TENSOR PRODUCT AND IRREGULARITY FOR HOLONOMIC $\mathcal{D}$-MODULES

by

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Introduction

Let X be a complex variety and let $D^b_{\mathrm{hol}}(\mathcal{D}_X)$ be the derived category of complexes of $\mathcal{D}_X$-modules with bounded holonomic cohomology. It is known [**Meb04**, 6.2-4] that for a regular complex [1] $\mathcal{M} \in D^b_{\mathrm{hol}}(\mathcal{D}_X)$, the derived tensor product $\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{M}$ is regular. The goal of this note is to prove the following

Theorem 1. — *Let $\mathcal{M} \in D^b_{\mathrm{hol}}(\mathcal{D}_X)$ and suppose that $\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{M}$ is regular. Then $\mathcal{M}$ is regular.*

The technique used in this text is similar to that used in [**Tey14**], and proceed by recursion on the dimension of X. The main tool is a sheaf-theoretic measure of irregularity [**Meb90**].

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0.1. For every morphism $f : Y \longrightarrow X$ with X and Y complex varieties, we denote by $f^+ : D^b_{\mathrm{hol}}(\mathcal{D}_X) \longrightarrow D^b_{\mathrm{hol}}(\mathcal{D}_Y)$ and $f_+ : D^b_{\mathrm{hol}}(\mathcal{D}_Y) \longrightarrow D^b_{\mathrm{hol}}(\mathcal{D}_X)$ the inverse image and direct image functors for $\mathcal{D}$-modules. We define $f^\dagger := f^+[\dim Y - \dim X]$.

0.2. If Z is a closed analytic subspace of X, we denote by $\mathrm{Irr}^*_Z(\mathcal{M})$ the irregularity sheaf of $\mathcal{M}$ along Z [**Meb90**].

1. that is, a complex with regular cohomology modules.

1. The proof

1.1. The 1-dimensional case. — We suppose that X is a neighbourhood of the origin $0 \in \mathbb{C}$ and we prove the following

Proposition 1.1.1. — *Let $\mathcal{M} \in D^b_{\mathrm{hol}}(\mathcal{D}_X)$ so that $\mathcal{H}^k\mathcal{M}$ is a smooth connexion away from 0 for every $k \in \mathbb{N}$. If $\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{M}$ is regular, then $\mathcal{M}$ is regular.*

The complex

$$(\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{M})(*0) \simeq \mathcal{M}(*0) \otimes_{\mathcal{O}_X} \mathcal{M}(*0)$$

is regular. Since we are in dimension one, the regularity of $\mathcal{H}^k\mathcal{M}$ is equivalent to the regularity of $\mathcal{H}^k\mathcal{M}(*0)$. Thus, we can suppose that $\mathcal{M}$ is localized at 0. In particular, the $\mathcal{H}^k\mathcal{M}$ are flat $\mathcal{O}_X$-modules, so the only possibly non zero terms in the Künneth spectral sequence

$$E_2^{pq} = \bigoplus_{i+j=q} \mathrm{Tor}^p_{\mathcal{O}_X}(\mathcal{H}^i\mathcal{M}, \mathcal{H}^j\mathcal{M}) \Longrightarrow \mathcal{H}^{p+q}(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{M}) \tag{1.1.2}$$

sit on the line $p = 0$. Hence, (1.1.2) degenerates at page 2 and induces a canonical identification

$$\mathcal{H}^k(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{M}) \simeq \bigoplus_{i+j=k} (\mathcal{H}^i\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{H}^j\mathcal{M})$$

for every k. In particular, the module $\mathcal{H}^i\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{H}^i\mathcal{M}$ is regular for every i. Hence, one can suppose that $\mathcal{M}$ is a germ of meromorphic connexions at 0. By looking formally at 0, one can further suppose that $\mathcal{M}$ is a $\mathbb{C}((x))$-differential module. In this case, the regularity of $\mathcal{M}$ is a direct consequence of the Levelt-Turrittin decomposition theorem [**Sv00**].

