Thus, as immediate consequence of Corollary A.5, we obtain:

Corollary 3.23. *The functor* $\mathbb{T}\mathrm{riv}^{\otimes}$ *has a right adjoint*

$$\Sigma^{\mathrm{ccrt},\otimes} \colon \mathbf{PrFib}^{\mathrm{L},\otimes} \longrightarrow \mathbf{Pr}^{\mathrm{L},\otimes} \times \mathbf{Cat}_{\infty}^{\mathrm{op}}$$

relative to $\mathbf{Cat}_{\infty}^{\mathrm{op}}$*. Furthermore,* $\Sigma^{\mathrm{ccrt},\otimes}$ *is lax symmetric monoidal relative to* $\mathbf{Cat}_{\infty}^{\mathrm{op}}$*.*

Variant 3.24. Starting with the cocartesian fibration $\mathbb{B} \colon \mathbf{Cat}_{\infty}^{[1]\star} \to \mathbf{Cat}_{\infty}^{\mathrm{op}}$ and repeating the above reasoning, we find a symmetric monoidal functor

$$\mathbb{T}\mathrm{riv}^{\otimes} \colon \mathbf{Cat}_{\infty}^{\times} \times \mathbf{Cat}_{\infty}^{\mathrm{op}} \longrightarrow \mathbf{Cat}_{\infty}^{[1]\star,\times}$$

relative to $\mathbf{Cat}_{\infty}^{\mathrm{op}}$. Here we consider $\mathbf{Cat}_{\infty}^{[1]\star}$ equipped with the relative cartesian symmetric monoidal structure (that can for instance be obtained via Proposition A.7 and [6, §2.4.1]). We also obtain a right adjoint

$$\Sigma^{\otimes} \colon \mathbf{Cat}_{\infty}^{[1]\star,\times} \longrightarrow \mathbf{Cat}_{\infty}^{\times} \times \mathbf{Cat}_{\infty}^{\mathrm{op}} .$$

Unraveling the definitions, we see that this functor sends a functor $\mathcal{A} \to \mathcal{X}$ to the ∞-category of sections

$$\Sigma(\mathcal{A} \to \mathcal{X}) \simeq \mathrm{Fun}_{/\mathcal{X}}(\mathcal{X}, \mathcal{A}) ,$$

with the obvious functoriality with respect to morphisms of the form (3.3).

Warning 3.25. The functors $\Sigma^{\mathrm{ccrt},\otimes}$ and $\Sigma^{\otimes}$ do not preserve cocartesian edges. In other words, given a pullback square

$$\begin{array}{ccc} \mathcal{B} & \xleftarrow{u} & \mathcal{B}_{\mathcal{X}} \\ \downarrow & & \downarrow \\ \mathcal{Y} & \xleftarrow{f} & \mathcal{X} \end{array}$$

seen as a morphism from $\mathcal{B} \to \mathcal{Y}$ to $\mathcal{B}_{\mathcal{X}} \to \mathcal{X}$ in $\mathbf{Cat}_\infty^{[1]\star}$, the induced map

$$\mathrm{Fun}_{/\mathcal{Y}}(\mathcal{Y}, \mathcal{B}) \longrightarrow \mathrm{Fun}_{/\mathcal{X}}(\mathcal{X}, \mathcal{B}_{\mathcal{X}})$$

is not necessarily an equivalence.

Remark 3.26. The functor $\Sigma^{\mathrm{ccrt},\otimes}$ induces a well defined functor

$$\Sigma^{\mathrm{ccrt}} \colon \mathrm{CAlg}(\mathbf{PrFib}^{\mathrm{L},\otimes}/\mathbf{Cat}_\infty^{\mathrm{op}}) \longrightarrow \mathrm{CAlg}(\mathbf{Pr}^{\mathrm{L},\otimes} \times \mathbf{Cat}_\infty^{\mathrm{op}}/\mathbf{Cat}_\infty^{\mathrm{op}}) \simeq \mathrm{CAlg}(\mathbf{Pr}^{\mathrm{L},\otimes}) \times \mathbf{Cat}_\infty^{\mathrm{op}} .$$

Composing with the projection, we obtain a canonical functor

$$\mathrm{CAlg}(\mathbf{PrFib}^{\mathrm{L},\otimes}/\mathbf{Cat}_\infty^{\mathrm{op}}) \longrightarrow \mathrm{CAlg}(\mathbf{Pr}^{\mathrm{L},\otimes}) .$$

We will commit a slight abuse of notation and still denote by Σ^{ccrt} this last functor.

4. Day's convolution product

We now exploit the general constructions of the previous section to define a relative version of Day's convolution product. We begin with an explicit computation of the sections of the exponential construction of Variant 3.22, which allows to obtain the most basic version of the relative Day's convolution. After explaining in which sense this generalizes the usual construction, we turn to the more exotic case where the base is a pro-∞-category.

4.1. Sections of the exponential construction.

Before getting to Day's convolution product and its generalizations, we still need one last preliminary discussion. Fix an ∞-category $\mathcal{X}$ and let $\mathcal{E} \in \mathrm{CAlg}(\mathbf{Pr}^{\mathrm{L}})$ be a presentably symmetric monoidal ∞-category. In Variant 3.22, we obtained a functor

$$\exp_{\mathcal{E},\mathcal{X}} \colon \mathbf{CoCart}_{\mathcal{X}} \longrightarrow \mathbf{PrFib}^{\mathrm{L}}_{\mathcal{X}} .$$

Concretely, this takes a cocartesian fibration $\mathcal{A} \to \mathcal{X}$ to the presentable (cocartesian) fibration $\exp_{\mathcal{E},\mathcal{X}}(\mathcal{A}) \to \mathcal{X}$ classifying the functor

$$\mathrm{Fun}_!(\Phi_{\mathcal{A}}(-), \mathcal{E}) \colon \mathcal{X} \longrightarrow \mathbf{Pr}^{\mathrm{L}} ,$$

where $\Phi_{\mathcal{A}}$ denotes the straightening of $\mathcal{A}$. On the other hand, we also observed the existence of a functor

$$\Sigma_{\mathcal{X}} \colon \mathbf{PrFib}^{\mathrm{L}}_{\mathcal{X}} \longrightarrow (\mathbf{Cat}_\infty)_{/\mathcal{X}} \longrightarrow \mathbf{Cat}_\infty$$

taking a presentable (cocartesian) fibration over $\mathcal{X}$ to its ∞-category of sections. We now have:

Proposition 4.1. *There is a canonical equivalence*

$$\mathrm{Fun}(\mathcal{A}, \mathcal{E}) \simeq \Sigma_{\mathcal{X}}(\exp_{\mathcal{E},\mathcal{X}}(\mathcal{A})) \simeq \mathrm{Fun}_{/\mathcal{X}}(\mathcal{X}, \exp_{\mathcal{E},\mathcal{X}}(\mathcal{A})) .$$

Proof. Using the equivalence $\mathbf{Pr}^{\mathrm{L}} \simeq (\mathbf{Pr}^{\mathrm{R}})^{\mathrm{op}}$, we see that the presentable fibration $\exp_{\mathcal{E},\mathcal{X}}(\mathcal{A}) \to \mathcal{X}$ is at the same time a cocartesian and a cartesian fibration. Seen as a cartesian fibration, it classifies the functor

$$\mathrm{Fun}(\Phi_{\mathcal{A}}(-), \mathcal{E}) \colon \mathcal{X}^{\mathrm{op}} \longrightarrow \mathbf{Pr}^{\mathrm{R}} .$$

We use this second description to compute the sections of $\exp_{\mathcal{E},\mathcal{X}}(\mathcal{A})$. Invoking [2, Proposition 7.1], we find the following chain of natural equivalences:

$$\begin{aligned}\mathrm{Fun}_{/\mathcal{X}}(\mathcal{X},\exp_{\mathcal{E},\mathcal{X}}(\mathcal{A})) &\simeq \lim_{\mathrm{Tw}(\mathcal{X})^{\mathrm{op}}} \mathrm{Fun}(\mathcal{X}_{-/},\mathrm{Fun}(\Phi_{\mathcal{A}}(-),\mathcal{E}))\\ &\simeq \lim_{\mathrm{Tw}(\mathcal{X})^{\mathrm{op}}} \mathrm{Fun}(\Phi_{\mathcal{A}}(-)\times\mathcal{X}_{-/}\times,\mathcal{E})\\ &\simeq \mathrm{Fun}\big(\operatorname*{colim}_{\mathrm{Tw}(\mathcal{X})} \Phi_{\mathcal{A}}(-)\times\mathcal{X}_{-/},\mathcal{E}\big)\\ &\simeq \mathrm{Fun}(\mathcal{A},\mathcal{E})\ .\end{aligned}$$

Here $\mathrm{Tw}(\mathcal{X})$ denotes the ∞-category of twisting arrows of $\mathcal{X}$ (see [6, §5.2.1]), and the last equivalence is consequence of [2, Theorem 7.4]. This proves the first half of the statement. □

Notation 4.2. We write $\mathrm{Fun}^{\mathrm{ccrt}}(\mathcal{A},\mathcal{E})$ the full subcategory of $\mathrm{Fun}(\mathcal{A},\mathcal{E})$ spanned by those functors that are sent to cocartesian sections of $\exp_{\mathcal{E},\mathcal{X}}(\mathcal{A})$ under the equivalence of Proposition 4.1, and we refer to functors in $\mathrm{Fun}^{\mathrm{ccrt}}(\mathcal{A},\mathcal{E})$ as *cocartesian functors.*

Remark 4.3. Cocartesian functors play a prominent role in the companion paper [**?**], where they are thoroughly studied and characterized (see §8 in *loc. cit.*). They should not be confused with functors $F\colon\mathcal{A}\to\mathcal{E}$ that take cocartesian edges in $\mathcal{A}$ to equivalences in $\mathcal{E}$. See Warning 8.2.5 in *loc. cit.*

