NEARBY SLOPES AND BOUNDEDNESS FOR ℓ-ADIC SHEAVES IN POSITIVE CHARACTERISTIC

by

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Abstract. — The goal of this paper is to motivate a boundedness conjecture for nearby slopes of ℓ-adic sheaves in positive characteristic, and to prove it for smooth curves. For a constructible ℓ-adic sheaf, we prove the finiteness of the set of nearby slopes associated to a given morphism.

Introduction

Let S be an excellent strictly henselian trait of equal characteristic $p > 0$. As usual, s denotes the closed point of S, k its residue field, $\eta = \operatorname{Spec} K$ the generic point of S, $\overline{K}$ an algebraic closure of K and $\overline{\eta} = \operatorname{Spec} \overline{K}$. Let $f : X \longrightarrow S$ be a morphism of finite type, $\ell \neq p$ a prime number, $\mathcal{F}$ an object of the derived category $D^b_c(X_\eta, \overline{\mathbb{Q}}_\ell)$ of ℓ-adic complexes with bounded and constructible cohomology.

Let $\psi^t_f : D^b_c(X_\eta, \overline{\mathbb{Q}}_\ell) \longrightarrow D^b_c(X_s, \overline{\mathbb{Q}}_\ell)$ be the moderate nearby cycle functor. We say that $r \in \mathbb{R}_{\geqslant 0}$ is a *nearby slope of $\mathcal{F}$ associated to f* if one can find $N \in \operatorname{Sh}_c(\eta, \overline{\mathbb{Q}}_\ell)$ with slope r such that $\psi^t_f(\mathcal{F} \otimes f^*N) \neq 0$. We denote by $\operatorname{Sl}^{\mathrm{nb}}_f(\mathcal{F})$ the set of nearby slopes of $\mathcal{F}$ associated to f. By projection formula, $\operatorname{Sl}^{\mathrm{nb}}_f(\mathcal{F})$ is also the set of slopes of the germs of $\psi_f \mathcal{F}$.

The main result of [**Tey15**] is a boundedness theorem for the set of nearby slopes of a complex holonomic $\mathcal{D}$-module. The goal of the present paper is to give some motivation for an analogue of this theorem for ℓ-adic sheaves in positive characteristic.

For complex holonomic $\mathcal{D}$-modules, regularity is preserved by push-forward. On the other hand, for a morphism $C' \longrightarrow C$ between smooth curves over k, a tame constructible sheaf on C' may acquire wild ramification by push-forward. If $0 \in C$ is a closed point, the failure of $C' \longrightarrow C$ to preserve tameness above 0 is accounted for by means of the ramification filtration on the absolute Galois group of the function field of the strict henselianization C^{sh}_0 of C at 0. Moreover, the Swan conductor at 0 measures to which extent an ℓ-adic constructible sheaf on C fails to be tame at 0.

In higher dimension, both these measures of wild ramification (for a morphism and for a sheaf) are missing in a form that would give a precise meaning to the following question raised in [**Tey14**]

Question 1. — *Let $g : V_1 \longrightarrow V_2$ be a morphism between schemes of finite type over k, and $\mathcal{G} \in D^b_c(V_1, \overline{\mathbb{Q}}_\ell)$. Can one bound the wild ramification of $Rg_*\mathcal{G}$ in terms of the wild ramification of $\mathcal{G}$ and the wild ramification of $g_{|\operatorname{Supp}\mathcal{G}}$?*

Note that in an earlier formulation, "wild ramification of $g_{|\operatorname{Supp}\mathcal{G}}$" was replaced by "wild ramification of g", which cannot hold due to the following example that we owe to A. Beilinson: take $f : \mathbb{A}^1_S \longrightarrow S$, $P \in S[t]$ and $i_P : \{P = 0\} \hookrightarrow \mathbb{A}^1_S$. Then $i_{P*}\overline{\mathbb{Q}}_\ell$ is tame but $f_*(i_{P*}\overline{\mathbb{Q}}_\ell)$ has arbitrary big wild ramification as P runs through the set of Eisenstein polynomials.