1.2. Proof of theorem 1 in higher dimension. — We proceed by recursion on the dimension of X and suppose that $\dim X > 1$. For every point $x \in X$ taken away from a discrete set of points $S \subset X$, one can find a smooth hypersurface $i : Z \longrightarrow X$ passing through x which is non characteristic for $\mathcal{M}$. Since regularity is preserved by inverse image, the complex

$$i^+\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{O}_X} i^+\mathcal{M}$$

is regular. By recursion hypothesis, we deduce that $i^+\mathcal{M}$ is regular. From [**Tey14**, 3.3.2], we obtain

$$\mathrm{Irr}^*_x(\mathcal{M}) \simeq \mathrm{Irr}^*_x(i^+\mathcal{M}) \simeq 0$$

Since regularity can be punctually tested [**Meb04**, 6.2-6], we deduce that $\mathcal{M}$ is regular away from S. In what follows, one can thus suppose that X is a neighbourhood of the origin $0 \in \mathbb{C}^n$ and that $\mathcal{M}$ is regular away from 0.

Let us suppose that 0 is contained in an irreducible component of $\mathrm{Supp}\,\mathcal{M}$ of dimension > 1. Let Z be an hypersurface containing 0 and satisfying the conditions

(1) $Z \cap \mathrm{Supp}\,\mathcal{M}$ has codimension 1 in $\mathrm{Supp}\,\mathcal{M}$.

(2) The modules $\mathcal{H}^k\mathcal{M}$ are smooth [2] away from Z.

2. That is, $\mathrm{Supp}(\mathcal{H}^k\mathcal{M})$ is smooth away from Z and the characteristic variety of $\mathcal{H}^k\mathcal{M}$ away from Z is the conormal bundle of $\mathrm{Supp}(\mathcal{H}^k\mathcal{M})$ in X.

(3) $\dim \operatorname{Supp} R\Gamma_{[Z]}\mathcal{M} < \dim \operatorname{Supp} \mathcal{M}$.

Such an hypersurface always exists by [**Meb04**, 6.1-4]. According to the fundamental criterion for regularity [**Meb04**, 4.3-17], the complex $\mathcal{M}(*Z)$ is regular. From the local cohomology triangle

$$R\Gamma_{[Z]}\mathcal{M} \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}(*Z) \xrightarrow{+1}$$

we deduce that one is left to prove that $R\Gamma_{[Z]}\mathcal{M}$ is regular. There is a canonical isomorphism

$$R\Gamma_{[Z]}\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{O}_X} R\Gamma_{[Z]}\mathcal{M} \simeq R\Gamma_{[Z]}(\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{M}) \tag{1.2.1}$$

Since $R\Gamma_{[Z]}$ preserves regularity, the left hand side of (1.2.1) is regular. So one is left to prove theorem 1 for $R\Gamma_{[Z]}\mathcal{M}$, with $\dim \operatorname{Supp} R\Gamma_{[Z]}\mathcal{M} < \dim \operatorname{Supp} \mathcal{M}$. By iterating this procedure if necessary, one can suppose that the components of $\operatorname{Supp} \mathcal{M}$ containing 0 are curves. We note $C := \operatorname{Supp} \mathcal{M}$. At the cost of restricting the situation to a small enough neighbourhood of 0, one can suppose that C is smooth away from 0. Let $p : \widetilde{C} \longrightarrow X$ be the composite of normalization map for C and the canonical inclusion $C \longrightarrow X$. By Kashiwara theorem [**HTT00**, 1.6.1], the canonical adjunction [**Meb89**, 7.1]

$$p_+p^\dagger\mathcal{M} \longrightarrow \mathcal{M} \tag{1.2.2}$$

is an isomorphism away from 0. So the cone of (1.2.2) is supported at 0. Hence, it is regular. One is then left to show that $p_+p^\dagger\mathcal{M}$ is regular. Since regularity is preserved by proper direct image, we are left to prove that $p^\dagger\mathcal{M}$ is regular. There is a canonical isomorphism

$$p^\dagger\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{O}_{\widetilde{C}}} p^\dagger\mathcal{M} \simeq p^\dagger(\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{M}) \tag{1.2.3}$$

So the left hand side of (1.2.3) is regular and one can apply 1.1.1, which concludes the proof of theorem 1.

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