4.2. **Recovering the classical construction.** Fix an ∞-category $\mathcal{X}$ and a presentably symmetric monoidal ∞-category $\mathcal{E}\in\mathrm{CAlg}(\mathbf{Pr}^{\mathrm{L}})$. The constructions of Variant 3.22 and of Corollary 3.23 show that the functors

$$\exp_{\mathcal{E},\mathcal{X}}\colon \mathbf{CoCart}_{\mathcal{X}}\longrightarrow\mathbf{PrFib}^{\mathrm{L}}_{\mathcal{X}} \qquad\text{and}\qquad \Sigma_{\mathcal{X}}\colon\mathbf{PrFib}^{\mathrm{L}}_{\mathcal{X}}\longrightarrow\mathbf{Pr}^{\mathrm{L}}$$

admit natural lax symmetric monoidal enhancements

$$\exp^{\otimes}_{\mathcal{E},\mathcal{X}}\colon \mathbf{CoCart}^{\times}_{\mathcal{X}}\longrightarrow\mathbf{PrFib}^{\mathrm{L},\otimes}_{\mathcal{X}} \qquad\text{and}\qquad \Sigma^{\otimes}_{\mathcal{X}}\colon\mathbf{PrFib}^{\mathrm{L}}_{\mathcal{X}}\longrightarrow\mathbf{Pr}^{\mathrm{L},\otimes}\ .$$

Therefore, starting with an object

$$\mathcal{A}^{\otimes}\in\mathrm{CAlg}(\mathbf{CoCart}^{\times}_{\mathcal{X}})\ ,$$

we canonically obtain an presentably symmetric monoidal ∞-category

$$\Sigma^{\otimes}_{\mathcal{X}}(\exp^{\otimes}_{\mathcal{E},\mathcal{X}}(\mathcal{A}^{\otimes}))\in\mathrm{CAlg}(\mathbf{Pr}^{\mathrm{L},\otimes})\ ,$$

whose underlying presentable ∞-category is given by

$$\Sigma_{\mathcal{X}}(\exp_{\mathcal{E},\mathcal{X}}(\mathcal{A}))\simeq\mathrm{Fun}_{/\mathcal{X}}(\mathcal{X},\exp_{\mathcal{E},\mathcal{X}}(\mathcal{A}))\ ,$$

which Proposition 4.1 allows to further identify with $\mathrm{Fun}(\mathcal{A},\mathcal{E})$. Besides, replacing $\Sigma^{\otimes}_{\mathcal{X}}$ with $\Sigma^{\mathrm{ccrt},\otimes}_{\mathcal{X}}$, we deduce that the full subcategory $\mathrm{Fun}^{\mathrm{ccrt}}(\mathcal{A},\mathcal{E})$ introduced in Notation 4.2 inherits a symmetric monoidal structure in such a way that the functor

$$\mathrm{Fun}^{\mathrm{ccrt}}(\mathcal{A},\mathcal{E})\hookrightarrow\mathrm{Fun}(\mathcal{A},\mathcal{E})$$

is symmetric monoidal. Summarizing this discussion, we find:

Corollary 4.4. *Let* $\mathcal{A}^{\otimes}\in\mathrm{CAlg}(\mathbf{CoCart}^{\times}_{\mathcal{X}})$*. Then both* $\mathrm{Fun}(\mathcal{A},\mathcal{E})$ *and* $\mathrm{Fun}^{\mathrm{ccrt}}(\mathcal{A},\mathcal{E})$ *have canonical symmetric monoidal structures. Furthermore,* $\mathrm{Fun}^{\mathrm{ccrt}}(\mathcal{A},\mathcal{E})$ *is closed under tensor product in* $\mathrm{Fun}(\mathcal{A},\mathcal{E})$*.*

This recovers [4, §4.2.C]. Furthermore, when $\mathcal{X} = *$, this recovers the usual construction of Day's convolution product. Indeed, in this case the functor $\Sigma_{\mathcal{X}}$ is an equivalence, while $\exp^{\otimes}_{\mathcal{E},\mathcal{X}}$ is the lax symmetric monoidal functor

$$\mathrm{Fun}_!(-,\mathcal{E})\colon \mathbf{Cat}_\infty \longrightarrow \mathbf{Pr}^{\mathrm{L}} ,$$

(whose lax symmetric monoidal structure can be directly deduced from [6, Proposition 4.8.1.10]). In particular, when $\mathcal{A}^\otimes$ is a symmetric monoidal ∞-category, $\mathrm{Fun}(\mathcal{A},\mathcal{E})$ inherits a canonical symmetric monoidal structure, which is given by the classical *Day's convolution product*, as already observed in [6, Remark 4.8.1.13]. Let us make the connection more explicit:

Remark 4.5. For every ∞-category $\mathcal{C}$, write $y_{\mathcal{C}}\colon \mathcal{C} \longrightarrow \mathrm{PSh}(\mathcal{C})$ for the Yoneda embedding.

(1) Let $\mathcal{A}$ and $\mathcal{B}$ two ∞-categories. The functor

$$\mathcal{A}\times\mathcal{B} \longrightarrow \mathrm{PSh}(\mathcal{A})\times\mathrm{PSh}(\mathcal{B}) \longrightarrow \mathrm{PSh}(\mathcal{A})\otimes\mathrm{PSh}(\mathcal{B})$$

induces a cocontinuous functor

$$\mathrm{PSh}(\mathcal{A}\times\mathcal{B}) \longrightarrow \mathrm{PSh}(\mathcal{A})\otimes\mathrm{PSh}(\mathcal{B}) ,$$

which is the equivalence supplied by the symmetric monoidal structure on $\mathrm{PSh}(-)$ (as it follows from the proof of [6, Proposition 4.8.1.3] – see also [5, Proposition 5.3.6.11]). On the other hand, for a pair $(c,d)\in\mathcal{A}\times\mathcal{B}$ one has a canonical equivalence

$$y_{\mathcal{A}\times\mathcal{B}}(c,d) \simeq y_{\mathcal{A}}(c)\times y_{\mathcal{B}}(d) .$$

Under the above equivalence, the presheaf $y_{\mathcal{A}\times\mathcal{B}}(c,d)$ is sent to the elementary tensor $y_{\mathcal{A}}(c)\otimes y_{\mathcal{B}}(d)$. It follows that under the above equivalence, an elementary tensor $F\otimes G \in \mathrm{PSh}(\mathcal{A})\otimes\mathrm{PSh}(\mathcal{B})$ is sent to the presheaf informally defined by

$$(c,d)\mapsto F(c)\times F(d) .$$

(2) Let $\mathcal{A}^\otimes$ be a symmetric monoidal ∞-category, and let $\otimes\colon \mathcal{A}\times\mathcal{A}\to\mathcal{A}$ be the underlying tensor product of $\mathcal{A}$. Unraveling the definitions, we see that the underlying tensor product on $\mathrm{PSh}(\mathcal{A})$ can explicitly be computed as follows:

$$\circledast\colon\ \mathrm{Fun}(\mathcal{A}^{\mathrm{op}},\mathcal{S})\otimes\mathrm{Fun}(\mathcal{A}^{\mathrm{op}},\mathcal{S}) \simeq \mathrm{Fun}(\mathcal{A}^{\mathrm{op}}\times\mathcal{A}^{\mathrm{op}},\mathcal{S}) \xrightarrow{\otimes_!} \mathrm{Fun}(\mathcal{A}^{\mathrm{op}},\mathcal{S}) ,$$

where $\otimes_!$ denotes the left Kan extension along $\otimes$ and where the equivalence is the one discussed in point (1) of this remark. It follows that the tensor product of two presheaves F and G can be explicitly computed by the formula

$$F\circledast G(X) \simeq \operatorname*{colim}_{X\to X_1\otimes X_2} F(X_1)\times G(X_2) .$$

In other words, it coincides with Day's convolution product.

4.3. **Extension to pro-∞-categories.** We now generalize Corollary 4.4 to the case where the base $\mathcal{X}$ is a pro-∞-category. We refer to Section 5 for a discussion of a common example arising from stratified topological spaces.

Fix a presentably symmetric monoidal ∞-category $\mathcal{E}\in\mathrm{CAlg}(\mathbf{Pr}^{\mathrm{L},\otimes})$, and consider it as a cocartesian section of

$$\mathrm{CAlg}(\mathbf{PrFib}^{\mathrm{L},\otimes}/\mathbf{Cat}_\infty^{\mathrm{op}}) \longrightarrow \mathbf{Cat}_\infty^{\mathrm{op}} .$$

Consider the following commutative diagram:

$$\begin{array}{ccccc} \mathbf{CoCart}^\otimes & \xrightarrow{\exp^\otimes_{\mathcal{E}}} & \mathbf{PrFib}^{\mathrm{L},\otimes} & \xleftarrow{\mathbb{T}\mathrm{riv}^\otimes} & \mathbf{Pr}^{\mathrm{L},\otimes}\times\mathbf{Cat}_\infty^{\mathrm{op}} \\ & \searrow & \downarrow & \swarrow & \\ & & \mathrm{Fin}_*\times\mathbf{Cat}_\infty^{\mathrm{op}} & & \end{array} .$$

The vertical arrows are $\mathbf{Cat}_\infty^{\mathrm{op}}$-families of symmetric monoidal ∞-categories in the sense of Definition A.1. In particular, they are cocartesian fibrations. The functor $\mathbb{T}\mathrm{riv}^\otimes$ is symmetric monoidal relative to $\mathbf{Cat}_\infty^{\mathrm{op}}$, while $\exp_{\mathcal{E}}^\otimes$ is a $\mathbf{Cat}_\infty^{\mathrm{op}}$-lax symmetric monoidal functor. Via the straightening equivalence of Proposition A.7, we see that $\mathbf{CoCart}^\otimes$, $\mathbf{PrFib}^{\mathrm{L},\otimes}$ and $\mathbf{Pr}^{\mathrm{L},\otimes} \times \mathbf{Cat}_\infty^{\mathrm{op}}$ classify functors

$$\mathrm{A}, \mathrm{B}, \mathrm{C} \colon \mathbf{Cat}_\infty^{\mathrm{op}} \longrightarrow \mathrm{SMCat}_\infty^{\mathrm{lax}}$$

with values in the ∞-category of symmetric monoidal ∞-categories and lax monoidal functors between them. Concretely, A is the functor sending $\mathcal{X} \in \mathbf{Cat}_\infty^{\mathrm{op}}$ to the symmetric monoidal ∞-category $\mathbf{CoCart}_{\mathcal{X}}^\otimes$, while B sends $\mathcal{X}$ to $\mathbf{PrFib}_{\mathcal{X}}^{\mathrm{L},\otimes}$ and C is the constant functor associated to $\mathbf{Pr}^{\mathrm{L},\otimes}$. Using Proposition A.7 again, we see that the functors $\exp_{\mathcal{E}}^\otimes$ and $\mathbb{T}\mathrm{riv}^\otimes$ give rise to natural transformations

$$\alpha_{\mathcal{E}} \colon \mathrm{A} \longrightarrow \mathrm{B} \qquad \text{and} \qquad \tau \colon \mathrm{C} \longrightarrow \mathrm{B} .$$