If $f : X \longrightarrow S$ is proper, 2.2.1 shows that $\mathrm{Sl}^{\mathrm{nb}}_f(\mathcal{F})$ controls the slopes of $H^i(X_{\overline{\eta}}, \mathcal{F})$ for every i. It is thus tempting to take for "wild ramification of $\mathcal{G}$" the nearby slopes of $\mathcal{G}$. Let us note that one could instead use the characteristic cycle $\operatorname{Char}\mathcal{G}$ of $\mathcal{G}$ as constructed by A. Beilinson [**Bei15**] and T. Saito [**Sai15a**][**Sai15b**]. In that case, the theory of $\mathcal{D}$-modules suggests a relation between $\operatorname{Char}\mathcal{G}$ and $\operatorname{Char} Rg_*\mathcal{G}$ at least when g is proper.

Hence, Question 1 leads to the question of bounding nearby slopes of constructible ℓ-adic sheaves. This question has a negative answer since already the constant sheaf $\overline{\mathbb{Q}}_\ell$ has arbitrary big nearby slopes (1). This is actually good news since for curves, these nearby slopes keep track of the aforementioned ramification filtration (2). Hence, one can use them in higher dimension to quantify the wild ramification of a morphism and in Question 1 take for "wild ramification of $g_{|\operatorname{Supp}\mathcal{G}}$" the nearby slopes of $\overline{\mathbb{Q}}_\ell$ on $\operatorname{Supp}\mathcal{G}$ associated with $g_{|\operatorname{Supp}\mathcal{G}}$ (at least when V_2 is a curve).

To get a good boundedness statement, one has to correct the nearby slopes associated with a morphism by taking into account the maximal nearby slope of $\overline{\mathbb{Q}}_\ell$ associated with the same morphism. That such a maximal slope exists in general is a consequence of the following

Theorem 1. — *Let $f : X \longrightarrow S$ be a morphism of finite type and $\mathcal{F} \in D^b_c(X_\eta, \overline{\mathbb{Q}}_\ell)$. The set $\mathrm{Sl}^{\mathrm{nb}}_f(\mathcal{F})$ is finite.*

The proof of this theorem follows an argument due to Deligne [**Del77**, Th. finitude 3.7]. For a $\mathcal{D}$-module version, let us refer to [**Del07**]. Thus, $\operatorname{Max}\mathrm{Sl}^{\mathrm{nb}}_f(\overline{\mathbb{Q}}_\ell)$ makes sense if $\mathrm{Sl}^{\mathrm{nb}}_f(\overline{\mathbb{Q}}_\ell)$ is not empty. Otherwise, we set $\operatorname{Max}\mathrm{Sl}^{\mathrm{nb}}_f(\overline{\mathbb{Q}}_\ell) = +\infty$. Proposition 2.3.4 suggests and gives a positive answer to the following question for smooth curves

Question 2. — *Let V/k be a scheme of finite type and $\mathcal{F} \in D^b_c(V, \overline{\mathbb{Q}}_\ell)$. Is it true that the following set*

$$\{r/(1 + \operatorname{Max}\mathrm{Sl}^{\mathrm{nb}}_f(\overline{\mathbb{Q}}_\ell)),\ \textit{for } r \in \mathrm{Sl}^{\mathrm{nb}}_f(\mathcal{F}) \textit{ and } f \in \mathcal{O}_V\} \tag{0.0.1}$$

is bounded?

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1. as mentionned by T. Saito, it is however a consequence of [**Sai93**] that $\mathrm{Sl}^{\mathrm{nb}}_f(\overline{\mathbb{Q}}_\ell) = \{0\}$ for $f : X \longrightarrow S$ log-smooth.
2. see 2.1.2 (3) for a precise statement.

Let us explain what $\mathrm{Sl}_f^{\mathrm{nb}}(\mathcal{F})$ means in this global setting. A function $f \in \Gamma(U, \mathcal{O}_V)$ reads as $f : U \longrightarrow \mathbb{A}^1_k$. If S is the strict henselianization of $\mathbb{A}^1_k$ at a geometric point over the origin, we set $\mathrm{Sl}_f^{\mathrm{nb}}(\mathcal{F}) := \mathrm{Sl}_{f_S}^{\mathrm{nb}}(\mathcal{F}_{U_S})$ where the subscripts are synonyms of pull-back. For smooth curves, the main point of the proof of boundedness is the concavity of Herbrand φ functions.