Consider now the canonical embedding

$$\mathbf{Cat}_\infty^{\mathrm{op}} \hookrightarrow \mathrm{Pro}(\mathbf{Cat}_\infty)^{\mathrm{op}} .$$

We extend the functors A, B and C by left Kan extension along this embedding. This yields three functors

$$\widetilde{\mathrm{A}}, \widetilde{\mathrm{B}}, \widetilde{\mathrm{C}} \colon \mathrm{Pro}(\mathbf{Cat}_\infty)^{\mathrm{op}} \longrightarrow \mathrm{SMCat}_\infty^{\mathrm{lax}} ,$$

and two natural transformations

$$\widetilde{\alpha}_{\mathcal{E}} \colon \widetilde{\mathrm{A}} \longrightarrow \widetilde{\mathrm{B}} \qquad \text{and} \qquad \widetilde{\tau} \colon \widetilde{\mathrm{C}} \longrightarrow \widetilde{\mathrm{B}} .$$

Notice that the functor $\widetilde{\mathrm{C}}$ is once again the constant functor associated to $\mathbf{Pr}^{\mathrm{L},\otimes}$. Thus, applying once more Proposition A.7, we obtain a commutative diagram

$$\begin{array}{ccccc} \mathbf{proCoCart}^\otimes & \xrightarrow{\mathrm{proexp}_{\mathcal{E}}^\otimes} & \mathbf{proPrFib}^{\mathrm{L},\otimes} & \xleftarrow{\mathrm{pro}\mathbb{T}\mathrm{riv}^\otimes} & \mathbf{Pr}^{\mathrm{L},\otimes} \times \mathrm{Pro}(\mathbf{Cat}_\infty)^{\mathrm{op}} \\ & \searrow & \downarrow & \swarrow & \\ & & \mathrm{Fin}_* \times \mathrm{Pro}(\mathbf{Cat}_\infty)^{\mathrm{op}} . & & \end{array} \tag{4.6}$$

Remark 4.7. Fix $\mathfrak{X} \in \mathrm{Pro}(\mathbf{Cat}_\infty)$ and choose a presentation

$$\mathfrak{X} \simeq \text{“}\lim_{i \in I}\text{”}\, \mathcal{X}_i .$$

Unraveling the definitions, we find canonical equivalences

$$\mathbf{proCoCart}_{\mathfrak{X}}^\otimes \simeq \operatorname*{colim}_{i \in I^{\mathrm{op}}} \mathbf{CoCart}_{\mathcal{X}_i}^\otimes \qquad \text{and} \qquad \mathbf{proPrFib}_{\mathfrak{X}}^{\mathrm{L},\otimes} \simeq \operatorname*{colim}_{i \in I^{\mathrm{op}}} \mathbf{PrFib}_{\mathcal{X}_i}^{\mathrm{L},\otimes} ,$$

where both colimits are computed in $\mathbf{Cat}_\infty$. Objects in $\mathbf{proCoCart}_{\mathfrak{X}}$ can be represented as pairs $(i, \mathcal{A})$, where $i \in I$ and $\mathcal{A} \to \mathcal{X}_i$ is a cocartesian fibration. Notice that morphisms from $(i, \mathcal{A})$ to $(j, \mathcal{B})$ are given by a common refinement $k \to i$ and $k \to j$ together with a morphism

$$\mathcal{A} \times_{\mathcal{X}_i} \mathcal{X}_k \longrightarrow \mathcal{B} \times_{\mathcal{X}_j} \mathcal{X}_k$$

of cocartesian fibrations over $\mathcal{X}_k$. In particular, two objects $(i, \mathcal{A})$ and $(j, \mathcal{B})$ are equivalent if they become equivalent after base change to a (suitably large) common refinement of both i and j. For this reason, we denote objects in $\mathbf{proCoCart}_{\mathcal{X}}$ by $[(i, \mathcal{A})]$, and we obtain the formula

$$\mathrm{Map}_{\mathbf{proCoCart}_{\mathfrak{X}}}\big([(i,\mathcal{A})], [(j,\mathcal{B})]\big) \simeq \operatorname*{colim}_{h \in (I_{/k})^{\mathrm{op}}} \mathrm{Map}_{\mathbf{CoCart}_{\mathcal{X}_h}}(\mathcal{A} \times_{\mathcal{X}_i} \mathcal{X}_h, \mathcal{B} \times_{\mathcal{X}_j} \mathcal{X}_h) , \tag{4.8}$$

where k is any common refinement of i and j in I. Similar conventions apply to $\mathbf{proPrFib}_{\mathfrak{X}}^{\mathrm{L}}$.

Example 4.9. Keep the notation of Remark 4.7 and let $\mathcal{E} \in \mathbf{Pr}^{\mathrm{L}}$ be a presentable ∞-category. Then $\mathrm{pro}\mathbb{T}\mathrm{riv}_{\mathfrak{X}}(\mathcal{E}) \simeq [(i, \mathbb{T}\mathrm{riv}_{\mathcal{X}_i}(\mathcal{E}))] = [(i, \mathcal{E} \times \mathcal{X}_i)]$. Notice that for every $j \to i$ in I, one has

$$[(i, \mathcal{E} \times \mathcal{X}_i)] \simeq [(j, \mathcal{E} \times \mathcal{X}_j)] \ .$$

Remark 4.10. The tensor structure in both $\mathbf{proCoCart}_{\mathfrak{X}}$ and $\mathbf{proPrFib}^{\mathrm{L}}_{\mathfrak{X}}$ is informally defined as

$$[(i, \mathcal{A})] \otimes_{\mathfrak{X}} [(j, \mathcal{B})] \coloneqq \big[\big(k, (\mathcal{A} \times_{\mathcal{X}_i} \mathcal{X}_k) \otimes_{\mathcal{X}_k} (\mathcal{B} \times_{\mathcal{X}_j} \mathcal{X}_k)\big)\big] \ ,$$

where k is any object equipped with maps $k \to i$ and $k \to j$ (with the convention that in the case of $\mathbf{proCoCart}_{\mathfrak{X}}$ we write $\otimes_{\mathcal{X}_k}$ for the usual fiber product over $\mathcal{X}_k$).

Warning 4.11. The tensor structure is similarly defined; in particular to tensor two objects $(i, \mathcal{A})$ and $(j, \mathcal{B})$, one first finds a common index k mapping to both i and j, pulls back both $\mathcal{A}$ and $\mathcal{B}$ to $\mathcal{X}_k$, and finally tensors the results inside $\mathbf{CoCart}_{\mathcal{X}_k}$. In particular, we obtain a canonical map

$$\operatorname*{colim}_{i \in I^{\mathrm{op}}} \mathrm{CAlg}(\mathbf{CoCart}^{\otimes}_{\mathcal{X}_i}) \longrightarrow \mathrm{CAlg}(\mathbf{proCoCart}_{/\mathfrak{X}}) \ ,$$

which is *not* an equivalence.

Observe now that, by construction, the functor $\mathrm{pro}\mathbb{T}\mathrm{riv}^{\otimes}$ is symmetric monoidal relative to $\mathrm{Pro}(\mathbf{Cat}_\infty)^{\mathrm{op}}$. In order to mimic the construction of Day's convolution in the classical setting, we would like to know that $\mathrm{pro}\mathbb{T}\mathrm{riv}^{\otimes}$ admits a (lax monoidal) right adjoint relative to $\mathrm{Pro}(\mathbf{Cat}_\infty)^{\mathrm{op}}$. Invoking Corollary A.5, all we need to check is to verify that for a fixed $\mathfrak{X} \in \mathrm{Pro}(\mathbf{Cat}_\infty)^{\mathrm{op}}$, the functor

$$\mathrm{pro}\mathbb{T}\mathrm{riv}_{\mathfrak{X}} \colon \mathbf{Pr}^{\mathrm{L}} \longrightarrow \mathbf{proPrFib}^{\mathrm{L}}_{\mathfrak{X}}$$

has a right adjoint. Unfortunately, this turns out to be false. On the other hand, we have:

Lemma 4.12. *The functor* $\mathrm{pro}\mathbb{T}\mathrm{riv}_{\mathfrak{X}}$ *commutes with finite colimits. In particular, it admits an ind-right adjoint.*

Proof. Let $J \to \mathbf{Pr}^{\mathrm{L}}$ be a finite diagram, written $j \mapsto \mathcal{E}_j$. Set

$$\mathcal{E} \coloneqq \operatorname*{colim}_{j \in J} \mathcal{E}_j \ .$$

Fix a presentation $\mathfrak{X} \simeq$ "lim"$_{i \in I} \mathcal{X}_i$ and let $(i, \mathcal{A}) \in \mathbf{proPrFib}^{\mathrm{L}}_{\mathfrak{X}}$. By Remark 4.7, we find:

$$\begin{aligned} \mathrm{Map}_{\mathbf{proPrFib}^{\mathrm{L}}_{\mathfrak{X}}}(\mathrm{pro}\mathbb{T}\mathrm{riv}_{\mathfrak{X}}(\mathcal{E}), (i, \mathcal{A})) &\simeq \operatorname*{colim}_{k \in (I_{/i})^{\mathrm{op}}} \mathrm{Map}_{\mathbf{PrFib}^{\mathrm{L}}_{\mathcal{X}_i}}(\mathcal{E} \times \mathcal{X}_j, \mathcal{A} \times_{\mathcal{X}_i} \mathcal{X}_j) \\ &\simeq \operatorname*{colim}_{k \in (I_{/i})^{\mathrm{op}}} \lim_{j \in J} \mathrm{Map}_{\mathbf{PrFib}^{\mathrm{L}}_{\mathcal{X}_i}}(\mathcal{E}_i \times \mathcal{X}_j, \mathcal{A} \times_{\mathcal{X}_i} \mathcal{X}_j) \\ &\simeq \lim_{j \in J} \operatorname*{colim}_{k \in (I_{/i})^{\mathrm{op}}} \mathrm{Map}_{\mathbf{PrFib}^{\mathrm{L}}_{\mathcal{X}_i}}(\mathcal{E}_i \times \mathcal{X}_j, \mathcal{A} \times_{\mathcal{X}_i} \mathcal{X}_j) \\ &\simeq \lim_{j \in J} \mathrm{Map}_{\mathbf{proPrFib}^{\mathrm{L}}_{\mathfrak{X}}}(\mathrm{pro}\mathbb{T}\mathrm{riv}_{\mathfrak{X}}(\mathcal{E}_i), (i, \mathcal{A})) \ , \end{aligned}$$

whence the conclusion. □

Let us denote

$$\Sigma^{\mathrm{ccrt}}_{\mathfrak{X}} \colon \mathrm{Ind}(\mathbf{proPrFib}^{\mathrm{L}}_{\mathfrak{X}}) \longrightarrow \mathrm{Ind}(\mathbf{Pr}^{\mathrm{L}})$$

the ind-right adjoint to $\mathrm{pro}\mathbb{T}\mathrm{riv}_{\mathfrak{X}}$ supplied by the above lemma. We have:

Lemma 4.13. *Fix a presentation* $\mathfrak{X} \simeq$ "lim"$_{i \in I} \mathcal{X}_i$ *for* $\mathfrak{X}$. *For an object* $(i, \mathcal{A}) \in \mathbf{proPrFib}^{\mathrm{L}}_{\mathfrak{X}} \simeq \mathrm{colim}_{k \in I^{\mathrm{op}}} \mathbf{PrFib}^{\mathrm{L}}_{\mathcal{X}_k}$ *there is a canonical equivalence*

$$\Sigma^{\mathrm{ccrt}}_{\mathfrak{X}}(i, \mathcal{A}) \simeq \text{“}\operatorname*{colim}_{k \in (I_{/i})^{\mathrm{op}}}\text{”} \mathrm{Fun}^{\mathrm{ccrt}}_{/\mathcal{X}_k}(\mathcal{X}_k, \mathcal{A} \times_{\mathcal{X}_i} \mathcal{X}_k)$$

in $\mathrm{Ind}(\mathbf{Pr}^{\mathrm{L}})$.

Proof. Fix $\mathcal{E} \in \mathbf{Pr}^{\mathrm{L}}$. By Remark 4.7 we find:

$$\begin{aligned}\mathrm{Map}_{\mathrm{Ind}(\mathbf{Pr}^{\mathrm{L}})}\big(\mathcal{E}, \text{“}\operatorname*{colim}_{k\in(I_{/i})^{\mathrm{op}}}\text{”}\,\mathrm{Fun}^{\mathrm{ccrt}}_{/\mathcal{X}_k}(\mathcal{X}_k, \mathcal{A}\times_{\mathcal{X}_i}\mathcal{X}_k)\big) &\simeq \operatorname*{colim}_{k\in(I_{/i})^{\mathrm{op}}} \mathrm{Map}_{\mathbf{Pr}^{\mathrm{L}}}\big(\mathcal{E}, \mathrm{Fun}^{\mathrm{ccrt}}_{/\mathcal{X}_k}(\mathcal{X}_k, \mathcal{A}\times_{\mathcal{X}_i}\mathcal{X}_k)\big)\\ &\simeq \operatorname*{colim}_{k\in(I_{/i})^{\mathrm{op}}} \mathrm{Map}_{\mathbf{CoCart}_{\mathcal{X}_k}}\big(\mathbb{T}\mathrm{riv}_{\mathcal{X}_k}(\mathcal{E}), \mathcal{A}\times_{\mathcal{X}_i}\mathcal{X}_k\big)\\ &\simeq \mathrm{Map}_{\mathbf{proCoCart}_{\mathfrak{X}}}\big(\mathrm{pro}\mathbb{T}\mathrm{riv}_{\mathfrak{X}}(\mathcal{E}), (i,\mathcal{A})\big)\ ,\end{aligned}$$

so the conclusion follows from the Yoneda lemma. □

This suggests to further upgrade (4.6) as follows. Consider the induced diagrams

$$\mathrm{Ind}(\widetilde{\mathrm{A}}), \mathrm{Ind}(\widetilde{\mathrm{B}}), \mathrm{Ind}(\widetilde{\mathrm{C}}) \colon \mathrm{Pro}(\mathbf{Cat}_\infty)^{\mathrm{op}} \longrightarrow \mathrm{SMCat}_\infty^{\mathrm{lax}}\ ,$$

obtained composing the functors $\widetilde{\mathrm{A}}$, $\widetilde{\mathrm{B}}$ and $\widetilde{\mathrm{C}}$ with the symmetric monoidal functor Ind. Applying once more Proposition A.7, we obtain an analogue of (4.6) which we denote as follows:
(4.14)

$$\begin{array}{ccccc} \mathbf{indproCoCart}^{\otimes} & \xrightarrow{\mathrm{indproexp}^{\otimes}_{\mathcal{E}}} & \mathbf{indproPrFib}^{\mathrm{L},\otimes} & \xleftarrow{\mathrm{indpro}\mathbb{T}\mathrm{riv}^{\otimes}} & \mathrm{Ind}(\mathbf{Pr}^{\mathrm{L}})^{\otimes}\times\mathrm{Pro}(\mathbf{Cat}_\infty)^{\mathrm{op}} \\ & \searrow & \downarrow & \swarrow & \\ & & \mathrm{Fin}_*\times\mathrm{Pro}(\mathbf{Cat}_\infty)^{\mathrm{op}}\ . & & \end{array}$$

Remark 4.15.

(1) An object $\mathfrak{A} \in \mathbf{indproCoCart}_{\mathfrak{X}}$ can be represented as an ind-system

$$\text{“}\operatorname*{colim}_{j\in J}\text{”}\,[(i_j, \mathcal{A}_j)]\ ,$$

where $i_j \in I$ is an index and $\mathcal{A}_j \to \mathcal{X}_{i_j}$ is a cocartesian fibration. Notice that for $j \to \ell$ in J, the transition morphism

$$[(i_j, \mathcal{A}_j)] \longrightarrow [(i_\ell, \mathcal{A}_\ell)]$$

is understood as in Remark 4.7-(1).

(2) Fix two objects $\mathfrak{A}, \mathfrak{B}$ in either $\mathbf{indproCoCart}_{\mathfrak{X}}$ or $\mathbf{indproPrFib}^{\mathrm{L}}$ with presentations

$$\mathfrak{A} \simeq \text{“}\operatorname*{colim}_{j\in J}\text{”}\,[(i_j, \mathcal{A}_j)] \qquad \text{and} \qquad \mathfrak{B} \simeq \text{“}\operatorname*{colim}_{\ell\in L}\text{”}\,[(i_\ell, \mathcal{B}_\ell)]\ .$$

Their tensor product can be explicitly represented as

$$\mathfrak{A}\otimes_{\mathfrak{X}}\mathfrak{B} \simeq \text{“}\operatorname*{colim}_{(j,\ell)\in J\times L}\text{”}\,[(i_j, \mathcal{A}_i)]\otimes_{\mathfrak{X}}[(i_\ell, \mathcal{B}_\ell)]$$

where the tensor product on the right hand side is understood as in Remark 4.7-(2). Notice however that when $\mathfrak{B} = \mathfrak{A}$, we can also write

$$\mathfrak{A}\otimes_{\mathfrak{X}}\mathfrak{A} \simeq \text{“}\operatorname*{colim}_{j\in J}\text{”}\,[(i_j, \mathcal{A}_j)]\otimes_{\mathfrak{X}}[(i_j, \mathcal{A}_j)] \simeq \text{“}\operatorname*{colim}_{j\in J}\text{”}\,[(i_j, \mathcal{A}_j\otimes_{\mathcal{X}_{i_j}}\mathcal{A}_j)]\ .$$

Indeed, since J is filtered, the diagonal $J \to J\times J$ is cofinal, and therefore the two presentations for $\mathfrak{A}\otimes_{\mathfrak{X}}\mathfrak{A}$ agree.

Combining Lemma 4.12 and Corollary A.5, we deduce that $\mathrm{indpro}\mathbb{T}\mathrm{riv}^{\otimes}$ has a lax symmetric monoidal right adjoint $\Sigma^{\mathrm{ccrt},\otimes}_{\mathrm{indpro}}$.

Variant 4.16. Working with $\mathbf{Cat}_\infty^{[1]\star}$ instead of $\mathbf{PrFib}^{\mathrm{L}}$, one obtain similar results, where the functor $\Sigma^{\mathrm{ccrt},\otimes}_{\mathrm{indpro}}$ is replaced by its non-cocartesian variant $\Sigma^{\otimes}_{\mathrm{indpro}}$.

It follows that the composites $\Sigma^{\otimes}_{\mathrm{indpro}} \circ \mathrm{indproexp}^{\otimes}_{\mathcal{E}}$ and $\Sigma^{\mathrm{ccrt},\otimes}_{\mathrm{indpro}} \circ \mathrm{indproexp}^{\otimes}_{\mathcal{E}}$ induce well defined functors

$$\mathrm{CAlg}(\mathbf{indproCoCart}^{\otimes}/\mathrm{Pro}(\mathbf{Cat}_\infty)^{\mathrm{op}}) \longrightarrow \mathrm{Ind}(\mathbf{Pr}^{\mathrm{L},\otimes}) .$$

Summing up the discussion so far, we obtain the main result of this paper:

Corollary 4.17. *Let* $\mathfrak{X} \in \mathrm{Pro}(\mathbf{Cat}_\infty)$. *Let* $\mathfrak{A}^{\otimes} \in \mathrm{CAlg}(\mathbf{indproCoCart}^{\otimes}_{\mathfrak{X}})$. *Fix compatible presentations*

$$\mathfrak{X} \simeq \text{“}\lim_{i\in I}\text{”}\, \mathcal{X}_i \qquad \text{and} \qquad \mathfrak{A} \simeq \text{“}\operatorname*{colim}_{j \in J}\text{”}\, [(i_j, \mathcal{A}_j)] \ .$$