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1. Notations

1.1. For a general reference on wild ramification in dimension 1, let us mention [**Ser68**]. Let η_t be the point of S corresponding to the tamely ramified closure K_t of K in $\overline{K}$ and $P_K := \mathrm{Gal}(\overline{K}/K_t)$ the wild ramification group of K. We denote by $(G_K^r)_{r \in \mathbb{R}_{\geqslant 0}}$ the *upper-numbering ramification filtration on* G_K and define

$$G_K^{r+} := \overline{\bigcup_{r'>r} G_K^{r'}}$$

If L/K is a finite extension, we denote by S_L the normalization of S in L and v_L the valuation on L associated with the maximal ideal of S_L.

If moreover L/K is separable, we denote by $q : G_K \longrightarrow G_K/G_L$ the quotient morphism and define a decreasing separated $\mathbb{R}_{\geqslant 0}$-filtration on the set G_K/G_L by $(G_K/G_L)^r := q(G_K^r)$. We also define $(G_K/G_L)^{r+} := q(G_K^{r+})$.

In case L/K is Galois, this filtration is the upper numbering ramification filtration on $\mathrm{Gal}(L/K)$. If L/K is non separable trivial, the *jumps* of L/K are the $r \in \mathbb{R}_{\geqslant 0}$ such that $(G_K/G_L)^{r+} \subsetneq (G_K/G_L)^r$. If L/K is trivial, we say by convention that 0 is the only jump of $\mathrm{Gal}(L/K)$.

1.2. For $M \in D_c^b(\eta, \overline{\mathbb{Q}}_\ell)$, we denote by $\mathrm{Sl}(M) \subset \mathbb{R}_{\geqslant 0}$ the set of *slopes* of M as defined in [**Kat88**, Ch 1]. We view M in an equivalent way as a continuous representation of G_K.

1.3. Let $f : X \longrightarrow S$ be a morphism of finite type and $\mathcal{F} \in D_c^b(X_\eta, \overline{\mathbb{Q}}_\ell)$. Consider the following diagram with cartesian squares

$$\begin{array}{ccccc} X_s & \xrightarrow{\ i\ } & X & \xleftarrow{\ \bar{j}\ } & X_{\bar{\eta}} \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ s & \longrightarrow & S & \longleftarrow & \bar{\eta} \end{array}$$

Following [**DK73**, XIII], we define the *nearby cycles of* $\mathcal{F}$ as

$$\psi_f \mathcal{F} := i^* R\bar{j}_* \bar{j}^* \mathcal{F}$$

By [**Del77**, Th. finitude 3.2], the complex $\psi_f\mathcal{F}$ is an object of $D^b_c(X_s,\overline{\mathbb{Q}}_\ell)$ endowed with a continuous G_K-action. Define $X_t := X\times_S \eta_t$ and $j_t : X_t \longrightarrow X$ the projection. Following [**Gro72**, I.2], we define the *moderate nearby cycles of* $\mathcal{F}$ as

$$\psi^t_f\mathcal{F} := i^*Rj_{t*}j_t^*\mathcal{F}$$

It is a complex in $D^b_c(X_s,\overline{\mathbb{Q}}_\ell)$ endowed with a continuous G/P_K-action. Since P_K is a pro-p group, we have a canonical identification

$$\psi^t_f\mathcal{F} \simeq (\psi_f\mathcal{F})^{P_K}$$

Note that by proper base change [**AGV73**, XII], ψ^t_f and ψ_f are compatible with proper push-forward.

1.4. If X is a scheme, $x \in X$ and if $\overline{x}$ is a geometric point of X lying over X, we denote by X^{sh}_x the strict henselization of X at x.