Then for every presentably symmetric monoidal ∞*-category* $\mathcal{E}^{\otimes} \in \mathrm{CAlg}(\mathbf{Pr}^{\mathrm{L},\otimes})$*, the ind-objects*

$$\text{“}\operatorname*{colim}_{j\in J}\text{”}\ \text{“}\operatorname*{colim}_{k \in (I_{/i_j})^{\mathrm{op}}}\text{”}\ \mathrm{Fun}(\mathcal{A}_j \times_{\mathcal{X}_{i_j}} \mathcal{X}_k, \mathcal{E})$$

and

$$\text{“}\operatorname*{colim}_{j\in J}\text{”}\ \text{“}\operatorname*{colim}_{k \in (I_{/i_j})^{\mathrm{op}}}\text{”}\ \mathrm{Fun}^{\mathrm{ccrt}}(\mathcal{A}_j \times_{\mathcal{X}_{i_j}} \mathcal{X}_k, \mathcal{E})$$

can be canonically lifted to objects in $\mathrm{CAlg}(\mathrm{Ind}(\mathbf{Pr}^{\mathrm{L}})^{\otimes})$*, and the natural morphism*

$$\text{“}\operatorname*{colim}_{j\in J}\text{”}\ \text{“}\operatorname*{colim}_{k \in (I_{/i_j})^{\mathrm{op}}}\text{”}\ \mathrm{Fun}^{\mathrm{ccrt}}(\mathcal{A}_j \times_{\mathcal{X}_{i_j}} \mathcal{X}_k, \mathcal{E}) \longrightarrow \text{“}\operatorname*{colim}_{j\in J}\text{”}\ \text{“}\operatorname*{colim}_{k \in (I_{/i_j})^{\mathrm{op}}}\text{”}\ \mathrm{Fun}(\mathcal{A}_j \times_{\mathcal{X}_{i_j}} \mathcal{X}_k, \mathcal{E})$$

can be canonically lifted to a morphism in $\mathrm{CAlg}(\mathrm{Ind}(\mathbf{Pr}^{\mathrm{L}})^{\otimes})$*. In particular, both*

$$\mathrm{Fun}(\mathfrak{A}, \mathcal{E}) \coloneqq \operatorname*{colim}_{j\in J}\ \operatorname*{colim}_{k \in (I_{/i_j})^{\mathrm{op}}}\ \mathrm{Fun}(\mathcal{A}_j \times_{\mathcal{X}_{i_j}} \mathcal{X}_k, \mathcal{E}) \in \mathbf{Pr}^{\mathrm{L}} \tag{4.18}$$

and

$$\mathrm{Fun}^{\mathrm{ccrt}}(\mathfrak{A}, \mathcal{E}) \coloneqq \operatorname*{colim}_{j\in J}\ \operatorname*{colim}_{k \in (I_{/i_j})^{\mathrm{op}}}\ \mathrm{Fun}^{\mathrm{ccrt}}(\mathcal{A}_j \times_{\mathcal{X}_{i_j}} \mathcal{X}_k, \mathcal{E}) \in \mathbf{Pr}^{\mathrm{L}}$$

have a presentably symmetric monoidal structure, and the natural inclusion

$$\mathrm{Fun}^{\mathrm{ccrt}}(\mathfrak{A}, \mathcal{E}) \hookrightarrow \mathrm{Fun}(\mathfrak{A}, \mathcal{E})$$

is symmetric monoidal. Furthermore these constructions are functorial in the pair $(\mathfrak{X}, \mathfrak{A}^{\otimes})$ *seen as an object of* $\mathrm{CAlg}(\mathbf{indproCoCart}^{\otimes}/\mathrm{Pro}(\mathbf{Cat}_\infty)^{\mathrm{op}})$.

Remark 4.19. Assume that $\mathfrak{X}$ admits a presentation

$$\mathfrak{X} \simeq \text{“}\lim_{i\in I}\text{”}\, \mathcal{X}_i$$

where all the transition maps $\mathcal{X}_i \to \mathcal{X}_j$ are localizations, and in particular they are limit-final (see e.g. [?, Lemma 1.6] or [?, Proposition 7.1.10]). In particular, if $\mathcal{A} \to \mathcal{X}_i$ is a cocartesian fibration, then [5, Proposition 3.3.3.1] and [?, Corollary 8.2.4] show that the induced functor

$$\mathrm{Fun}^{\mathrm{ccrt}}(\mathcal{A}, \mathcal{E}) \longrightarrow \mathrm{Fun}^{\mathrm{ccrt}}(\mathcal{A} \times_{\mathcal{X}_i} \mathcal{X}_j, \mathcal{E})$$

is an equivalence. In particular, under this assumption, the statement of Corollary 4.17 simplifies to say that

$$\operatorname*{colim}_{j\in J} \mathrm{Fun}^{\mathrm{ccrt}}(\mathcal{A}_j, \mathcal{E})$$

has an induced symmetric monoidal structure.

5. Pro-categories arising from stratified spaces

To construct symmetric monoidal ∞-categories via Corollary 4.17 one needs as a starting input an object $\mathfrak{A}^{\otimes} \in \mathrm{CAlg}(\mathbf{indproCoCart}_{\mathfrak{X}})$. At first glance this ∞-category looks so complicated that it is legitimate to wonder whether Corollary 4.17 can actually be of any practical use. The goal of this section is to show that in the situation where the pro-∞-category $\mathfrak{X}$ arises from the study of stratified topological spaces, this ∞-category simplifies quite a bit, making the task of constructing explicit objects inside much more manageable. The key ingredient is the exodromy equivalence proven in [**?**, Theorem 5.4.1].

Let X be a locally weakly contractible topological space. Recall that a stratification on X is the given of a poset P and a continuous map $X \to P$, where P is equipped with the Alexandroff topology, where opens are exactly the closed upwards subsets [**?**, Recollection 2.1.1]. Stratifications assemble into a category $\mathsf{Strat}(X)$. We let $\mathsf{Strat}^{\mathsf{ex}}(X)$ be the full subcategory consisting of *exodromic stratifications*, in the sense of [**?**, Definition 1.5.6], see also Example 1.5.8-(1) in loc. cit. Let

$$\mathcal{P} \colon I \longrightarrow \mathsf{Strat}^{\mathsf{ex}}(X)$$

be a cofiltered diagram, and write $P_i \coloneqq \mathcal{P}(i)$. We set

$$\Pi_\infty(X, \mathcal{P}) \coloneqq \text{“}\lim_{i \in I}\text{”}\, \Pi_\infty(X, P_i) \in \mathrm{Pro}(\mathbf{Cat}_\infty) \ .$$

We also introduce the following:

Definition 5.1. We say that a hypersheaf $F \in \mathrm{Sh}^{\mathrm{hyp}}(X; \mathbf{Cat}_\infty)$ is *$\mathcal{P}$-hyperconstructible* if it is P_i-hyperconstructible for some $i \in I$. We let $\mathrm{Cons}^{\mathrm{hyp}}_{\mathcal{P}}(X; \mathbf{Cat}_\infty)$ denote the full subcategory of $\mathrm{Sh}^{\mathrm{hyp}}(X; \mathbf{Cat}_\infty)$ spanned by $\mathcal{P}$-hyperconstructible sheaves.

Remark 5.2. In other words, we have

$$\mathrm{Cons}^{\mathrm{hyp}}_{\mathcal{P}}(X; \mathbf{Cat}_\infty) \simeq \operatorname*{colim}_{i \in I^{\mathrm{op}}} \mathrm{Cons}^{\mathrm{hyp}}_{P_i}(X; \mathbf{Cat}_\infty),$$

as all the transition functors are fully faithful and compatible with the natural inclusion in $\mathrm{Sh}^{\mathrm{hyp}}(X; \mathbf{Cat}_\infty)$.

As a consequence of the exodromy equivalence [**?**], we find:

Lemma 5.3. *There is a symmetric monoidal equivalence*

$$\mathbf{proCoCart}^{\otimes}_{\Pi_\infty(X,\mathcal{P})} \simeq \mathrm{Cons}^{\mathrm{hyp}}_{\mathcal{P}}(X; \mathbf{Cat}_\infty)^{\times} \ .$$

In particular there is a symmetric monoidal and fully faithful embedding

$$\mathbf{proCoCart}^{\otimes}_{\Pi_\infty(X,\mathcal{P})} \hookrightarrow \mathrm{Sh}^{\mathrm{hyp}}(X; \mathbf{Cat}_\infty)^{\times}$$

Proof. Using Remark 4.7, we can compute

$$\mathbf{proCoCart}^{\otimes}_{\Pi_\infty(X,\mathcal{P})} \simeq \operatorname*{colim}_{i \in I^{\mathrm{op}}} \mathbf{CoCart}^{\otimes}_{\Pi_\infty(X,P_i)} \ .$$

On the other hand, combining the straightening and the exodromy equivalence ([**?**, Theorem 5.4.1]), we obtain the following chain of symmetric monoidal equivalences:

$$\mathbf{CoCart}^{\otimes}_{\Pi_\infty(X,P_i)} \simeq \mathrm{Fun}(\Pi_\infty(X, P_i), \mathbf{Cat}_\infty)^{\times} \simeq \mathrm{Cons}^{\mathrm{hyp}}_{P_i}(X; \mathbf{Cat}_\infty)^{\times} \ ,$$

so the conclusion follows from Remark 5.2. □

Corollary 5.4. *There is a symmetric monoidal equivalence*

$$\mathbf{indproCoCart}^{\otimes}_{\Pi_\infty(X,\mathcal{P})} \simeq \mathrm{Ind}(\mathrm{Cons}^{\mathrm{hyp}}_{\mathcal{P}}(X;\mathbf{Cat}_\infty))^\times \ .$$

In particular, there is an induced symmetric monoidal and fully faithful embedding

$$\mathbf{indproCoCart}^{\otimes}_{\Pi_\infty(X,\mathcal{P})} \hookrightarrow \mathrm{Ind}(\mathrm{Sh}^{\mathrm{hyp}}(X;\mathbf{Cat}_\infty)) \ .$$

Corollary 5.5. *There is a fully faithful embedding*

$$\mathrm{CAlg}\big(\mathbf{indproCoCart}^{\otimes}_{\Pi_\infty(X,\mathcal{P})}\big) \hookrightarrow \mathrm{CAlg}\big(\mathrm{Ind}(\mathrm{Sh}^{\mathrm{hyp}}(X;\mathbf{Cat}_\infty))^\times\big) \ ,$$

and the essential image consists of those $\mathfrak{A}^\otimes \in \mathrm{CAlg}(\mathrm{Ind}(\mathrm{Sh}^{\mathrm{hyp}}(X;\mathbf{Cat}_\infty))^\times)$ *whose underlying ind-object* $\mathfrak{A} \in \mathrm{Ind}(\mathrm{Sh}^{\mathrm{hyp}}(X;\mathbf{Cat}_\infty))$ *admits a presentation*

$$\mathfrak{A} \simeq \text{“}\operatorname*{colim}_{j\in J}\text{”}\, \mathcal{A}_j \ ,$$

where each $\mathcal{A}_j$ *is a* $\mathcal{P}$*-hyperconstructible hypersheaf with values in* $\mathbf{Cat}_\infty$*.*

Appendix A. Relative symmetric monoidal structures

In [6, §2.3.2] Lurie introduced the notion of family of ∞-operads (see more precisely Definition 2.3.2.10 in *loc. cit.*). In this note, we will need the milder notion of family of symmetric monoidal ∞-categories.