2. Nearby slopes in dimension one

2.1. We show here that nearby slopes associated with the identity morphism are the usual slopes as in [**Kat88**, Ch 1].

Lemma 2.1.1. — *For every* $M \in \mathrm{Sh}_c(\eta,\overline{\mathbb{Q}}_\ell)$, *we have*

$$\mathrm{Sl}^{\mathrm{nb}}_{\mathrm{id}}(M) = \mathrm{Sl}(M)$$

Proof. — We first remark that ψ^t_{id} is just the "invariant under P" functor. Suppose that $r \in \mathrm{Sl}(M)$. Then M has a non zero quotient N purely of slope r. The dual $N^\vee$ has pure slope r. Since N is non zero, the canonical map

$$N\otimes N^\vee \longrightarrow \overline{\mathbb{Q}}_\ell$$

is surjective. Since taking P-invariants is exact, we obtain that the maps in

$$(M\otimes N^\vee)^P \longrightarrow (N\otimes N^\vee)^P \longrightarrow \overline{\mathbb{Q}}_\ell$$

are surjective. Hence $(M\otimes N^\vee)^P \neq 0$, so $r \in \mathrm{Sl}^{\mathrm{nb}}_{\mathrm{id}}(M)$.

If r is not a slope of M, then for any N of slope r, the slopes of $M\otimes N$ are non zero. This is equivalent to $(M\otimes N)^P = 0$. □

We deduce the following

Lemma 2.1.2. — *Let* $f : X \longrightarrow S$ *be a finite morphism with* X *local and* $\mathcal{F} \in \mathrm{Sh}_c(X_\eta,\overline{\mathbb{Q}}_\ell)$.

(1) $\mathrm{Sl}^{\mathrm{nb}}_f(\mathcal{F}) = \mathrm{Sl}(f_*\mathcal{F})$.

(2) Suppose that X *is regular connected and let* L/K *be the extension of function fields induced by* f. *Suppose that* L/K *is separable. Then* $\mathrm{Max}\,\mathrm{Sl}^{\mathrm{nb}}_f(\overline{\mathbb{Q}}_\ell)$ *is the highest jump in the ramification filtration on* G_K/G_L.

(3) Suppose further in (2) *that* L/K *is Galois and set* $G := \mathrm{Gal}(L/K)$. *Then* $\mathrm{Sl}^{\mathrm{nb}}_f(\overline{\mathbb{Q}}_\ell)$ *is the union of* $\{0\}$ *with the set of jumps in the ramification filtration on* G.

Proof. — Point (1) comes from 2.1.1 and the compatibility of ψ_f^t with proper push-forward. From point (1) and $f_*\overline{\mathbb{Q}}_\ell \simeq \overline{\mathbb{Q}}_\ell[G_K/G_L]$, we deduce

$$\mathrm{Sl}_f^{\mathrm{nb}}(\overline{\mathbb{Q}}_\ell) = \mathrm{Sl}(\overline{\mathbb{Q}}_\ell[G_K/G_L])$$

If L/K is trivial, (2) is true by our definition of jumps in that case. If L/K is non trivial, $r_{\max} = \mathrm{Max}\,\mathrm{Sl}(\overline{\mathbb{Q}}_\ell[G_K/G_L])$ is characterized by the property that $G_K^{r_{\max}}$ acts non trivially on $\overline{\mathbb{Q}}_\ell[G_K/G_L]$ and $G_K^{r_{\max}+}$ acts trivially. On the other hand, the highest jump r_0 in the ramification filtration on G_K/G_L is such that $q(G_K^{r_0}) \neq \{G_L\}$ and $q(G_K^{r_0+}) = \{G_L\}$, that is $G_K^{r_0} \not\subset G_L$ and $G_K^{r_0+} \subset G_L$. The condition $G_K^{r_0} \not\subset G_L$ ensures that $G_K^{r_0}$ acts non trivially on $\overline{\mathbb{Q}}_\ell[G_K/G_L]$. If $h \in G_K^{r_0+}$, then for every $g \in G_K$

$$h \cdot (gG_L) = hgG_L = gg^{-1}hgG_L = gG_L$$

where the last equality comes from the fact that since $G_K^{r_0+}$ is a normal subgroup in G_K, we have $g^{-1}hg \in G_K^{r_0+} \subset G_L$. So (2) is proved.

Let S be the union of $\{0\}$ with the set of jumps in the ramification filtration of G. To prove (3), we have to prove $\mathrm{Sl}(\overline{\mathbb{Q}}_\ell[G]) = S$. If $r \in \mathbb{R}_{\geqslant 0}$ does not belong to S, we can find an open interval J containing r such that $G^{r'} = G^r$ for every $r' \in J$. In particular, the image of $G_K^{r'}$ by $G_K \longrightarrow \mathrm{GL}(\overline{\mathbb{Q}}_\ell[G])$ does not depend on r' for every $r' \in J$. So r is not a slope of $\overline{\mathbb{Q}}_\ell[G]$.

Reciprocally, $\overline{\mathbb{Q}}_\ell[G]$ contains a copy of the trivial representation, so $0 \in \mathrm{Sl}(\overline{\mathbb{Q}}_\ell[G])$. Let $r \in S\backslash\{0\}$. The projection morphism $G \longrightarrow G/G^{r+}$ induces a surjection of G_K-representations

$$\overline{\mathbb{Q}}_\ell[G] \longrightarrow \overline{\mathbb{Q}}_\ell[G/G^{r+}] \longrightarrow 0$$

So $\mathrm{Sl}(\overline{\mathbb{Q}}_\ell[G/G^{r+}]) \subset \mathrm{Sl}(\overline{\mathbb{Q}}_\ell[G])$. Note that G^{r+} acts trivially on $\overline{\mathbb{Q}}_\ell[G/G^{r+}]$. By definition $G^{r+} \subsetneq G^r$, so G^r acts non trivially on $\overline{\mathbb{Q}}_\ell[G/G^{r+}]$. So $r = \mathrm{Max}\,\mathrm{Sl}(\overline{\mathbb{Q}}_\ell[G/G^{r+}])$ and point (3) is proved.

□

2.2. Let us draw a consequence of 2.1.1. We suppose that $f : X \longrightarrow S$ is proper. Let $\mathcal{F} \in D_c^b(X_\eta, \overline{\mathbb{Q}}_\ell)$. The G_K-module associated to $R^k f_*\mathcal{F} \in D_c^b(\eta, \overline{\mathbb{Q}}_\ell)$ is $H^k(X_{\overline{\eta}}, \mathcal{F})$. From 2.1.1, we deduce

$$\begin{aligned}\mathrm{Sl}(H^k(X_{\overline{\eta}}, \mathcal{F})) &= \mathrm{Sl}_{\mathrm{id}}^{\mathrm{nb}}(R^k f_*\mathcal{F}) \\ &\subset \mathrm{Sl}_{\mathrm{id}}^{\mathrm{nb}}(Rf_*\mathcal{F})\end{aligned}$$

where the inclusion comes from the fact that taking P_K-invariants is exact. For every $N \in \mathrm{Sh}_c(\eta, \overline{\mathbb{Q}}_\ell)$, the projection formula and the compatibility of ψ_f^t with proper push-forward gives

$$\begin{aligned}\psi_{\mathrm{id}}^t(Rf_*\mathcal{F} \otimes N) &\simeq \psi_{\mathrm{id}}^t(Rf_*(\mathcal{F} \otimes f^*N)) \\ &\simeq Rf_*\psi_f^t(\mathcal{F} \otimes f^*N)\end{aligned}$$

Hence we have proved the following

Proposition 2.2.1. — *Let $f : X \longrightarrow S$ be a proper morphism, and let $\mathcal{F} \in D^b_c(X_\eta, \overline{\mathbb{Q}}_\ell)$. For every $i \in \mathbb{Z}$, we have*

$$\mathrm{Sl}(H^i(X_{\overline{\eta}}, \mathcal{F})) \subset \mathrm{Sl}^{\mathrm{nb}}_f(\mathcal{F})$$

2.3. Boundedness. — We first need to see that the upper-numbering filtration is unchanged by purely inseparable base change. This is the following

Lemma 2.3.1. — *Let K'/K be a purely inseparable extension of degree p^n. Let L/K be finite Galois extension, $L' := K' \otimes_K L$ the associated Galois extension of K'. Then, the isomorphism*

$$\mathrm{Gal}(L/K) \xrightarrow{\sim} \mathrm{Gal}(L'/K') \tag{2.3.2}$$

$$g \longrightarrow \mathrm{id} \otimes g \tag{2.3.3}$$

is compatible with the upper-numbering filtration.

Proof. — Note that for every $g \in \mathrm{Gal}(L/K)$, $\mathrm{id} \otimes g \in \mathrm{Gal}(L'/K')$ is determined by the property that its restriction to L is g.