A.1. **Several equivalent definitions.** Fix an ∞-category $\mathcal{C}$. Write $\mathbf{CoCart}_{\mathcal{C}}$ for the (non full) subcategory of $(\mathbf{Cat}_\infty)_{/\mathcal{C}}$ spanned by cocartesian fibrations, and whose morphisms are morphisms over $\mathcal{C}$ preserving cocartesian arrows. Using [5, Proposition 2.4.4.3], we see that $\mathbf{CoCart}_{\mathcal{C}}$ has finite products, so it is inherits a (cartesian) symmetric monoidal structure. Via the cocartesian straightening / unstraightening equivalence

$$\mathrm{St}^{\mathrm{co}}_{\mathcal{C}} \colon \mathbf{CoCart}_{\mathcal{C}} \leftrightarrows \mathrm{Fun}(\mathcal{C},\mathbf{Cat}_\infty) \colon \mathrm{Un}^{\mathrm{co}}_{\mathcal{C}} \ ,$$

this symmetric monoidal structure corresponds to the objectwise monoidal structure on $\mathrm{Fun}(\mathcal{C},\mathbf{Cat}_\infty)$. In particular, the above equivalence lifts to an equivalence

$$\mathrm{CAlg}(\mathbf{CoCart}_{\mathcal{C}}) \simeq \mathrm{Fun}(\mathcal{C},\mathrm{CAlg}(\mathbf{Cat}_\infty)) \ .$$

These ∞-categories can be realized in a yet different way, which sometimes is more convenient:

Definition A.1. Let $\mathcal{C}$ be an ∞-category. A *$\mathcal{C}$-family of symmetric monoidal ∞-categories* consists of a cocartesian fibration

$$p \colon \mathcal{D}^\otimes \longrightarrow \mathrm{Fin}_* \times \mathcal{C}$$

such that for every $c \in \mathcal{C}$ the induced cocartesian fibration

$$p_c \colon \mathcal{D}^\otimes_c \longrightarrow \mathrm{Fin}_* \times \{c\}$$

is a symmetric monoidal ∞-category.

Let $\mathcal{C}$ be an ∞-category and let $\mathcal{D}^\otimes$ be a $\mathcal{C}$-family of symmetric monoidal ∞-categories. Unraveling the definitions, we see that $\mathcal{D}^\otimes$ is in particular a $\mathcal{C}$-family of ∞-operads in the sense of [6, Definition 2.3.2.10]. In particular, Proposition 2.3.2.11 in *loc. cit.* allows to see $\mathcal{D}^\otimes$ as a generalized ∞-operad.

Notation A.2. Let $\mathcal{C}$ be an ∞-category and let $\mathcal{D}^\otimes$ be a $\mathcal{C}$-family of symmetric monoidal ∞-categories. We will write $\mathcal{D} \to \mathcal{C}$ for the fiber of $\mathcal{D}^\otimes \to \mathrm{Fin}_* \times \mathcal{C}$ at $\{\langle 1\rangle\} \times \mathcal{C}$. Similarly, if $f^\otimes \colon \mathcal{D}^\otimes \to \mathcal{E}^\otimes$ is a functor over $\mathrm{Fin}_* \times \mathcal{C}$, we write $f \colon \mathcal{D} \to \mathcal{E}$ for the fiber at $\{\langle 1\rangle\} \times \mathcal{C}$.

Definition A.3. Let $\mathcal{C}$ be an ∞-category and let $\mathcal{D}^\otimes$ and $\mathcal{E}^\otimes$ be two $\mathcal{C}$-families of symmetric monoidal ∞-categories. We say that a functor $f^\otimes \colon \mathcal{D}^\otimes \to \mathcal{E}^\otimes$ over $\mathrm{Fin}_* \times \mathcal{C}$ is:

(1) *lax symmetric monoidal* if it is a morphism of generalized ∞-operads in the sense of [6, Definition 2.3.2.2] (that is, it preserves inert morphisms);

(2) $\mathcal{C}$*-lax symmetric monoidal* if it is lax symmetric monoidal and the underlying functor $f\colon \mathcal{D} \to \mathcal{E}$ is morphism in $\mathbf{CoCart}_{\mathcal{C}}$ (that is, it preserves cocartesian edges);

(3) $\mathcal{C}$*-symmetric monoidal functor* if it is preserves cocartesian morphisms.

We write $(\mathrm{SMCat}_\infty)^{\mathrm{lax}}_{/\mathcal{C}}$ for the full subcategory of $(\mathbf{Cat}_\infty)_{/\mathrm{Fin}_* \times \mathcal{C}}$ spanned by $\mathcal{C}$-families of symmetric monoidal ∞-categories and $\mathcal{C}$-lax symmetric monoidal functors between them. We write $(\mathrm{SMCat}_\infty)_{/\mathcal{C}}$ for the (non full) subcategory of $(\mathrm{SMCat}_\infty)^{\mathrm{lax}}_{/\mathcal{C}}$ having the same objects and whose morphisms are $\mathcal{C}$-symmetric monoidal functors.

Lemma A.4. *Let $\mathcal{C}$ be an ∞-category and let $\mathcal{D}^\otimes$ and $\mathcal{E}^\otimes$ be two $\mathcal{C}$-families of symmetric monoidal ∞-categories. A functor $f^\otimes\colon \mathcal{D}^\otimes \to \mathcal{E}^\otimes$ over $\mathrm{Fin}_* \times \mathcal{C}$ is lax symmetric monoidal if and only if for every $c \in \mathcal{C}$, the induced functor $f^\otimes_c\colon \mathcal{D}^\otimes_c \to \mathcal{E}^\otimes_c$ is a map of ∞-operads in the sense of* [6, Definition 2.1.2.7].

Proof. Let $p\colon \mathcal{D}^\otimes \to \mathrm{Fin}_* \times \mathcal{C}$ be the structural map and let $\pi\colon \mathrm{Fin}_* \times \mathcal{C} \to \mathrm{Fin}_*$ be the natural projection. Set $q \coloneqq \pi \circ p$ for the composite. By definition a morphism $f\colon d \to d'$ in $\mathcal{D}^\otimes$ is inert if and only if $q(f)$ is inert and f is q-cocartesian. Homotopy uniqueness of cocartesian lifts implies that $p(f)$ is a π-cocartesian morphism in $\mathrm{Fin}_* \times \mathcal{C}$, and therefore that the projection to $\mathcal{C}$ is an equivalence. At this point, the conclusion follows simply unraveling the definitions. □

Corollary A.5. *Let $\mathcal{C}$ be an ∞-category and let $\mathcal{D}^\otimes$ and $\mathcal{E}^\otimes$ be $\mathcal{C}$-families of symmetric monoidal ∞-categories. Let $f^\otimes\colon \mathcal{D}^\otimes \to \mathcal{E}^\otimes$ be a symmetric monoidal functor relative to $\mathcal{C}$. Assume that for every $c \in \mathcal{C}$, the functor $f_c\colon \mathcal{D}_c \to \mathcal{E}_c$ has a right adjoint. Then $f^\otimes$ has a right adjoint $g^\otimes$ relative to $\mathrm{Fin}_* \times \mathcal{C}$, and additionally $g^\otimes$ is lax symmetric monoidal relative to $\mathcal{C}$.*

Proof. Fix an object $c \in \mathcal{C}$ and an element $\langle n \rangle \in \mathrm{Fin}_*$. The morphisms $\{\rho^n_i\colon \langle n \rangle \to \langle 1 \rangle\}_{i=1,2,\ldots,n}$ induce canonical identifications

$$\mathcal{D}_{\langle n \rangle, c} \simeq \mathcal{D}_c^{\times n} \qquad \text{and} \qquad \mathcal{E}_{\langle n \rangle, c} \simeq \mathcal{E}_c^{\times n} \ .$$

With respect to these identifications, $f_{\langle n \rangle, c}$ is identified with $f_c^{\times n}$. In particular, it follows that every fiber $f_{\langle n \rangle, c}$ has a right adjoint. Therefore, [6, Proposition 7.3.2.6] guarantees that $f^\otimes$ has a right adjoint $g^\otimes$ relative to $\mathrm{Fin}_* \times \mathcal{C}$. At this point, the conclusion follows combining Proposition 7.3.2.5 and 7.3.2.7 in *loc. cit.* with the criterion provided by Lemma A.4. □

Lemma A.6. *Let $\mathcal{C}$ be an ∞-category and let $f^\otimes\colon \mathcal{D}^\otimes \to \mathcal{E}^\otimes$ be a $\mathcal{C}$-lax symmetric monoidal functor between $\mathcal{C}$-families of symmetric monoidal ∞-categories. Write $p\colon \mathcal{D}^\otimes \to \mathcal{C}$ and $q\colon \mathcal{E}^\otimes \to \mathcal{C}$ be the two structural morphisms. Then $f^\otimes$ takes p-cocartesian morphisms to q-cocartesian ones.*

Proof. Consider the commutative triangle

$$\begin{array}{ccc} \mathcal{D}^\otimes & \xrightarrow{p'} & \mathrm{Fin}_* \times \mathcal{C} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & \mathcal{C} & \end{array}$$

By assumption p' is a cocartesian fibration. It follows that p' takes p-cocartesian edges to π-cocartesian ones. Let $\alpha\colon d \to d'$ be a p-cocartesian morphism in $\mathcal{D}^\otimes$. Its image under p' is of the form $(\mathrm{id}_{\langle n \rangle}, \beta)\colon (\langle n \rangle, c) \to (\langle n \rangle, c')$. Thus, [5, Proposition 2.4.1.3] guarantees that α is p'-cocartesian. For $i = 1, 2, \ldots, n$ choose inert morphisms

$$r_i\colon d \longrightarrow d_i \qquad \text{and} \qquad r'_i\colon d' \longrightarrow d'_i \ .$$