Let π be a uniformizer of S and π_L a uniformizer of S_L. We have $K \simeq k((\pi))$ and $L \simeq k((\pi_L))$. Since k is perfect and since K'/K and L'/L are purely inseparable of degree p^n, we have $K' = k((\pi^{1/p^n}))$ and $L' = k((\pi_L^{1/p^n}))$. So π_L^{1/p^n} is a uniformizer of $S_{L'}$. For every $\sigma \in \mathrm{Gal}(L'/K')$ we have

$$(\sigma(\pi_L^{1/p^n}) - \pi_L^{1/p^n})^{p^n} = \sigma_{|L}(\pi_L) - \pi_L$$

so

$$\begin{aligned} v_{L'}(\sigma(\pi_L^{1/p^n}) - \pi_L^{1/p^n}) &= \frac{1}{p^n} v_{L'}(\sigma_{|L}(\pi_L) - \pi_L) \\ &= v_L(\sigma_{|L}(\pi_L) - \pi_L) \end{aligned}$$

So (2.3.2) commutes with the lower-numbering filtration. Hence, (2.3.2) commutes with the upper-numbering filtration and lemma 2.3.1 is proved. □

Boundedness in case of smooth curves over k is a consequence of the following

Proposition 2.3.4. — *Let S_0 be an henselian trait over k, let $\eta_0 = \mathrm{Spec}\, K_0$ be the generic point of S_0 and $M \in \mathrm{Sh}_c(\eta_0, \overline{\mathbb{Q}}_\ell)$. There exists a constant $C_M \geqslant 0$ depending only on M such that for every finite morphism $f : S_0 \longrightarrow S$, we have*

$$\mathrm{Sl}^{\mathrm{nb}}_f(M) \subset [0, \mathrm{Max}(C_M, \mathrm{Max}\, \mathrm{Sl}^{\mathrm{nb}}_f(\overline{\mathbb{Q}}_\ell))] \tag{2.3.5}$$

In particular, the quantity

$$\mathrm{Max}\, \mathrm{Sl}^{\mathrm{nb}}_f(M)/(1 + \mathrm{Max}\, \mathrm{Sl}^{\mathrm{nb}}_f(\overline{\mathbb{Q}}_\ell))$$

is bounded uniformely in f.

Proof. — By 2.1.2 (1), we have to bound $\mathrm{Sl}(f_* M)$ in terms of $\mathrm{Max}\, \mathrm{Sl}(f_* \overline{\mathbb{Q}}_\ell)$. Using [**Kat88**, I 1.10], we can replace $\overline{\mathbb{Q}}_\ell$ by $\mathbb{F}_\lambda$, where $\lambda = \ell^n$. Hence, G_{K_0} acts on M via a finite quotient $H \subset \mathrm{GL}_{\mathbb{F}_\lambda}(M)$. Let L/K_0 be the corresponding finite Galois extension and $f_M : S_L \longrightarrow S_0$ the induced morphism. We have $H = \mathrm{Gal}(L/K_0)$. Let us denote by r_M the highest jump in the ramification filtration of H. Using Herbrand functions

[**Ser68**, IV 3], we will prove that the constant $C_M := \psi_{L/K_0}(r_M)$ does the job.

Using 2.3.1, we are left to treat the case where K_0/K is separable. The adjunction morphism

$$M \longrightarrow f_{M*}f_M^* M$$

is injective. Since $f_M^* M \simeq \mathbb{F}_\lambda^{\mathrm{rg}\, M}$, we obtain by applying f_* an injection

$$f_* M \longrightarrow \mathbb{F}_\lambda[\mathrm{Gal}(L/K)]^{\mathrm{rg}\, M}$$

So we are left to bound the slopes of $\mathbb{F}_\lambda[\mathrm{Gal}(L/K)]$ viewed as a G_K-representation, that is by 2.1.2 (2) the highest jump in the upper-numbering ramification filtration of $\mathrm{Gal}(L/K)$. By 2.1.2 (2), $r_0 := \mathrm{Max}\,\mathrm{Sl}_f^{\mathrm{nb}}(\overline{\mathbb{Q}}_\ell)$ is the highest jump in the ramification filtration of $\mathrm{Gal}(L/K)/H$. Choose $r > \mathrm{Max}(r_0, \varphi_{L/K}\psi_{L/K_0}(r_M))$. We have

$$\begin{aligned}\mathrm{Gal}(L/K)^r &= H \cap \mathrm{Gal}(L/K)^r \\ &= H \cap \mathrm{Gal}(L/K)_{\psi_{L/K}(r)} \\ &= H_{\psi_{L/K}(r)} \\ &= H^{\varphi_{L/K_0}\psi_{L/K}(r)} \\ &= \{1\}\end{aligned}$$

The first equality comes from $r > r_0$. The third equality comes from the compatibility of the lower-numbering ramification filtration with subgroups. The last equality comes from the fact that $r > \varphi_{L/K}\psi_{L/K_0}(r_M)$ is equivalent to $\varphi_{L/K_0}\psi_{L/K}(r) > r_M$. Hence,

$$\mathrm{Sl}_f^{\mathrm{nb}}(M) \subset [0, \mathrm{Max}(r_0, \varphi_{L/K}\psi_{L/K_0}(r_M))]$$

Since $\varphi_{L/K} : [-1, +\infty[\longrightarrow \mathbb{R}$ is concave, satisfies $\varphi_{L/K}(0) = 0$ and is equal to the identity on $[-1, 0]$, we have

$$\varphi_{L/K}\psi_{L/K_0}(r_M) \leqslant \psi_{L/K_0}(r_M)$$

and we obtain (2.3.5) by setting $C_M := \psi_{L/K_0}(r_M)$. □

3. Proof of Theorem 1

3.1. Preliminary. — Let us consider the affine line $\mathbb{A}^1_S \longrightarrow S$ over S. Let s' be the generic point of $\mathbb{A}^1_s$ and S' the strict henselianization of $\mathbb{A}^1_S$ at s'. We denote by $\overline{S}$ the normalization of S in $\overline{\eta}$, by κ the function field of the strict henselianization of $\mathbb{A}^1_{\overline{S}}$ at s', and by $\overline{\kappa}$ an algebraic closure of κ. We have $\kappa \simeq K' \otimes_K \overline{K}$ and

$$G_K \simeq \mathrm{Gal}(\kappa/K') \tag{3.1.1}$$

Let L/K be a finite Galois extension of K in $\overline{K}$. Set $L' := K' \otimes_K L$. At finite level, (3.1.1) reads

$$\mathrm{Gal}(L/K) \xrightarrow{\sim} \mathrm{Gal}(L'/K') \tag{3.1.2}$$

$$g \longrightarrow \mathrm{id} \otimes g \tag{3.1.3}$$

Since a uniformizer in S_L is also a uniformizer in $S'_{L'}$, we deduce that (3.1.2) is compatible with the lower-numbering ramification filtration on $\mathrm{Gal}(L/K)$ and $\mathrm{Gal}(L'/K')$. Hence, (3.1.2) is compatible with the upper-numbering ramification filtration on

$\mathrm{Gal}(L/K)$ and $\mathrm{Gal}(L'/K')$. We deduce that through (3.1.1), the canonical surjection $G_{K'} \longrightarrow G_K$ is compatible with the upper-numbering ramification filtration.

3.2. The proof. — We can suppose that $\mathcal{F}$ is concentrated in degree 0. In case $\dim X = 0$, there is nothing to prove. We first reduce the proof of Theorem 1 to the case where $\dim X = 1$ by arguing by induction on $\dim X$.

Since the problem is local on X, we can suppose that X is affine. We thus have a digram

$$\begin{array}{ccccc} X & \longrightarrow & \mathbb{A}^n_S & \hookrightarrow & \mathbb{P}^n_S \\ & \searrow & \downarrow & \swarrow & \\ & & S & & \end{array} \tag{3.2.1}$$

Let $\overline{X}$ be the closure of X in $\mathbb{P}^n_S$ and let $j : X \hookrightarrow \overline{X}$ be the associated open immersion. Replacing $(X, \mathcal{F})$ by $(\overline{X}, j_!\mathcal{F})$, we can suppose X/S projective. Then Theorem 1 is a consequence of the following assertions

(A) There exists a finite set $E_A \subset \mathbb{R}_{\geqslant 0}$ such that for every $N \in \mathrm{Sh}_c(\eta, \overline{\mathbb{Q}}_\ell)$ with slope not in E_A, the support of $\psi^t_f(\mathcal{F} \otimes f^*N)$ is punctual.