Notice that the image of $p'(d_i)$ is of the form $(\langle 1\rangle, c)$, and $p'(d_i')$ is of the form $(\langle 1\rangle, c')$. Applying once again [5, Proposition 2.4.1.3], we see that these morphisms are also p'-cocartesian. Thus, we find p'-cocartesian morphisms

$$\alpha_i \colon d_i \longrightarrow d_i'$$

making the diagram

$$\begin{array}{ccc} d & \xrightarrow{\alpha} & d' \\ {\scriptstyle r_i}\downarrow & & \downarrow{\scriptstyle r_i'} \\ d_i & \xrightarrow{\alpha_i} & d_i' \end{array}$$

commutative in $\mathcal{D}^\otimes$. Applying $f^\otimes$, we find a commutative diagram

$$\begin{array}{ccc} f^\otimes(d) & \xrightarrow{f^\otimes(\alpha)} & f^\otimes(d') \\ {\scriptstyle f^\otimes(r_i)}\downarrow & & \downarrow{\scriptstyle f^\otimes(r_i')} \\ f^\otimes(d_i) & \xrightarrow{f^\otimes(\alpha_i)} & f^\otimes(d_i') \end{array} .$$

By assumption, the vertical morphisms are again inert, and every $f^\otimes(\alpha_i)$ is q-cocartesian. At this point, the conclusion follows from the fact that since $\mathcal{E}^\otimes$ is a $\mathcal{C}$-family of symmetric monoidal ∞-categories, cocartesian lifts of inert morphisms determine an equivalence

$$\mathcal{E}^\otimes_{\langle n\rangle} \longrightarrow \mathcal{E} \times_{\mathcal{C}} \mathcal{E} \times_{\mathcal{C}} \cdots \times_{\mathcal{C}} \mathcal{E} .$$

□

Write $\mathrm{SMCat}_\infty^{\mathrm{lax}}$ for the full subcategory of Op_∞ spanned by symmetric monoidal ∞-categories. Then we have:

Proposition A.7. *Let $\mathcal{C}$ be an ∞-category. The cocartesian straightening / unstraightening equivalence over $\mathcal{C}$ lifts to equivalences*

$$(\mathrm{SMCat}_\infty)^{\mathrm{lax}}_{/\mathcal{C}} \simeq \mathrm{Fun}(\mathcal{C}, \mathrm{SMCat}_\infty^{\mathrm{lax}}) \qquad \text{and} \qquad (\mathrm{SMCat}_\infty)_{/\mathcal{C}} \simeq \mathrm{CAlg}(\mathbf{CoCart}_{\mathcal{C}}) .$$

Proof. Write $\pi \colon \mathrm{Fin}_* \times \mathcal{C} \to \mathcal{C}$ for the canonical projection. Then there is an obvious forgetful functor

$$(\mathrm{SMCat}_\infty)^{\mathrm{lax}}_{/\mathcal{C}} \longrightarrow ((\mathbf{Cat}_\infty)_{/\mathcal{C}})_{/\pi} ,$$

and Lemma A.6 shows that it canonically factors through $(\mathbf{CoCart}_{\mathcal{C}})_{/\pi}$. Write $\underline{\mathrm{Fin}_*}$ for the constant functor $\mathcal{C} \to \mathbf{Cat}_\infty$ associated to Fin_*. The straightening / unstraightening equivalence provides us with an equivalence

$$(\mathbf{CoCart}_{\mathcal{C}})_{/\pi_{\mathcal{C}}} \simeq \mathrm{Fun}(\mathcal{C}, \mathbf{Cat}_\infty)_{/\underline{\mathrm{Fin}_*}}$$

Write $\mathrm{Fun}(\mathcal{C}, \mathbf{Cat}_\infty)^{\mathrm{lsm}}_{/\underline{\mathrm{Fin}_*}}$ for the (non full) subcategory having:

- as objects natural transformations $\Phi \to \underline{\mathrm{Fin}_*}$ such that for every $c \in \mathcal{C}$ the induced functor $\Phi(c) \to \mathrm{Fin}_*$ is a symmetric monoidal ∞-category in the sense of [6, Definition 2.0.0.7];
- as 1-morphisms transformations $\Phi \to \Psi$ over $\underline{\mathrm{Fin}_*}$ such that for every $c \in \mathcal{C}$ the functor $\Phi(c) \to \Psi(c)$ is a morphism of ∞-operads (that is, it preserves inert morphisms).

Then the above equivalence restricts by definition to an equivalence

$$(\mathrm{SMCat}_\infty)^{\mathrm{lax}}_{/\mathcal{C}} \simeq \mathrm{Fun}(\mathcal{C}, \mathbf{Cat}_\infty)^{\mathrm{lsm}}_{/\underline{\mathrm{Fin}_*}}$$

Next notice that there is a canonical equivalence

$$\mathrm{Fun}(\mathcal{C}, \mathbf{Cat}_\infty)_{/\underline{\mathrm{Fin}_*}} \simeq \mathrm{Fun}(\mathcal{C}, (\mathbf{Cat}_\infty)_{/\mathrm{Fin}_*}) .$$

Reviewing Op_∞ as a (non full) subcategory of $(\mathbf{Cat}_\infty)_{/\mathrm{Fin}_*}$, we see that the above equivalence restricts to

$$\mathrm{Fun}(\mathcal{C}, \mathbf{Cat}_\infty)^{\mathrm{lsm}}_{/\underline{\mathrm{Fin}}_*} \simeq \mathrm{Fun}(\mathcal{C}, \mathrm{SMCat}^{\mathrm{lax}}_\infty) \ .$$

This proves the fist half of the statement. For the second half, consider the (non full) subcategory $\mathrm{Fun}(\mathcal{C}, \mathbf{Cat}_\infty)^{\mathrm{sm}}_{/\underline{\mathrm{Fin}}_*}$ of $\mathrm{Fun}(\mathcal{C}, \mathbf{Cat}_\infty)^{\mathrm{lsm}}_{/\underline{\mathrm{Fin}}_*}$ having the same objects and whose morphisms are natural transformations $\Phi \to \Psi$ over $\underline{\mathrm{Fin}}_*$ such that for every $c \in \mathcal{C}$ the functor $\Phi(c) \to \Psi(c)$ is a symmetric monoidal functor. Then a direct check shows that the above equivalences restrict to

$$(\mathrm{SMCat}_\infty)_{/\mathcal{C}} \simeq \mathrm{Fun}(\mathcal{C}, \mathbf{Cat}_\infty)^{\mathrm{sm}}_{/\underline{\mathrm{Fin}}_*} \simeq \mathrm{Fun}(\mathcal{C}, \mathrm{SMCat}_\infty) \ .$$

The conclusion follows. □

A.2. **Relative algebras and modules.** Let $\mathcal{C}$ be an ∞-category and let $\mathcal{D}^\otimes$ be a $\mathcal{C}$-family of symmetric monoidal ∞-categories. Reviewing $\mathcal{D}^\otimes$ as a generalized ∞-operad, we can form the ∞-category

$$\mathrm{CAlg}(\mathcal{D}^\otimes) \coloneqq \mathrm{Alg}_{\mathrm{Comm}^\otimes}(\mathcal{D}^\otimes) \ ,$$

where the right hand side is defined as in [6, Definition 2.3.2.2]. Proposition 2.3.2.9-(1) in *loc. cit.* shows that $\mathrm{Fin}_* \times \mathcal{C}$ is also a generalized ∞-operad. In particular, we obtain a canonical map

$$p \colon \mathrm{CAlg}(\mathcal{D}^\otimes) \longrightarrow \mathrm{CAlg}(\mathrm{Fin}_* \times \mathcal{C}) \ ,$$

and the proof of Proposition 2.3.2.9-(3) in *loc. cit.* shows that

$$\mathrm{CAlg}(\mathrm{Fin}_* \times \mathcal{C}) \simeq \mathcal{C} \ .$$

We have:

Lemma A.8. *The functor $p \colon \mathrm{CAlg}(\mathcal{D}^\otimes) \to \mathcal{C}$ is a cocartesian fibration, and the fiber at $c \in \mathcal{C}$ is canonically equivalent to $\mathrm{CAlg}(\mathcal{D}^\otimes_c)$.*

Proof. To begin with, observe that the square

$$\begin{array}{ccc} \mathrm{Fun}_{/\mathrm{Fin}_*}(\mathrm{Fin}_*, \mathcal{D}^\otimes) & \longrightarrow & \mathrm{Fun}(\mathrm{Fin}_*, \mathcal{D}^\otimes) \\ \downarrow{\scriptstyle \overline{p}} & & \downarrow \\ \mathrm{Fun}_{/\mathrm{Fin}_*}(\mathrm{Fin}_*, \mathrm{Fin}_* \times \mathcal{C}) & \longrightarrow & \mathrm{Fun}(\mathrm{Fin}_*, \mathrm{Fin}_* \times \mathcal{C}) \ . \end{array}$$

is a pullback of simplicial sets. Besides, since $\mathcal{D}^\otimes \to \mathrm{Fin}_* \times \mathcal{C}$ is a cocartesian fibration, it is in particular a categorical fibration. Therefore, the right vertical map is a categorical fibration as well, so this square is a homotopy fiber product. Notice that there is an equivalence

$$\mathcal{C} \simeq \mathrm{Fun}_{/\mathrm{Fin}_*}(\mathrm{Fin}_*, \mathrm{Fin}_* \times \mathcal{C}) \ ,$$

under which an object c corresponds to the associated constant section $s_c \colon \mathrm{Fin}_* \to \mathrm{Fin}_* \times \mathcal{C}$. This allows to identify the fiber of $\overline{p}$ at c with

$$\{s_c\} \times_{\mathrm{Fun}(\mathrm{Fin}_*, \mathrm{Fin}_* \times \mathcal{C})} \mathrm{Fun}(\mathrm{Fin}_*, \mathcal{D}^\otimes) \simeq \mathrm{Fun}(\mathrm{Fin}_*, \mathcal{D}^\otimes_c) \ .$$

Unraveling the definitions, we therefore find a canonical equivalence

$$\mathrm{CAlg}(\mathcal{D}^\otimes)_c \simeq \mathrm{CAlg}(\mathcal{D}^\otimes_c) \ .$$

Let $f \colon c \to c'$ be a morphism in $\mathcal{C}$. There is an induced symmetric monoidal functor $F \colon \mathcal{D}^\otimes_c \to \mathcal{D}^\otimes_{c'}$, which induces a well defined functor

$$F \colon \mathrm{CAlg}(\mathcal{D}^\otimes_c) \longrightarrow \mathrm{CAlg}(\mathcal{D}^\otimes_{c'}) \ .$$

If $d \in \mathrm{CAlg}(\mathcal{D}^\otimes)_c \simeq \mathrm{CAlg}(\mathcal{D}^\otimes_c)$, then there is a canonically induced morphism $d \to F(d)$ inside $\mathrm{CAlg}(\mathcal{D}^\otimes)$, and a straightforward verification reveals that this is a p-cocartesian lift of f. □

Remark A.9. Let $\Phi_{\mathcal{D}^\otimes} \colon \mathcal{C} \to \mathrm{CAlg}(\mathbf{Cat}_\infty)$ be the straightening of $\mathcal{D}^\otimes$ supplied by Proposition A.7. Then the above lemma identifies $\mathrm{CAlg}(\mathcal{D}^\otimes)$ with the straightening of the induced functor $\mathcal{C} \to \mathbf{Cat}_\infty$ informally given by $c \mapsto \mathrm{CAlg}(\mathcal{D}^\otimes_c)$. For this reason, we will often write $\mathrm{CAlg}(\mathcal{D}^\otimes/\mathcal{C})$ instead of $\mathrm{CAlg}(\mathcal{D}^\otimes)$.