(B) There exists a finite set $E_B \subset \mathbb{R}_{\geqslant 0}$ such that for every $N \in \mathrm{Sh}_c(\eta, \overline{\mathbb{Q}}_\ell)$ with slope not in E_B, we have
$$R\Gamma(X_s, \psi^t_f(\mathcal{F} \otimes f^*N)) \simeq 0$$

Let us prove (A). This is a local statement on X, so we can suppose X to be a closed subset in $\mathbb{A}^n_S$ and consider the factorisations

$$\begin{array}{ccc} X & \xrightarrow{p_i} & \mathbb{A}^1_S \\ & {\scriptstyle f}\searrow & \downarrow \\ & & S \end{array}$$

where p_i is the projection on the i-th factor of $\mathbb{A}^n_S$. Using the notations in 3.1, let X'/S' making the upper square of the following diagram

$$\begin{array}{ccc} X' & \xrightarrow{\lambda} & X \\ {\scriptstyle p'_i}\downarrow & & \downarrow{\scriptstyle p_i} \\ S' & \longrightarrow & \mathbb{A}^1_S \\ & {\scriptstyle h}\searrow & \downarrow \\ & & S \end{array}$$

cartesian. Let us set $\mathcal{F}' := \lambda^*\mathcal{F}$ and $N' := h^*N$. From [**Del77**, Th. finitude 3.4], we have

$$\lambda^*\psi_f(\mathcal{F} \otimes f^*N) \simeq \psi_{hp'_i}(\mathcal{F}' \otimes p'^*_i N') \simeq \psi_{p'_i}(\mathcal{F}' \otimes p'^*_i N')^{G_\kappa} \tag{3.2.2}$$

where G_κ is a pro-p group sitting in an exact sequence

$$1 \longrightarrow G_\kappa \longrightarrow G_{K'} \longrightarrow G_K \longrightarrow 1$$

In particular, G_κ is a subgroup of the wild-ramification group $P_{K'}$ of $G_{K'}$. So applying the $P_{K'}$-invariants on (3.2.2) yields

$$\lambda^* \psi_f^t(\mathcal{F} \otimes f^* N) \simeq \psi_{p'_i}^t(\mathcal{F}' \otimes p_i'^* N') \tag{3.2.3}$$

If N has pure slope r, we know from 3.1 that N' has pure slope r as a sheaf on η'. Applying the recursion hypothesis gives a finite set $E_i \subset \mathbb{R}_{\geqslant 0}$ such that the right-hand side of (3.2.3) is 0 for N of slope not in E_i. The union of the E_i for $1 \leqslant i \leqslant n$ is the set E_A sought for in (A).

To prove (B), we observe that the compatibility of ψ_f^t with proper morphisms and the projection formula give

$$R\Gamma(X_s, \psi_f^t(\mathcal{F} \otimes f^* N)) \simeq \psi_{\mathrm{id}}^t(Rf_*\mathcal{F} \otimes N)$$

By 2.1.1, the set $E_B := \mathrm{Sl}(Rf_*\mathcal{F})$ has the required properties.

We are thus left to prove Theorem 1 in the case where $\dim X = 1$. At the cost of localizing, we can suppose that X is local and maps surjectively on S. Let x be the closed point of X. Note that $k(x)/k(s)$ is of finite type but may not be finite. Choosing a transcendence basis of $k(x)/k(s)$ yields a factorization $X \longrightarrow S' \longrightarrow S$ satisfying $\mathrm{trdeg}_{k(s')} k(x) = \mathrm{trdeg}_{k(s)} k(x) - 1$.

So we can further suppose that $k(x)/k(s)$ is finite. Since $k(s)$ is algebraically closed, we have $k(x) = k(s)$. If $\widehat{S}$ denotes the completion of S at s, we deduce that $X \times_S \widehat{S}$ is finite over $\widehat{S}$. By faithfully flat descent [**Gro71**, VIII 5.7], we obtain that X/S is finite. We conclude the proof of Theorem 1 with 2.1.2 (1).

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