Thanks to [6, Proposition 2.3.2.11], we know that the map $\mathcal{D}^\otimes \to \mathrm{Fin}_* \times \mathcal{C}$ is a fibration of generalized ∞-operads. Combining Remark 2.3.2.4-(2) and Proposition 2.3.2.9-(1) in *loc. cit.*, we see that the map $\mathrm{Fin}_* \times \mathcal{C} \to \mathrm{Fin}_*$ is also a fibration of generalized ∞-operads. Therefore, the composite map $\mathcal{D}^\otimes \to \mathrm{Fin}_*$ is also a fibration of generalized ∞-operads. We can thus apply [6, Definition 3.3.3.8] (with $\mathcal{O}^\otimes = \mathrm{Comm}^\otimes = \mathrm{Fin}_*$) to obtain a canonical map of generalized ∞-operads

$$\mathbf{Mod}(\mathcal{D}^\otimes)^\otimes \longrightarrow \mathrm{CAlg}(\mathcal{D}^\otimes/\mathcal{C}) \times \mathrm{Fin}_* \ .$$

Assume that under the equivalence of Proposition A.7, the functor $\Phi_{\mathcal{D}^\otimes} \colon \mathcal{C} \to \mathrm{CAlg}(\mathbf{Cat}_\infty)$ factors through $\mathrm{CAlg}(\mathbf{Cat}_\infty(\mathbf{\Delta}^{\mathrm{op}}))$. Here $\mathbf{Cat}_\infty(\mathbf{\Delta}^{\mathrm{op}})$ is the ∞-category of ∞-categories admitting geometric realizations of simplicial objects, and whose morphisms are functors that commute with geometric realizations. Then [6, Theorem 4.5.3.1] guarantees that the above map is a cocartesian fibration. Moreover, [6, Theorem 4.5.2.1-(1)] implies that $\mathbf{Mod}(\mathcal{D}^\otimes)^\otimes$ is a $\mathrm{CAlg}(\mathcal{D}^\otimes/\mathcal{C})$-family of symmetric monoidal ∞-categories.

Any cocartesian section $R\colon \mathcal{C} \to \mathrm{CAlg}(\mathcal{D}^\otimes/\mathcal{C})$ allows to define the ∞-category $\mathbf{Mod}_R(\mathcal{D}^\otimes/\mathcal{C})$ as the fiber product

$$\begin{array}{ccc} \mathbf{Mod}_R(\mathcal{D}^\otimes/\mathcal{C})^\otimes & \longrightarrow & \mathbf{Mod}(\mathcal{D}^\otimes/\mathcal{C})^\otimes \\ \big\downarrow & & \big\downarrow \\ \mathcal{C} \times \mathrm{Fin}_* & \xrightarrow{\ R\ } & \mathrm{CAlg}(\mathcal{D}^\otimes/\mathcal{C}) \times \mathrm{Fin}_* \ . \end{array}$$

By construction, $\mathbf{Mod}_R(\mathcal{D}^\otimes/\mathcal{C})$ is again a cocartesian fibration over $\mathcal{C}$, and inspection reveals that it classifies the functor sending $c \in \mathcal{C}$ to $\mathbf{Mod}_{R(c)}(\mathcal{D}^\otimes_c)$. Finally, given two sections $R, R' \colon \mathcal{C} \to \mathrm{CAlg}(\mathcal{D}^\otimes/\mathcal{C})$ and a natural transformation $\alpha\colon R' \to R$ between them, we naturally obtain a lax symmetric monoidal adjunction relative to $\mathcal{C}$:

$$\alpha^{*,\otimes} \colon \mathbf{Mod}_{R'}(\mathcal{D}^\otimes/\mathcal{C})^\otimes \leftrightarrows \mathbf{Mod}_R(\mathcal{D}^\otimes/\mathcal{C})^\otimes \colon \alpha_*^\otimes \ .$$

In particular, taking R' to be the tensor unit section and $\alpha\colon R' \to R$ the canonical map, we obtain a lax symmetric monoidal functor relative to $\mathcal{C}$:

$$R \otimes - \coloneqq \alpha^* \circ \alpha_* \colon \mathcal{D}^\otimes \longrightarrow \mathcal{D}^\otimes \ . \tag{A.10}$$

Notice that the underlying functors

$$\alpha^* \colon \mathbf{Mod}_{R'}(\mathcal{D}^\otimes/\mathcal{C}) \longrightarrow \mathbf{Mod}_R(\mathcal{D}^\otimes/\mathcal{C}) \qquad \text{and} \qquad \alpha_* \colon \mathbf{Mod}_R(\mathcal{D}^\otimes/\mathcal{C}) \longrightarrow \mathbf{Mod}_{R'}(\mathcal{D}^\otimes/\mathcal{C})$$

preserve cocartesian edges. Thus, it follows that both functors in the adjunction (A.10) are $\mathcal{C}$-lax symmetric monoidal functors.

Appendix B. A lemma on relative adjunctions

Consider a commutative triangle

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\ g\ } & \mathcal{B} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & \mathcal{X} & \end{array}$$

in $\mathbf{Cat}_\infty$. We refer to [6, Definition 7.3.2.2] for the definition of relative adjunction. The goal of this appendix is to record the following mild variation of Proposition 7.3.2.6 in *loc. cit.*:

Proposition B.1. *Assume that:*

(1) both p and q are are cocartesian fibrations, and g takes p-cocartesian morphisms to q-cocartesian ones;

(2) for every morphism $\gamma\colon x \to y$ in $\mathcal{X}$, the square

$$\begin{array}{ccc} \mathcal{A}_x & \xrightarrow{g_x} & \mathcal{B}_x \\ \downarrow & & \downarrow \\ \mathcal{A}_y & \xrightarrow{g_y} & \mathcal{B}_y \end{array}$$

is horizontally left adjointable.

Then g admits a left adjoint $f\colon \mathcal{B} \to \mathcal{A}$ relative to $\mathcal{X}$.

Proof. Condition (2) applied to the identity $\mathrm{id}_x\colon x \to x$ guarantees that $g_x\colon \mathcal{A}_x \to \mathcal{B}_x$ has a left adjoint. Let us denote it by f_x. We start by showing that $g\colon \mathcal{A} \to \mathcal{B}$ has a left adjoint. For this, it is enough to show that for every $b \in \mathcal{B}$ there exists an object $a \in \mathcal{A}$ together with a morphism $\alpha\colon b \to g(a)$ such that for every $a' \in \mathcal{A}$ the induced map

$$\theta_\alpha\colon \operatorname{Map}_{\mathcal{A}}(a, a') \longrightarrow \operatorname{Map}_{\mathcal{B}}(g(a), g(a')) \xrightarrow{\alpha^*} \operatorname{Map}_{\mathcal{B}}(b, g(a'))$$

is an equivalence. Set $x \coloneqq q(x)$ and let $a \coloneqq f_x(b)$. This is an object in $\mathcal{A}_x$, which we review as an object in $\mathcal{A}$. By construction, the unit of the adjunction $f_x \dashv g_x$ provides a canonical transformation $\alpha\colon b \to g_x(f_x(b)) = g(a)$ in $\mathcal{B}_x$, which we review as a morphism in $\mathcal{B}$ lying over id_x. Fix an element $a' \in \mathcal{A}$, and set $y \coloneqq p(a')$. Consider the following commutative triangle:

$$\begin{array}{ccc} \operatorname{Map}_{\mathcal{A}}(a, a') & \xrightarrow{\quad\theta_\alpha\quad} & \operatorname{Map}_{\mathcal{B}}(b, g(a')) \\ & \searrow \quad \swarrow & \\ & \operatorname{Map}_{\mathcal{X}}(x, y)\ . & \end{array}$$

Since the functors p and q are inner fibrations, [5, 2.4.4.1] ensures that the vertical maps of the above diagram are Kan fibrations. Thus, to check that θ_α is an equivalence, it is enough to check that for every morphism $\gamma\colon x \to y$, the map θ_α induces an equivalence above γ. Choose a q-cocartesian lift $b \to \overline{b}$ of γ. From assumption (2), γ admits a p-cocartesian lift of the form $a \to f_y(\overline{b})$. Therefore, applying [5, Proposition 2.4.4.2], we can identify the fiber of θ_α above γ with the morphism

$$\operatorname{Map}_{\mathcal{A}_y}(f_y(\overline{b}), a') \longrightarrow \operatorname{Map}_{\mathcal{B}_y}(\overline{b}, a')$$

induced by the unit of the adjunction $f_y \dashv g_y$. As it is an equivalence by construction, the conclusion follows. To complete the proof, it is enough to observe that by construction for every $x \in \mathcal{X}$ and every $b \in \mathcal{B}_x$, the unit $b \to g(f(b))$ is a morphism in $\mathcal{B}_x$, and hence lies over the identity of x. In other words, the adjunction $f \dashv g$ is an adjunction relative to $\mathcal{X}$. □

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